48-th Mathematical Olympiad in Poland

First Round, September–December, 1996

1. Solve the system of equations:

$$\begin{cases} x \cdot |x| + y \cdot |y| = 1 \\ [x] + [y] = 1\,. \end{cases}$$

Note: $[t]$ denotes the biggest integer not bigger than t.

2. Point P is an interior point of a parallelogram $ABCD$, such $\angle ABP = \angle ADP$. Prove that $\angle PAB = \angle PCB$.

3. Prove that for any real numbers $a, b \geq 1$, $c \geq 0$ and for any integer $n \geq 1$ the inequality $(ab+c)^n - c \leq ((b+c)^n - c)a^n$ holds.

4. Prove that a natural number $n \geq 2$ is composite if and only if there exist natural numbers $a, b, x, y \geq 1$ with $a + b = n$ and $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

5. The angle bisectors of the internal angles A, B, C of triangle ABC intersect the opposite sides at points D, E, F, respectively and the circumcircle of ABC at K, L, M, respectively. Prove that

$$\frac{AD}{DK} + \frac{BE}{EL} + \frac{CF}{FM} \geq 9\,.$$

6. Polynomial $P(x)$ of degree n satisfies the following condition:

$$P(k) = \frac{1}{k} \qquad \text{for} \quad k = 1, 2, 4, 8, \ldots, 2^n\,.$$

Find $P(0)$.

7. Find the supremum of volumes of tetrahedra contained in a ball of given radius, whose one edge is a diameter of the ball.

8. Let a_n denote the number of all nonempty subsets of $\{1, 2, \ldots, 6n\}$, whose sum of elements gives the remainder 5 upon division by 6; and let b_n be the number of all nonempty subsets of $\{1, 2, \ldots, 7n\}$, whose product of elements gives the remainder 5 upon division by 7. Find the ratio a_n/b_n.

9. Determine all functions $f\colon [1;\infty) \to [1;\infty)$ satisfying the following two conditions:

(i) $f(x+1) = \dfrac{(f(x))^2 - 1}{x}$ for $x \geq 1$; (ii) the function $g(x) = f(x)/x$ is bounded.

10. Points P, Q lie inside an acute triangle ABC and satisfy $\angle ACP = \angle BCQ$ and $\angle CAP = \angle BAQ$. Points D, E, F are the orthogonal projections from P onto the lines BC, CA, AB, respectively. Prove that the angle DEF is right if and only if Q is the orthocenter of triangle BDF.

11. Given a natural number $m \geq 1$ and a polynomial $P(x)$ of positive degree with integer coefficients. Prove that if $P(x)$ has at least three distinct integer roots, then $P(x) + 5^m$ has at most one integer root.

12. A group consisting of n people noticed that, for some period of time, three of them might be going for a dinner together, each pair meeting at exactly one dinner. Prove that n gives the remainder 1 or 3 upon division by 6.