Romanian Math Olympiad 1994

First Examination - Arad

PROBLEM 1

If x and y are positive integers with no common prime factors and n is a square free integer prove that the number $x^n + y^n$ is not divisible by $(x+y)^3$.

PROBLEM 2

Let n be a positive integer greater than 4. Find the maximum area of an n-gon inscribed in a circle of radius 1 that has two diagonals which are perpendicular.

PROBLEM 3

Take $\mathfrak{M}$ the set of all maps f defined on the set $\{1, 2, ...11\}$ with values in the set of all sets consisiting of 5 positive integers such that $f(i)$ and $f(j)$ are not disjoint for all $i, j = 1...11$. Find the value of

$$\min_{f \in \mathfrak{M}} \max_{n \in \mathbb{Z}} a_n(f)$$

where $a_n(f)$ is the number of elements of the set $\{1 \leq i \leq 11 \mid n \in f(i)\}$

PROBLEM 3

Let $A_1A_2A_3A_4$ be a thetraedron and M a point inside. Define N_{ij} the symmetric of M in respect with the middlepoint of the edge A_iA_j. A "special" point X is a point whose all projections on the thetraedron's edges are in the same plane which we denote by $p(X)$. Prove that if all N_{ij} are special then M belongs to all the planes $p(N_{ij})$.

Second Examination - Bucharest

PROBLEM 1

Take $n > 1$ and $X = \{1, 2, ...n\}$. For $m : X \to X$ an arbitrary map of X into X define $m(A) = \sum_{a \in A} m(a)$. A set A is called odd if $m(A)$ is odd. A set that is not odd is called even. By convention $m(\emptyset) = 0$.
a)Prove that the number of odd and even sets is the same.
b)Prove that the sum of $m(A)$ when A is odd and the sum of $m(A)$ when A is even is the same. What is the value of this sum?

PROBLEM 2

Prove that for any prime integer p,

$$(p-1)^p + 1 \mid p\left((p-1)^{(p-1)^p+1} + 1\right)$$

Is the property true for all integeres p?

PROBLEM 3

Prove that the sequence $a_n = 3^n - 2^n$ does not contain three terms in geometric progresion.

PROBLEM 4

Given a triangle ABC find the maximum area of all equilateral triangles inscribed in ABC i.e. of all triangles XYZ such that $X \in (BC)$, $Y \in (AC)$ and $Z \in (AB)$

Third Examination - Bucharest

PROBLEM 1

Define the sequence (a_n) such that $a_{n+1} = a_n + 7$ if a_n is odd and $a_{n+1} = \frac{a_n}{2}$ otherwise. If $a_1 = 1993^{1994^{1995}}$ find the least term in the sequence.

PROBLEM 2

Consider three spheres $\Sigma_1, \Sigma_2, \Sigma_3$ having the radii a, b, c such that Σ_2 is externally tangent to Σ_1 and Σ_3 and their centers lie on a line l. Moreover assume that Σ_1 and Σ_3 have no common points. The line s is tangent to all three spheres. Find the angle between s and l.

PROBLEM 3

Let $n > 2$ be an integer. Consider n numbers all equal to 0 or 1 and write them down in a row. Under that row we write another one. Between two 0's or two 1's we write down 0 and between a 0 and a 1 we write down 1. We get a row of $n-1$ 0's and 1's. If we carry until we have the row with one element we get a triangular table with 0's and 1's. Find the maximum number of 1's in that table.

PROBLEM 4

The quadrilater $ABCD$ is inscribed in a circle of radius R. The quadrilater $A_1B_1C_1D_1$ is inscribed in a circle of radius R_1 concentric with the first circle such that $R_1 > R$. Moreover A_1, B_1, C_1, D_1 lie on the halflines $(AB, (BC, (CD, (DA$. Prove that

$$\frac{\sigma(A_1B_1C_1D_1)}{\sigma(ABCD)} \geq \left(\frac{R_1}{R}\right)^2$$

Fourth Examination - Bucharest

PROBLEM 1

Let p be a prime number and $a_1, a_2, \ldots, a_{p+1}$ be real numbers. The set $\{a_1, a_2, \ldots, a_{p+1}\}$ has the property that no matter what element of the set we leave out, the remaining p elements can be partitioned into several disjoint subsets such that the mean of elements in all these subsets the same. Prove that $a_1 = a_2 = \cdots = a_{p+1}$.

PROBLEM 2

Let $n > 1$ be a positive integer. Find the number of polynomials f with coefficients in the set $\{0, 1, 2, 3\}$ such that $f(2) = n$.

PROBLEM 3

Solve the equation $x^n + y^n = 1994$ where x and y are integers and n is a positive integer.

The Balkan Olympiad 1994, Novi Sad

PROBLEM 1

Consider P a point inside the angle XOY. Construct,using only the ruler and the compasses, a line d passing through P and intersecting OX and OY in A, B such that the area of OAB is OP^2.

PROBLEM 2

Prove that the polynomial $x^4 - 1993x^3 + (1993 + m)x^2 - 11x + m$ has at most one integer root where m is an integer.

PROBLEM 3

Consider n a positive integer. For all the permutations $(a_1, a_2, ...a_n)$ of the set $\{1, 2, ...n\}$, find the maximum value of the sum :

$$\sum_{i=1}^{n-1} \mid a_i - a_{i+1} \mid$$

PROBLEM 4

Find the least integer $n > 4$ such that there exists a set of n persons with the following propreties:
1) any two persons that know each other have no common acquiatances
2) any two persons that do not know each other have exactly two common friends. It is assumed that if A has B as his friend then also B has A as his friend.