49th Mathematical Olympiad in Poland

Problems of the first round, September – December 1997

1. Solve the system of equations:

$$\begin{cases} |x-y| - \dfrac{|x|}{x} = -1 \\ |2x-y| + |x+y-1| + |x-y| + y - 1 = 0. \end{cases}$$

2. The lines containing the sides of triangle ABC, inscribed in a circle of center O, intersect at H. It is known that $AO = AH$. Find the measure of the angle CAB.

3. The sequences (a_n), (b_n), (c_n) are given by the conditions: $a_1 = 4$, $a_{n+1} = a_n(a_n - 1)$, $2^{b_n} = a_n$, $2^{n-c_n} = b_n$ for $n = 1, 2, 3, \ldots$. Prove that the sequence (c_n) is bounded.

4. Given a positive number a. Determine all real numbers c with the following property: for any pair of positive numbers x, y the following inequality holds

$$(c-1)x^{a+1} \le (cy - x)y^a.$$

5. Given is an integer $n \ge 1$. Solve the equation:

$$|\tan^n x - \cot^n x| = 2n|\cot 2x|.$$

6. In triangle ABC with $AB > AC$, D is the midpoint of the side BC; E lies on the side AC. Points P and Q are the feet of the perpendiculars from B and E, respectively to the line AD. Prove that $BE = AE + AC$ if and only if $AD = PQ$.

7. Given positive integers m, n. Set $A = \{1, 2, 3, \ldots, n\}$. Determine the number of functions $f\colon A \to A$ attaining exactly m values and satisfying the condition

$$\text{if } k, \ell \in A,\ k \le \ell, \text{ then } f(f(k)) = f(k) \le f(\ell).$$

8. Determine if there exists a convex polyhedron having exactly k edges and a plane not passing through any of its vertices and cutting r edges with $3r > 2k$.

9. Let $a_0 = 0{,}91$ and $a_k = 0{,}\underbrace{99\ldots9}_{2^k}\underbrace{00\ldots0}_{2^k-1}1$ for $k = 1, 2, 3, \ldots$. Compute

$$\lim_{n\to\infty} (a_0 a_1 \ldots a_n).$$

10. The medians AD, BE, CF of triangle ABC intersect at G. The quadrilaterals $AFGE$ and $BDGF$ are cyclic. Prove that the triangle ABC is equilateral.

11. In a tennis tournament n players take part. Everyone has played against everyone else, there were no draws. Prove that there exists a player A who won with every player B directly or indirectly, i.e. either A won with B or A won with some player C, who had won with B.

12. Let $g(k)$ be the biggest prime divisor of an integer k if $|k| \ge 2$, and let $g(-1) = g(0) = g(1) = 1$. Determine if there exists a polynomial W of positive degree with integer coefficients for which the set of the numbers of the form $g(W(x))$ (x — an integer) is finite.