Mathematical Olympiad in Poland
Second Round, February 26–27, 1999

First Day

1. Let $f\colon \langle 0,1\rangle \to \mathbb{R}$ be a given function, such that $f(\frac{1}{n}) = (-1)^n$ for $n = 1, 2, \ldots$. Prove that there do not exist increasing functions $g\colon \langle 0,1\rangle \to \mathbb{R}$, $h\colon \langle 0,1\rangle \to \mathbb{R}$, such that $f = g - h$.

2. The cube S with edge length 2 consists of 8 unit cubes. We will call the cube S with one unit cube removed *a piece*. The cube T with edge length 2^n consists of $(2^n)^3$ unit cubes. Prove that if one unit cube is removed from T, then the remaining solid can be built with pieces.

3. The convex quadrilateral $ABCD$ is inscribed in a circle. The points E and F lie on the sides AB and CD respectively and satisfy the condition $AE : EB = CF : FD$. The point P lies on the straight segment EF and satisfies $EP : PF = AB : CD$. Prove that the ratio of the areas of the triangles APD and BPC does not depend on the choice of the points E and F.

Second Day

4. Point P lies inside the triangle ABC and satisfies $\sphericalangle PAB = \sphericalangle PCA$ and $\sphericalangle PAC = \sphericalangle PBA$. The point O is the center of the circumcircle of the triangle ABC. Prove that $\sphericalangle APO$ is a right angle if $O \neq P$.

5. Let $S = \{1, 2, 3, 4, 5\}$. Find out how many functions $f\colon S \to S$ exist with the following property: $f^{50}(x) = x$ for all $x \in S$.

Note: $f^{50}(x) = \underbrace{f \circ f \circ \ldots \circ f}_{50}(x)$.

6. Let the positive integer $k \geq 2$ and the integers $a_1, a_2, \ldots, a_n$ be given and have the following properties:

$$a_1 + 2^i a_2 + 3^i a_3 + \ldots + n^i a_n = 0\,, \qquad \text{for } \ i = 1, 2, \ldots, k-1.$$

Prove that the number $a_1 + 2^k a_2 + 3^k a_3 + \ldots + n^k a_n$ is divisible by $k!$.