INTRODUCTION

The project is about Tits-buildings. A Tits-building is an abstract object, either a simplicial complex endowed with a family of thin sub-complexes, the apartments [T5], or a chamber system [T6] endowed with a Weyl distance function satisfying some axioms similar to the ones of a real metric space [T7]. Tits-buildings were created by Jacques Tits to better understand semisimple Lie groups of exceptional type, and they are the geometric interpretations of the semisimple algebraic groups. A Tits-building can be viewed as a geometry in several ways. However, it is sometimes convenient to see a Tits-building, or one of its associated geometries, sitting inside a projective space, as a kind of representation. In the most preferred cases, this makes the action of the corresponding group (an algebraic group in the broad sense) more concrete. Also, it interacts in the obvious way with (modular) representation theory. It is also geometrically more satisfying in that structural properties and properties of substructures often become better visible and easier to understand (and find!). As an example we mention that geometric hyperplanes can be obtained from projective hyperplanes, and that this idea is the base to classify polar spaces (which are Tits-buildings of the types B_n, C_n and D_n, $n\geq3$); roughly, the geometric hyperplanes of the polar space are identified with the points of a geometry which is proved to be a projective space, thus providing a representation for an abstract polar space (for an extensive account on this, see [Sh]). Recovering the algebraic data and showing uniqueness has then become rather straightforward. Hence representations of geometries are useful objects. A vast piece of the literature is about ways to embed the geometries associated to Tits-buildings in projective spaces. In many cases one axiomatizes the properties that a given embedding should have in order to admit the expected automorphism group (giving rise to a (modular) projective representation of that group). We shall not survey all known embedding results, rather we shall point out the ones that triggered this project.

The first one is a general axiomatic characterization of Segre varieties and certain degenerate analogues over an arbitrary field [SV2]. It grew out of the curiosity of what would happen if one replaces "ovoid" with ruled quadrics in the axiomatic system (based on the approach of Mazzocca and Melone [MM] to finite quadric Veronesean varieties) that describes Hermitian Veronesean varieties (see [SV1]). Surprisingly we found Segre varieties (for hyperbolic quadrics) and the line-Grassmannian of a certain representation of Hjelmslev planes of level 2 due to Artmann [Ar] (for ovoidal cones). Despite the fact that Mazzocca and Melone developed their ideas merely over finite fields, and that counting techniques seemed to play a crucial role, we managed to classify the axiomatically defined objects over an arbitrary field. The three classes of objects that come out of the axioms can be referred to as (1) the split form of the Freudenthal-Tits Magic Square (MS), see [F,T3], (2) the nonsplit form of the MS, (3) a degenerate form of the MS not apparent in the literature. However, noting the fact that Hjelmslev planes of level 2 are basically the spheres of radius 2 in certain affine Tits-buildings, the degenerate form would be linked with an affine approach to the MS.
The second one is a description and characterization of the universal representations of certain dual polar spaces amongst which Tits' non-embeddable ones, solving a long-standing open problem in incidence geometry about the spin embeddings of polar spaces [DV]. These are

representations of polar spaces in projective space where the maximal singular subspaces of the polar space are points of the projective space, and next-to-maximal ones lines. Again this is linked with representations of the corresponding automorphism groups, as a deep result by Kasikova & Shult [KS] implies that any polar space which admits at least one spin embedding, admits a universal one, and this universal one admits "all" automorphisms. But not all polar spaces admit a spin embedding. For the non-classical polar spaces, the so-called non-embeddable ones, the question was settled by Tits himself in [T1] and by Mühlherr [M1]. However, in [DV] we showed that the spin embedding of a non-embeddable polar space is essentially unique as a homogeneous embedding, this e.g. implies that it is the universal embedding. Our results also hold for the other classes of rank 3 polar spaces parameterized by a quadratic alternative division algebra (which is only non-associative in the non-embeddable case), amongst which the one associated to groups of mixed type (for example certain pseudo-reductive algebraic groups). In the latter case, the associated dual polar spaces admit other homogeneous representations and they are all classified in [DV].
These two results at first seemed isolated from each other, but then I noticed that the Segre varieties are varieties over an arbitrary field belonging to the second cell of the second row of the Magic Square in the split version, and the representations of the dual polar spaces except for the mixed case are the varieties of the third row of the MS in the nonsplit version. The search for a deeper and more direct connection between these two separate results (than their mere common appearance in the MS), and partly also inspired by Zak's result on the classification of Severi varieties [Z], lead to the current project.

SHORT DESCRIPTION AND POSSIBLE DIRECTIONS

The project in full is about representations of Tits-buildings in projective spaces. Contrary to the existing results on "embeddings of geometries", which form a collection of separate theorems with hypotheses that suit the respective specific geometry, we aim at a more global theory inspired by the characterization of finite Veronese surfaces by Mazzocca & Melone in 1984. This characterization has been used recently to characterize the split and nonsplit geometries attached to the second row of the Freudenthal-Tits Magic Square (FTMS for short). Besides the Veronese representations of projective planes over quadratic division algebras, this comprises also some Segre varieties, the line Grassmannian of projective 5-space and 26-dimensional standard embedding of the $E_{6,1}$ geometry, everything over arbitrary fields. The Mazzocca-Melone axioms (henceforth for short MM axioms) have a kind of “functional” nature in that a certain class of objects in some projective space can be varied. In the original MM axioms, this class consisted of the class of finite conics, but it is meaningful to consider any set of points with well defined tangent space at any point. If, for example, one considers the class of all ovoids (or quadrics of Witt index 1), then the Veronese embeddings f projective planes come out; if we use split quadrics, then the split geometries of the second row of the FTMS appear.

The PhD project aims at further investigation of this “MM function”. Some possible directions, depending on the taste of the applicant, include the following ideas:

(1) Extend the MM axioms to the first, third and/or fourth row of the FTMS. The first row seems particularly interesting, since it plays an exceptional role within the FTMS.
(2) Complete the conjecture stated in [SV3], i.e., prove that the only quadrics giving a non-trivial geometry under the MM function have maximal or minimal Witt index.
(3) Complete the MM approach of the second row of the FTMS to the higher dimensional split case by including arbitrary Segre varieties and arbitrary line Grassmannians.
(4) Try to fit the exceptional quadrangles of type F_4 into the MM approach by exhibiting a Veronese embedding for these objects. This way, also a projective representation of the metasymplectic spaces in characteristic 2 will be characterized. Can we lift the restriction on the characteristic here?
(5) Try to fit the exceptional quadrangles of type E into the MM approach. The quadrangle of type E_6 seems promising in that the MM axioms should here be used with a Hermitian surface of Witt index 1 instead of a quadric.
(6) The success of the MM approach is partly due to the fact that in all the above examples, the anisotropic part of the geometry, viewed as a "form" of a group of algebraic origin and depicted as a Tits diagram, lies "between" the isotropic nodes. This is not the case for Moufang hexagons. It is a challenge to set up an M approach to these geometries and corresponding embeddings.
(7) The MM approach proved additional insight in the projective representation of various geometries obtained from spherical buildings. In particular, it hinted at a purely geometric construction of the $E_{6,1}$ geometry (not yet published). Can one apply the same ideas to the third row of the FTMS and hence construct in a purely geometric way the standard $E_{7,7}$ geometry?

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All my own published papers can be found on my homepage http://cage.ugent.be/~hvm

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