54th Mathematical Olympiad in Poland

Problems of the first round, September – December 2002

1. Determine all pairs of positive integers x, y satisfying the equation $(x+y)^2 - 2(xy)^2 = 1$.

2. A real number a_1 is given. The sequence (a_n) is defined by $a_{n+1} = a_n^2 - a_n + 1$ for $n \geq 1$. Prove that for all positive integers n the following inequality
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{1}{a_1 - 1} \quad \text{holds.}$$

3. Three different points A, B, C lie on a circle o. The tangent lines to o at the points A and B intersect in the point P. The tangent line to o at C intersects the line AB in the point Q. Prove that
$$PQ^2 = PB^2 + QC^2 .$$

4. Consider the set containing all sequences of the length k with values in the set $\{1, 2, \ldots, m\}$. From each sequence the smallest value is chosen and all these chosen numbers are summed together. Prove that the sum is equal to $1^k + 2^k + 3^k + \cdots + m^k$.

5. A positive integer n_1 in the decimal expansion contains 333 digits each of them does not equal to zero. For $i = 1, 2, \ldots, 332$ the positive integer n_{i+1} is obtained from n_i by moving the last digit of n_i to the beginning. Prove that either 333 divides all the numbers $n_1, n_2, \ldots, n_{333}$ or 333 does not divide any of these numbers.

6. Points A, B, C, D lie in this order on a circle o. Let M be the midpoint of the arc AB of o which does not contain the points C and D, and N be the midpoint of the arc CD of o which does not contain the points A and B. Prove that
$$\frac{AN^2 - BN^2}{AB} = \frac{DM^2 - CM^2}{CD} .$$

7. On a meeting at aunt Renia met n persons (counting the aunt too). Each person gave at least one other person at least one present. Each person, except aunt Renia, obtained three times less presents as he gave out. The aunt obtained six times more presents as she gave out. Determine the smallest number of presents which aunt Renia could have obtained.

8. In a tetrahedron $ABCD$ the points M and N are the midpoints of the edges AB and CD, respectively. The point P lies on the segment MN and satisfies: $MP = CN$ and $NP = AC$. The point O is the center of the circumsphere of $ABCD$. Prove that if $O \neq P$ then $OP \perp MN$.

9. Find all polynomials W with real coefficients possessing the following property: if $x + y$ is a rational number, then the number $W(x) + W(y)$ is rational as well.

10. A deck of 52 cards labeled by the numbers $1, 2, \ldots, 52$ is given. A permutation $\pi : \{1, 2, \ldots, 52\} \to \{1, 2, \ldots, 52\}$ of these cards is called a shuffle if there exists a positive integer $m \leq 51$ such that $\pi(i) < \pi(i+1)$ for all $i < m$ and for all $i \in \{m+1, m+1, \ldots, 51\}$. Prove or disprove that starting from a fixed order of the cards every other permutation can be obtained after 5 shuffles.

11. A convex quadrilateral $ABCD$ is given. The points P and Q different from the corners of $ABCD$, lie on the sides BC and CD, respectively, and satisfy the condition $\sphericalangle BAP = \sphericalangle DAQ$. Prove that the triangles ABP and ADQ have the same areas if and only if their orthocenters lie on a line perpendicular to AC.

12. For positive real numbers a, b, c, d let us denote $A = a^3 + b^3 + c^3 + d^3$, $B = bcd + cda + dab + abc$. Prove that
$$(a + b + c + d)^3 \leq 4A + 24B .$$