49-th Mathematical Olympiad in Poland

Final Round, April 24–25, 1998

First Day

1. Find all integers (a, b, c, x, y, z) which satisfy the system of equations

$$\begin{cases} a+b+c = xyz \\ x+y+z = abc \end{cases}$$

and the conditions $a \geq b \geq c \geq 1\,, x \geq y \geq z \geq 1\,.$

2. The Fibonacci sequence (F_n) is given by:

$$F_0 = F_1 = 1\,, \quad F_{n+2} = F_{n+1} + F_n \quad \text{for } n = 0, 1, 2 \ldots.$$

Determine all pairs (k, m) of integers, with $m > k \geq 0$, for which the sequence (x_n) defined by

$$x_0 = \frac{F_k}{F_m}\,, \quad x_{n+1} = \begin{cases} \dfrac{2x_n - 1}{1 - x_n} & \text{for } x_n \neq 1\,, \\ 1 & \text{for } x_n = 1 \end{cases} \qquad (n = 0, 1, 2 \ldots)$$

contains the number 1.

3. The convex pentagon $ABCDE$ is the base of the pyramid $ABCDES$. A plane intersects the edges SA, SB, SC, SD, SE in points A', B', C', D', E' respectively which differ from the vertices of the pyramid. Prove that the intersection points of the diagonals of the quadrangles $ABB'A'$, $BCC'B'$, $CDD'C'$, $DEE'D'$, $EAA'E'$ are coplanar.

Second Day

4. Prove that the sequence (a_n) defined by

$$a_1 = 1\,, \quad a_n = a_{n-1} + a_{[n/2]} \quad \text{for } n = 2, 3, 4 \ldots$$

contains infinitely many integers divisible by 7.

Note: $[n/2]$ denotes the biggest integer not bigger than $n/2$.

5. Points D, E lie on the side AB of the triangle ABC and satisfy

$$\frac{AD}{DB} \cdot \frac{AE}{EB} = \left(\frac{AC}{CB}\right)^2.$$

Prove that $\angle ACD = \angle BCE$.

6. Consider unit squares in the plane whose vertices have integer coordinates. Let S be the chessboard which contains all unit squares lying entirely inside the circle $x^2 + y^2 \leq 1998^2$. In all the squares of the chessboard S we write $+1$. A move consists of reversing the signs of a row or of a column or of a diagonal of the chessboard S. (Diagonals are formed by the squares of S whose centers lie on the lines intersecting rows and columns under the angle $45°$). The goal is to reach -1 in exactly one unit square of S. Find out whether it is possible.