48-th Mathematical Olympiad in Poland

Final Round, April 4–5, 1997

First Day

1. Positive integers x_1, x_2, x_3, x_4, x_5, x_6 x_7 satisfy the conditions:

$$x_6 = 144 \qquad \text{and} \qquad x_{n+3} = x_{n+2}(x_{n+1} + x_n) \qquad \text{for} \quad n = 1, 2, 3, 4.$$

Determine x_7.

2. Find all triples of real numbers x, y, z satisfying the system of equations

$$\begin{cases} 3(x^2 + y^2 + z^2) = 1 \\ x^2y^2 + y^2z^2 + z^2x^2 = xyz(x + y + z)^3. \end{cases}$$

3. The medians of the lateral faces ABD, ACD, BCD of a pyramid $ABCD$ taken from the vertex D make equal angles with the edges they were led to. Prove that the area of each lateral face is less than the sum of the areas of the remaining lateral faces.

Second Day

4. The sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 0, \qquad a_n = a_{[n/2]} + (-1)^{n(n+1)/2} \qquad \text{for} \quad n > 1.$$

For each integer $k \geq 0$ determine the number of subscripts n satisfying the conditions

$$2^k \leq n < 2^{k+1}\,, \qquad a_n = 0\,.$$

Note: $[n/2]$ denotes the biggest integer not bigger than $n/2$.

5. Given is a convex pentagon $ABCDE$ with $DC = DE$ and $\angle DCB = \angle DEA = 90^\circ$. Let F be the interior point of segment AB determined by the condition $AF : BF = AE : BC$. Prove that $\angle FCE = \angle ADE$ and $\angle FEC = \angle BDC$.

6. On a circle of radius 1, n distinct points are given. Let q be the number of the line segments whose length is greater than $\sqrt{2}$ and whose endpoints lie in the given points. Prove that $3q \leq n^2$.