ROMANIAN MATH OLYMPIAD - 1995

11 TH GRADE

Time allowed 4 hours. Best score 21.5 out of 28. The most difficult problem 2. Because I actually took this contest a year ago some problems may be inaccurate. Sorry for any inconveniences. If something seems fishy to you please send me mail. I'll be happy to improve this page.

PROBLEM 1

Consider f a surjective real function such that for any sequence of real numbers $\{x_n\}_{n>0}$ such that $\{f(x_n)\}_{n>0}$ converges, $\{x_n\}_{n>0}$ itself should also converge. Prove that f is continuous.

PROBLEM 2

Consider k and n positive integers and $A_1, A_2, \ldots A_k$ real $n \times n$ square matrices. Prove that

$$det(A_1 \cdot A_1^t + A_2 \cdot A_2^t + \ldots A_k \cdot A_k^t) \geq 0$$

PROBLEM 3

Let A be a 3×2 matrix and B be 2×3 matrix such that

$$AB = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Find the possible values of $det(BA)$.

PROBLEM 4

Consider f a real continuous function such that it exists $M > 0$ such that for any real values of x and y

$$\|f(x+y) - f(x) - f(y)\| < M$$

a) Prove that for any reals x the limit $\lim_{n\to\infty} \frac{f(nx)}{n}$ exists and it is finite.
b) Call this the above limit $g(x)$. Prove that g is continuous in 0.
c) Prove that $\lim_{x\to\infty} \frac{f(x)}{x}$ exits.

FIRST EXAMINATION FOR IMO 1995

PROBLEM 1

If AD is the altitude of a triangle ABC and E, F are the centers of the inscribed circles in the triangles ABD and ACD, denote by K and L the intersections of EF with AB and AC. Prove that $AK = AL$ iff $m(A) = 90$ or if

$AB = AC$.

PROBLEM 2

Find all positive integers x, y, z, t such that $gcd(x, y) = gcd(y, z) = gcd(z, x) = 1$ and

$$(x + y)(y + z)(z + x) = txyz$$

PROBLEM 3

Consider $n > 6$ and $0 < p < \frac{n}{2}$ two integers. We paint the vertices of a regular n-gon in red and blue such that p vertices are red and the others are blue. Prove that we can find two congruent p-gons having the vertices among the vertices of the n-gon such that the p-gons have the all the vertices colored in red and in blue respectively.

PROBLEM 4

Consider n and m positive integers. How many polynomials of degree $2n-1$ having the coefficients in the set $\{1, 2, 3, ..n\}$ are divisible by $x^{n-1}+x^{n-2}+\cdots+1$?

SECOND EXAMINATION FOR IMO 1995

PROBLEM 1

Find all the positive integers a for which there exists a sequence of integers a_n such that $a_1 = 1, a_2 = a$ and for $n > 1$ we have $a_n = (2n+1)a_{n-1}-(n^2-1)an - 2$ and moreover a_i divides a_j for $i < j$

PROBLEM 2

Consider a convex poligon having the area and poligons of area placed inside the bigger poligon. Prove that there exist two of the smaller poligons intersecting area is at least 1.

PROBLEM 3

Let $ABCD$ be a convex quadrilater and M, N, P, Q points on the sides AB, BC, CD, DA such that $AM = BN = CP = DQ$. Prove that if $MNPQ$ is a square then $ABCD$ is also a square

PROBLEM 4

Consider p and n positive integers and a coloring of the plane in p colors. Prove that one can find two congruent regular n-gons having the vertices of the same color.

THIRD EXAMINATION FOR IMO 1995

PROBLEM 1

Consider $a_1, a_2, \ldots, a_n$ a sequence of positive distinct integers. Prove the inequality

$$\left(\sum_{i=1}^{n} a_i^5\right) + \left(\sum_{i=1}^{n} a_i^7\right) \geq 2\left(\sum_{i=1}^{n} a_i^3\right)^2$$

Prove that equality can be achieved.

PROBLEM 2

A cube is filled with parallelepipeds having the edges parallel to the edges of the cube. If the sum of all the volumes of the spheres circumscribed to the given parallelepipeds is the volume of the sphere circumscribed to the cube, prove that all the given parallelepipeds are cubes.

PROBLEM 3

Consider an irreducible monic polynomial with integer coefficients of odd degree greater than 1 and whose roots have the absolute value equal to 1. If $f(0)$ is square-free, prove that $f(X^3)$ is also irreducible.

PROBLEM 4

Prove or disprove that there exists a map from the positive integers to the positive integers such that for any positive integer n we have
a) $f(n) < n^8$
b) for any $a_1, a_2, \ldots, a_k$ distinct positive integers, $f(n) \neq f(a_1) + f(a_2) + \ldots + f(a_k)$

THE FOURTH EXAMINATION FOR IMO 1995

PROBLEM 1

Consider a regular n-gon whose vertices are painted in p colors such that no two neighbouring vertices have the same color. How many such such colorings exist up to rotations.

PROBLEM 2

For any positive integer n define $f(n)$ as the least common multiple of $1, 2, ...n$. Consider $k > 1$.

a) Find the sets X of consecutive positive integers with the largest cardinal such that f is strictly increasing on X.

b) Prove that there exists a set X of k consecutive positive integers such that f is constant on X.

PROBLEM 3

Find all the triangles having a prime number as the radius of the inscribed circle and the altitudes expressed by positive integers.

PROBLEM 4

Consider the quadrilater $ABCD$ and consider M, N, P, Q the centers of the squares constructed on its vertices in the exterior of the quadrilater. If $MNPQ$ is a square, prove that $ABCD$ is a rhombus.