27th Russian Mathematics Olympiad

10 April 2001

(Communicated by Fedor Petrov)

9 FORM

1. The integers from 1 to 999999 are partioned into two groups: those integers for which the nearest perfect square is odd and those integers for which the nearest perfect square is even. For which group is the sum of all its numbers greater?

   (N. Agakhanov)

2. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ be two polynomials. It is known that $P(x)$ is negative if and only if $Q(x)$ is negative and that the set of all x for which they both take negative values is an interval of length greater than 2. Prove that there exists a real number t for which $P(t) < Q(t)$.

   (N. Agakhanov)

3. A point K inside parallelogram $ABCD$ satisfies the following conditions: the midpoint of AD is equidistant from K and C; the midpoint of CD is equidistant from K and A. Let N be the midpoint of BK. Prove that the angles $\angle NAK$ and $\angle NCK$ are equal.

   (S. Berlov)

4. In a convex 2000-gon P, no three diagonals intersect at the same point. Each diagonal is coloured in one of 999 colours. (The sides of the 2000-gon are not coloured.) Prove that there exists a triangle whose vertices are vertices of P or the points of intersection of the diagonals such that all its sides are coloured in the same colour.

   (Y. Lifshits)

5. Yura put 2001 coins in a row, each coin worth 1,2 or 3 kopecks. He noted that between any two 1 kopeck coins there was at least one coin, between any two 2 kopeck coins there were at least two coins, and between any two 3 kopeck coins there were at least three coins. How many 3 kopeck coins could be in the row?

(Y. Lifshits)

6. In a collection of $2n+1$ persons, for any group of n persons there exists a person, not in the group, who knows each person of the group. Prove that there exists a person, who knows everybody.

(S. Berlov)

7. A point N is chosen on the longest side AC of triangle ABC. Perpendicular bisectors of AN and NC intersect AB and BC at K and M respectively. Prove that the circumcentre O of ABC lies on the circle which passes through K, B, M.

(S. Berlov)

8. Find all odd positive integers $n > 1$ such that for any two coprime divisors a, b of n the number $a + b - 1$ is also a divisor of n.

(D. Djukic)

10 FORM

9. Same as 1.

10. Let 100 subsets $A_1, A_2, \ldots, A_{100}$ of a line be given, each of them is a union of 100 mutually non-intersecting closed segments. Prove that the intersection of $A_1, A_2, \ldots, A_{100}$ is a union of at most 9901 mutually non-intersecting closed segments (a single point is also considered to be a closed segment).

(R. Karasev)

11. Let two circles, touching internally at point N be given. A tangent to the internal circle through a point K of this circle intersects the external circle at points A and B. Let M be the midpoint of the arc AB, not containing N. Prove that the circumradius of triangle BMK does not depend on the choice of K.

(T. Emelyanova)

12. In a country with a number of towns, some of the towns are joined by roads so that for any two towns there exists a unique non-selfintersecting way joining them. It is known that there exist exactly 100 towns, from each only one road is originated. Prove that it is possible to construct 50 new roads such that after the construction of new roads, any two towns will be connected, even if to close any one road.

(D. Karpov)

13. A polynomial $P(x) = x^3 + ax^2 + bx + c$ has three different real roots but the polynomial $P(Q(x))$, where $Q(x) = x^2 + x + 2001$, does not have any real roots. Prove that $P(2001) > 1/64$.

(D. Tereshin)

14. In an $n \times n$ magic square all its n^2 cells are filled with the numbers $1, 2, \ldots, n^2$, and for any pair of cells, their centres are connected by an arrow which is directed from the cell with the smaller number to the cell with the larger one. Prove that the sum of all such vectors equals to zero.

(I. Bogdanov)

Comment: A magic square is a square table of numbers such that the sum of numbers in each row equals the sum of numbers in each column. See example below:

1	9	5
8	4	3
6	2	7

15. Points A_1, B_1, C_1 lie on the altitudes AA', BB', CC' of an acute-angled triangle ABC (those points lie strictly on the altitudes, not on their continuations). It is known that A_1, B_1, C_1 are different from the orthocentre H and

$$\text{area}(ABC_1) + \text{area}(BCA_1) + \text{area}(CAB_1) = \text{area}(ABC).$$

Prove that the points A_1, B_1, C_1, H are concyclic.

(S. Berlov)

16. Find all positive integers $n > 1$ such that for any two coprime divisors a, b of n the number $a + b - 1$ is also a divisor of n.

(D. Djukic)

11 FORM

17. The total weight of a collection of stones is $2S$. A positive integer k is called realizable, if there exist k stones in this collection with total weight S. Find the maximal number of realizable integers which can occur.

(D. Kuznecov)

18. Same as 11.

19. In the plane, there are two families of convex polygons P_1 and P_2. For any two polygons from different families, their intersection is not empty. Also, each of the two families contains a pair of disjoint polygons. Prove that there exists a line which intersects all the polygons in both families.

(V. Dolnikov)

20. Contestants of a multichoice competition had n questions to answer. A correct answer for the ith question is worth p_i points, where p_i is a positive integer. Any incorrect answer brings zero points and there are no partial credits for incorrect answers. For any contestant, her total score is the total number of points she gained for her correct answers. After the test was written and marked and the ranking of contestants was determined, the jury noticed that, with the given answers of contestants, the numbers $p_1, p_2, \ldots, p_n$ can be changed to achieve any other ranking. What is the largest number of contestants in this contest?

(S. Tokarev)

21. Two monic quadratic polynomials $f(x)$ and $g(x)$ take negative values on disjoint intervals. Prove that there exist real numbers a, b such that for any real x the inequality $af(x) + bg(x) > 0$ holds.

(S. Berlov, O. Podlipskiy)

22. Suppose that a and b are two distinct positive integers such that $ab(a+b)$ is a multiple of $a^2 + ab + b^2$. Prove that $|a - b| > \sqrt[3]{ab}$.

(S. Berlov)

23. There are 2001 towns in a country. For any town, there exists a road going out of it, and there does not exist a town directly connected by roads with all the rest. A set of towns D is said to be dominating if any town that does not belong to D is directly connected by a road with at least one town from D. It is given that any dominating set consists at least k towns. Prove that the country may be partioned into $2001 - k$ republics such that no two towns from the same republic will be joined by a road.

(V. Dolnikov)

24. A tetrahedron $SABC$ is given. The centre of a sphere lies in the plane ABC. The sphere passes through the points A, B, C and intersects edges SA, SB, SC in points A_1, B_1, C_1, different from the points A, B, C, respectively. The tangent planes to this sphere at points A_1, B_1, C_1 intersect at O. Prove that O is the circumcentre of the tetrahedron $SA_1B_1C_1D_1$.

(L. Emelyanov)