**ROMANIAN MATH OLYMPIAD 1996 BUZAU**
**12th Grade**

**PROBLEM 1**

Let $G$ be a group and 1 its identity. If there are exactly two different elements in $G \setminus \{1\}$ such that they commute prove that G is isomorphic to either $\mathbb{Z}_3$ or $S_3$

**PROBLEM2**

Let $f$ be a monotone function $f : [a,b] \to \mathbb{R}$ such that for any $x_1, x_2 \ in[a,b]$ there exists a real number $c$ such that $\int_{x_1}^{x_2} f(t)dt = (x_2 - x_1)f(c)$
a) Prove that f is continuous.
b) Is f necessarily continuous if we do not assume that f is monotone?

**PROBLEM 3**

Let $A$ be a commutative ring such that $0 \neq 1$ and for any $x \in A \setminus \{0\}$ there exist two positive integers $m$ and $n$ such that $(x^m + 1)^n = x$. Prove that any morphism from $A$ into $A$ is a bijective.

**PROBLEM 4**

Consider $f : [0,1) \to \mathbb{R}$ a monotone function. Prove that the limits below exist and are equal

$$\lim_{x\to 1} \int_0^x f(t)dt = \lim_{n\to\infty} \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + ... + f\left(\frac{n-1}{n}\right)\right]$$

**XI TH GRADE**

**PROBLEM 1**

Let $I$ be an interval on the real line and $f : I \to \mathbb{R}$ be a differentiable real-valued function. Define $\mathfrak{J}$ the set of all values of expresion $\frac{f(b)-f(a)}{b-a}$ when $a, b \in I$. Prove that $\mathfrak{J}$ is also an interval and $\mathfrak{J} \setminus f(I)$ has at most two elements. Deduce that the derivative of f has the mean value property.

**PROBLEM 2**

a) Let $f_1, f_2, f_3 \dots f_n$ be real valued periodic functions and define $f = f_1 + f_2 + \dots f_n$. If $\lim_{x\to\infty} f(x)$ exists prove that $f$ is constant.
b) If $a_1, a_2, a_3$ are real numbers such that $a_1 + \cos a_1 x + a_2 \cos a_2 x + a_3 \cos a_3 x \geq 0$ for any real number $x$ then $a_1 = a_2 = a_3 = 0$.

**PROBLEM 3**

Let A and B be $2 \times 2$ square real matrices such that $det(AB + BA) \leq 0$. Prove that $det(A^2 + B^2) > 0$.

**PROBLEM 4**

Let $A, B, C, D$ be real matrices such that for any strictly positive integer $k$ we have $A^k B = C^k D$. Suppose that $A$ and $C$ are invertible. Prove that $B = D$.

**THE FIRST EXAMINATION FOR IMO**

Each problem is worth 7 points. Time allowed 4 and a half hours. The best score was 28 out of 28. The first problem appeared unjustifiably the most difficult one.The first 30 scores qualify for the next examination. A training session for the second examination is organized afterwards.

**PROBLEM 1**

Let $n > 2$ be an arbitrary integer and $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon in the plane $A_1 A_2 \dots A_n$ we have $f(A_1) + f(A_2) + \dots + f(A_n) = 0$. Prove that $f = 0$.

**PROBLEM 2**

Find the greatest positive integer $n$ for which there exist $x_1, x_2 \dots x_n$ positive integers such that for any integers $e_1, e_2 \cdot e_n \in \{-1, 0, 1\}$ not all 0, $n^3$ does not divide $e_1 x_1 + e_2 x_2 + \cdot + e_n x_n$.

**PROBLEM 3**

Let $x$ and $y$ be real numbers such that the set $\{cos(nx); +cos(ny) \mid n$ positive integer$\}$ is finite. Prove that $x$ and $y$ are rational multiples of $\pi$.

**PROBLEM 4**

Let $A, B, C, D$ be four points on a circle. Consider $\mathfrak{M}$ the set of all 16 centers of the circles tangent to the sides of each of the triangles $ABC, BCD, CDA.DAB$ repectively. Prove that there exists two sets $\mathfrak{K}$ and $\mathfrak{L}$ each of them consisting of four parallel lines such that every line in $\mathfrak{K}$ or $\mathfrak{L}$ contains exactly four of the points of $\mathfrak{M}$

**THE SECOND EXAMINATION**

Each problem is worth 7 points. Time allowed 4 hours and a half. The best score was 22 out of 28. The last problem was the most diffiult one. No complete solution was found during the examination.

**PROBLEM 1**

Let $OA$ and $OB$ be perpendicular rays in the circle $C$. Circles $C_1$ and $C_2$ are internally tangent to $C$ in $A$ and $B$, and a third circle $C_3$ is tangent externally to $C_1$ and $C_2$ in $S$ and $T$, and internally in $M$ to $C$. Find the measure of the angle $SMT$.

**PROBLEM 2**

Let $AB$ be the diameter of a circle $K$ of center $O$ and let $d$ be a line cutting $AB$ in $M$ and $K$ in $C$ and $D$ such that $MB > MA$ and $MD > MC$. If the circles passing through $A, O, C$ and $B, O, D$ intersect again in $P$ prove that $OP$ and $PM$ are perpendicular.

**PROBLEM 3**

Let $a$ be a real number and $f_1, f_2 \cdot f_n$ be additive real functions i.e. for any real numbers $x$ and $y$ we have $f(x+y) = f(x)+f(y)$. If $f_1(x)f_2(x)\cdot f_n(x) = ax^n$ then one of the functions has the form $f(x) = bx$ where $b$ is a real number.

**PROBLEM 4**

For any integer $n > 1$ we define the number $a_n = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m}$ where $p_1, p_2 \ldots p_m$ are the prime divisors of n. Prove that for any $N > 1$ we have the inequality

$$a_2 + a_2a_3 + \ldots + a_2a_3 \ldots a_N < 1$$

**THE THIRD EXAMINATION**

Each problem is worth 7 points. Time allowed 4 1/2 hours. This was the easiest examination as two students got the best score.

**PROBLEM 1**

Let $n > 2$ be an arbitrary integer. Find the maximum value of the expresion $x_1 + 4x_2 + 9x_3 + \ldots + n^2x_n$ when $(x_1, x_2, ...x_n)$ are non-negative integers subject to the conditions $x_1+x_2+\ldots+x_n = n$ and $x_1+2x_2+3x_3+\ldots+nx_n = 2n-2$.

**PROBLEM 2**

Let $r$ and $n$ be positive integers and $A$ a set of points in plane with integer coordinates such that any disc of radius $r$ contains at least one point of $A$ inside. Prove that for any coloring of the plane in $n$ colors there exist four points in $A$

having the same color which are the vertices of a rectangle.

**PROBLEM 3**

Find all the pairs of prime positive integers $(p, q)$ such that for any positive integer $a$, $3pq$ divides $a^{3pq} - a$.

**PROBLEM 4**

Let $p$ and $n$ be positive integers, $p > 2n$, $p$ prime and let $M$ be a set of $n$ points in the plane such that any three points of $M$ are not collinear. The function $f : M \to \{0, 1, ...p - 1\}$ has the following properties:

a) $f^{-1}(0)$ has exactly one element

b) for any circle $C$ passing through at least three points of $M$, the

$$\sum_{P \in M \cap C} f(P) \equiv 0 (mod p)$$

Prove that there is a circle containing all the points of M.

**THE FOURTH EXAMINATION**

The best score was 27.5 out of 28. The fourth problem was the most difficult although its proof is very easy. The first two problems were indeed trivia.

**PROBLEM 1**

Let $x_1, x_2 \ldots x_n$ be positive real numbers and let $x_{n+1} = x_1 + x_2 + \ldots x_n$. Prove that

$$\sqrt{(}x_{n+1} \sum_{i=1}^{n} (x_{n+1} - x_i)) \geq \sum_{i=1}^{n} \sqrt{(}x_i (x_{n+1} - x_i))$$

**PROBLEM 2**

Let $x, y, z$ be real numbers. Prove the equivalence of the following statements

a) $x, y, z > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 1$

b) for any $a, b, c, d$ the lengths of the sides of a convex quadrilater we have $a^2x + b^2y + c^2z > d^2$

**PROBLEM 3**

Let $M$ be the union of the sets of points laying on $n$ concentric circles. Let $f : M \to M$ be a function such that $d(f(A), f(B)) \geq d(A, B)$ Prove that for any $A$ and $B$ in $M$ the equality $d(A, B) = d(f(A), f(B))$ holds.

**PROBLEM 4**

Let $n > 2$ be a positive integer and $X$ be a subset of $\{1, 2, ...n^3\}$ with $3n^2$ elements. Prove that there exists nine different numbers $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ in $X$ such that the equations $a_1x + b_1y + c_1z = 0$ , $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ have a common solution $(x, y, z)$ where $x, y, z$ are all nonzero integers.