Romanian Math Olympiad 1997 - 11th grade

Problem 1

Consider p a positive integer and a square $n \times n$ matrix $A = (a_{ij})$ with positive entries. Prove that for any permutation ϕ of $\{1, 2, ...n\}$ there exists a function $f : \{1, 2, ...n\} \longrightarrow \{0, 1\}$ such that the matrix $B = (b_{ij})$ has the determinant not divisible by p where B is obtained by replacing the elements $a_{i\phi(i)}$ in the matrix A with $a_{i\phi(i)} + f(i)$.

Problem 2

Consider A an invertible square matrix with integer odd entries with odd dimension. Prove that it is not possible that all the minors of the elements of an arbitrary row to have the same absolute value.

Problem 3

Let $\mathfrak{F}$ be the set of differentiable real functions f such that $f(x) \geq f(x+sinx)$ for all real numbers x. Prove that $\mathfrak{F}$ contains non-constant functions. Also if f is in $\mathfrak{F}$ then f' is zero on an infinite subset of $\mathbb{R}$

Problem 4

Let f and g be two continuous bijective real functions such that $f(g^{-1}(x)) + g(f^{-1}(x)) = 2x$ for all real values of x and there is a real number a such that $f(a) = g(a)$. Prove that $f = g$.

12th grade

Problem 1

Consider a complex non-rational number a such that the set $A = \{m+na \mid m$ and n are integers $\}$ is a ring with unity having exactly four invertible elements. Prove that A is isomorphic with $\mathbb{Z}[i]$.

Problem 2

If $f : [-1, 1] \longrightarrow \mathbb{R}$ is a continous real function prove that

$$2 \cdot (\int_{-1}^{1} f^2(x)\,dx) \geq (\int_{-1}^{1} f(x)\,dx)^2 + 3(\int_{-1}^{1} xf(x)\,dx)^2$$

. When does equality occur?

Problem 3

Let K be a finite field and f an irreducible polynomial in $K[X]$ of degree n. Prove that the polynomial $g - 1$ is divisible by f where g is the product of all non-constant polynomials in $K[X]$ of degree at most $n - 1$.

Problem 4

Consider $\{f_n\}$ a sequence of non-negative functions defined on the closed interval $[0, 1]$ such that f_0 is continuous and

$$f_{n+1}(x) = \int_0^x \frac{1}{1 + f_n(t)} dt$$

Prove that the sequence f_n converges pointwisely to a function f and find that function.

First Test for IMO - Suceava - March 1997

Problem 1

There are given a line d in the plane and three circles of centers A, B, C all tangent to d and any two of them are externally tangent. Prove that ABC has an obtuse angle and find the maximum value of that angle.

Problem 2

Find the number of all systems of 9 positive distinct integers such that any positive integer less or equal to 500 can be written as sum of some distinct numbers of the considered system.

Problem 3

Consider a set M of n points in the plane ($n \geq 3$) and a function $f : M \longrightarrow \mathbb{R}$ such that

a) not all points of M lie on a circle

b) every 3 points of M are not on the same line

c) for every circle C passing through at least three points of M,

$$\sum_{P \in M \cap C} f(P) = 0$$

Prove that f is the null function.

Problem 4

Let ABC be a triangle inscribed in the circle O and D an arbitrary point on the side BC of ABC. Consider the circles K and L tangent to O and to (AD), K also being tangent to the segment (BD) and L to the segment (DC). Prove that K and L are tangent iff AD bisects the angle BAC.

The Second Exam

Problem 1

Consider $A_1, A_2 ... A_n$ points in the same plane α and V this plane where $n \geq 4$. A plane π cuts the edges $VA_1, VA_2 \ldots VA_n$ in $B_1, B_2 \ldots B_n$ respectively. Prove that if $A_1 ... A_n$ and $B_1 ... B_n$ are similar n-gons then π is paralallel with α

Problem 2

Consider A the set of integers that can be written as $a^2 + 2b^2$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. If p is prime and $p^2 \in A$ then $p \in A$.

Problem 3

Let $p \geq 5$ be a prime number, $k \in \{0, 1 \ldots p-1\}$ and A the set of all positive integers that do not contain k in their base p expansion. Find the greatest number of terms that a non-constant arithmetic progression consisting of elements of A can have.

Problem 4

Let p, q, r pairwise distinct prime numbers > 2 and $A = \{p^a q^b r^c \mid 0 \leq a, b, c \leq 5, a, b, c \in \mathbb{Z}\}$ Find the lowest number n such that any subset of A with n elements contains two distinct elements x and y such that $x \mid y$.

The Third Exam

Problem 1

Consider two triangles ACE and BDF suh that their sides meet in M, N, P, Q, R, S as follows $\{M\} = AC \cap BF, \{N\} = AC \cap BD, \{P\} = CE \cap BD, \{Q\} = DF \cap CE, \{R\} = DF \cap AE$ and $\{S\} = AE \cap BF$. Assume that

$$\frac{MN}{AC} = \frac{PQ}{CE} = \frac{SR}{AE}$$

Prove that

$$\frac{NP}{BD} = \frac{QR}{DF} = \frac{MS}{BF}$$

Problem 2

Let $\mathfrak{P}$ be the set of all points in the plane and $\mathfrak{D}$ the set of all lines in the plane. Is there any bijection $f : \mathfrak{P} \to \mathfrak{D}$ that sends any three colinear points to three lines passing through the same point?

Problem 3

Find all maps $f : \mathbb{R} \to [0, \infty)$ such that for all x and y the following realtion holds

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$

Problem 4

Consider $f = a_0 + a_1 X + \ldots + a_n X^n$ is a monic polynomial with integer coefficients such that for all $0 \leq k \leq n$ we have $a_k = a_{n-k}$. Prove that there exists as infinitely many pairs (x, y) such that $x \mid f(y)$ and $y \mid f(x)$.

The Fourth Contest

Problem 1

Consider α and β two non-rational numbers and f, g irreducible polynomials with rational coefficients such that $f(\alpha) = 0$ and $g(\beta) = 0$. Assuming that $\alpha + \beta \in \mathbb{Q}$ prove that $f^2 - g^2$ has a rational root.

Problem 2

Let $a \geq 2$ be a positive integer. Prove that one can find an infinite subset of the set $A = \{a^n + a^{n-1} - 1 \mid n \text{ positive integer}\}$ whose elements are pairiwse coprime.

Problem 3

Find the number of colorings of a regular 12-gon using two colors such that any regular polygon having the vertices among the vertices of the 12-gon is not monochromatic.

Problem 4

Consider a circle C and a line l not intersecting it. We choose two points A and B on the line l and let for each point M on the circle $f(M)$ be the point on the line l such that the lines AM and $Bf(M)$ meet on the circle C. If the function f^k has some fixed point for some $k \geq 1$ then $f^k = id_C$.