The Examinations for the International Math Olympiad in 1993

First Examination Constanta 1993

Problem 1

Consider the sequence $a_n = (1+i)...(n+i)$. Prove that there exist infinitely many integers n such that Im $a_n > 0$ and infinitely many integers n such that Im $a_n > 0$.

Problem 2

Consider the triangle ABC inscribed in a circle of radius R and circumscribed to a circle of radius r and let $d = \frac{rR}{r+R}$. Prove that there exists some points D, E, F such that for all points M inside the triangle ABC there exists a point X on the sides of the triangle DEF such that $MX \leq d$.

Problem 3

Consider X = $\{1, 2, \ldots\ 2^n\ \}$. Prove that there exists a partition of X into two disjoint classes such that none of them contains an arithmetic progression of length 2n.

Problem 4

Prove that the equation $(x+y)^n = x^m + y^m$ has a unique integer solution (x_0, y_0, m_0, n_0) with $x_0 > y_0 > 0, m_0 > 1$ and $n_0 > 1$.

Second Examination Bucharest 1993

Problem 1

Find the greatest number A > 0 such that for any x, y, z positive real numbers we have

$$\frac{x}{\sqrt{y^2+z^2}} + \frac{y}{\sqrt{x^2+z^2}} + \frac{z}{\sqrt{x^2+y^2}} > A$$

Problem 2

For x, y, z integers such that $x^2 + y^2 + z^2 = 1993$ prove that $x + y + z$ is not a perfect square.

Problem 3

If $ABCDEF$ is a convex hexagon such that the diagonals AD, BE and CF divide the hexagon into polygons with the same area and perimeters prove that AD, BE, CF are concurrent. Is it true that the hexagon has a center of symmetry?

Problem 4

Consider an integer $n > 1$ and $X = \{1, 2...n\}$, Y the set of all subsets of X. Prove that for any function f: Y$\longrightarrow$ X there exists A and B subsets of X such that $f(A) = f(B) = max A\Delta B$ where $A\Delta B = (A\backslash B) \cup (B\backslash A)$ is the symmetric difference of A and B.

Third Examination Bucharest 1993

Problem 1

Consider a real function f defined on the set of positive real numbers strictly increasing and such that $f(\frac{x+y}{2}) < \frac{f(x)+f(y)}{2}$ for all x, y distinct positive real numbers. Consider the sequence $a_n = f(n)$. Prove that a_n does not contain an infinite arithmetic progression.

Problem 2

For $m > n > 1$ positive integers such that $gcd(m, n) = 1$ consider the polynomials $f = x^{m+n} - x^{m+1} - x + 1$ and $g = x^{m+n} + x^{n+1} - x + 1$ Assume that f and g have a common divisor of degree > 1. Find this divisor.

Problem 3

Find all $n > 1$ such that there exists a set $\mathfrak{P}$ of n points in the plane such that for any $A \in \mathfrak{P}$ there are three points $X, Y, Z \in \mathfrak{P}$ such that $AX = AY = AZ = 1$

Problem 4

For n > 3 find all quadruplets (n_1, n_2, n_3, n_4) that maximize the expression

when $n_1 + n_2 + n_3 + n_4 = n$ and $n_i > 0$ are positive integers.

Fourth Examination Bucharest 1993

Problem 1

The sequence x_n is defined as follows : the first term is 1, the next two are the next even numbers namely 2 and 4, the next three are the next odd numbers namely 5, 7, 9, the next four the next four even numbers and so on. Express x_n as a function of n.

Problem 2

Consider the triangle ABC and D, E, F points on its sides such that $BD = CE = AF$ and $m(BAD) = m(CBE) = m(ACF)$. Prove that ABC is equilateral.

Problem 3

Take $p > 3$ a prime integer and consider a partition of the set $\{1, 2, ...p-1\}$ into three disjoint sets A, B, C. Prove that there exist a, b, c each of them in each of the sets A, B, C such that $b + c \equiv a (modp)$.