1-twinnings of Buildings

Peter Abramenko[1*] Hendrik Van Maldeghem[2†]

[1]*Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D – 33501 Bielefeld,* abramenk@mathematik.uni-bielefeld.de

[2]*Department of Pure Mathematics and Computer Algebra, University of Ghent, Galglaan 2, B – 9000 Ghent,* hvm@cage.rug.ac.be

Abstract

In this paper, we characterize twinnings of buildings by 1-twinnings and one further condition concerning twin apartments. Specialized to the spherical case, we obtain new characterizations of the opposition relation in such buildings. We also give a new description of the standard twin building of type $\widetilde{A}_{n-1}$.

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1 Introduction and Statement of the Main Result

Twin buildings were introduced by Ronan and Tits (see [14]) as the natural geometries of groups of Kac-Moody type. Roughly speaking, a twin building consists of a pair of buildings (Δ_+, Δ_-) of the same type, which are linked by a certain function, i.e., a map δ^* associating to every pair of chambers $(c, d) \in (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+)$ an element of the Weyl group W corresponding with Δ_+ and Δ_-, and satisfying some additional conditions (for precise definitions, see below). When $\delta^*(c, d) = 1$, then we call c and d *opposite*. For classification purposes, this opposition relation is of crucial importance (it plays the same role as the opposition relation in spherical buildings; in fact it can be seen as a generalization of the latter, as every spherical building can be viewed in a canonical

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†The second author is a Research Director of the Fund for Scientific Research – Flanders (Belgium)

way as a twin building). However, a reasonable classification is only possible for those twin buildings which are 2-*spherical*, i.e. with only generalized polygons (and no trees) as rank 2 residues. The uniqueness part of the classification of 2-spherical twin buildings is settled by Tits [14], Mühlherr and Ronan [8], and Ronan [9].

However, in the classification programme outlined by Mühlherr in [7], the proof of existence of certain 2-spherical twin buildings is, unlike the spherical case, a major part of the job. To that end, one tries to find the most suitable definition of twin building, and in the context of the classification it turns out that axiomatization of the opposition relation rather than the full codistance function makes things easier. This was done by Mühlherr in [6], where a characterization of twinning was given by a local condition on opposition in rank 2 residues, the so-called 2-*twinning*. These 2-twinnings are stronger than 1-twinnings (see below for precise definitions), and a question of Mühlherr if 1-twinnings would already characterize twinnings was answered negatively by Abramenko (who constructed a counterexample for trees) and Van Maldeghem (who constructed counterexamples for generalized polygons). These counterexamples will be given in a forthcoming paper. The main purpose of the present paper is to give a characterization of twin buildings by means of 1-twinnings. This will be done by adding just one global condition involving (twin) apartments. This new characterization is less technical and complicated than those known before, and it allows to introduce twin buildings very directly even without presupposing any knowledge of the defining properties of the opposition relation in rank 2 twin buildings (in fact, this also yields a new characterization of twin trees, see Corollary 2.6 and the forthcoming paper [4]). In particular, some (standard) examples of twin buildings can be treated in a new "group-free" way, as was done by the first author in the course "Introduction to and Applications of Twin Buildings" at the international conference *(Moufang) Polygons and (Twin) Buildings* held in Gent (Belgium) in June 1999. We will demonstrate this again in Section 4 below.

Specialized to the spherical case, our results imply new characterizations of the natural opposition relation in spherical buildings (see Corollary 3.6, where they are stated in the language of twinnings).

It should be noted that the one-dimensionality of the apartments in the rank 2 case admits a couple of further variations and applications of our general results; these can be found in [4].

In order to state our Main Result, we need some preliminaries. We assume the reader is familiar with the notion of *buildings*, the *Weyl group* related to a building, and *Coxeter systems* (see [12]).

Let (W, S) be a Coxeter system and denote by $\ell : W \to \mathbb{N}$ the usual length function with respect to S. Let Δ_+ and Δ_- be two buildings of type (W, S), i.e., with apartments isomorphic to the standard Coxeter complex $\Sigma(W, S)$ associated to (W, S). Denote by $\mathcal{C}_\epsilon$ the set of chambers of Δ_ϵ, $\epsilon \in \{+, -\}$. Associated with Δ_ϵ there is a Weyl distance function $\delta_\epsilon : \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon \to W$ (see for instance [13]). Two chambers c_ϵ, d_ϵ of Δ_ϵ are s-adjacent, $s \in S$, precisely if $\delta_\epsilon(c_\epsilon, d_\epsilon) = s$. A *panel of type* $\{s\}$ is a maximal set of chambers which are pairwise s-adjacent or equal.

With this set-up, we first recall the definition of *twin building*, as stated in [14].

Definition 1.1 Let there be given a function $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$. Then $(\Delta_-, \Delta_+, \delta^*)$ is a *twin building* if the following three axioms are satisfied, for all $x_\epsilon \in \mathcal{C}_\epsilon$, and all $y_{-\epsilon}, z_{-\epsilon} \in \mathcal{C}_{-\epsilon}$, $\epsilon \in \{+, -\}$.

(Tw1) $\delta^*(x_\epsilon, y_{-\epsilon}) = \delta^*(y_{-\epsilon}, x_\epsilon)^{-1}$,

(Tw2) if $\delta^*(x_\epsilon, y_{-\epsilon}) = w \in W$, $\delta_{-\epsilon}(y_{-\epsilon}, z_{-\epsilon}) = s \in S$ and $\ell(ws) < \ell(w)$, then $\delta^*(x_\epsilon, z_{-\epsilon}) = ws$,

(Tw3) if $\delta^*(x_\epsilon, y_{-\epsilon}) = w \in W$ and $s \in S$, then there exists $z_{-\epsilon} \in \mathcal{C}_{-\epsilon}$ satisfying $\delta_{-\epsilon}(y_{-\epsilon}, z_{-\epsilon}) = s$ and $\delta^*(x_\epsilon, z_{-\epsilon}) = ws$.

The mapping δ^* is called the *codistance*.

Next, with the foregoing set-up, we repeat the definition of a 1-twinning, due to Mühlherr [6].

Definition 1.2 A non-empty symmetric relation $\mathcal{O} \subseteq (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$ is called a 1-*twinning* of (Δ_+, Δ_-) if the following axiom holds.

(1Tw) Given a pair of chambers $(c_+, c_-) \in \mathcal{O}$ and two panels P_+, P_- of the same type and belonging to Δ_+, Δ_-, respectively, such that $c_\epsilon \in P_\epsilon$, $\epsilon \in \{+, -\}$, then for any $\epsilon \in \{+, -\}$ and $x_\epsilon \in P_\epsilon$, there exists a unique $y_{-\epsilon} \in P_{-\epsilon}$ such that $(x_\epsilon, y_{-\epsilon}) \notin \mathcal{O}$.

We say that a 1-twinning $\mathcal{O}$ of (Δ_+, Δ_-) *induces* a twin building $(\Delta_+, \Delta_-, \delta^*)$ if $\mathcal{O}$ is precisely the set of pairs of chambers at codistance 1 from each other.

We now may state our Main Result as follows.

Main Result. *A* 1*-twinning* $\mathcal{O}$ *of* (Δ_+, Δ_-) *induces a (necessarily unique) twin building* $(\Delta_+, \Delta_-, \delta^*)$ *if and only if the following condition is satisfied for some* $\epsilon \in \{+, -\}$.

(TA) *There exists a chamber* $c_{-\epsilon} \in \mathcal{C}_{-\epsilon}$ *such that for any chamber* x_ϵ*, with* $(c_{-\epsilon}, x_\epsilon) \in \mathcal{O}$*, there is an apartment* Σ_ϵ *of* Δ_ϵ *satisfying* $\{x_\epsilon\} = \{y_\epsilon \in \mathcal{C}_\epsilon \,|\, y_\epsilon \in \Sigma_\epsilon$ *and* $(c_{-\epsilon}, y_\epsilon) \in \mathcal{O}\}$.

The proof of this theorem will be completed in Section 3 (cf. Theorem 3.5). Some applications will be discussed in Corollary 3.6 and in Section 4. Note that in Section 2, we collect and prove many statements that can be derived from a 1-twinning. We also make some efforts to include the case of non-thick buildings, which were excluded in the main result of [6].

2 1-twinnings and the Weyl Codistance

In this section, we are given two buildings Δ_+ and Δ_- of type (W, S) and a 1-twinning $\mathcal{O}$ of (Δ_+, Δ_-). We introduce some further notation and conventions.

If not explicitly stated otherwise, any definition and assertion containing the symbol ϵ must be read as: for ϵ equal to $+$ and $-$ respectively. If we mention chambers $c_\epsilon, d_\epsilon, \ldots$, we always automatically mean that they belong to $\mathcal{C}_\epsilon$. Also, we shall write δ for the Weyl distance functions δ_+ and δ_- whenever it is clear which of the two buildings Δ_+ or Δ_- we are considering. For any $T \subset S$ and any chamber c_ϵ, we set $W_T := \langle T \rangle \leq W$ and define the *T-residue of* c_ϵ by

$$R_T(c_\epsilon) := \{d_\epsilon \in \mathcal{C}_\epsilon \mid \delta(c_\epsilon, d_\epsilon) \in W_T\}.$$

The cardinality $|T|$ is called the *rank* of $R_T(c_\epsilon)$, while T itself is called the *type* of $R_T(c_\epsilon)$. A panel (see above) is then just a rank 1 residue.

For any pair $(c_\epsilon, c_{-\epsilon}) \in \mathcal{O}$, we say that the chambers c_ϵ and $c_{-\epsilon}$ are *opposite* and we use the notation $c_\epsilon \mathcal{O} c_{-\epsilon}$. We write $c_\epsilon \not{\mathcal{O}} d_{-\epsilon}$ if the chambers c_ϵ and $d_{-\epsilon}$ are not opposite. Two residues are *opposite* if they have the same type and contain opposite chambers. Condition (1Tw) of Definition 1.2 can be stated as follows: given two opposite panels P_+, P_-, any chamber $x_\epsilon \in P_\epsilon$ is opposite all chambers of $P_{-\epsilon}$ but (exactly) one.

For any chambers $c_\epsilon, d_{-\epsilon}$, we define

$$\begin{aligned}
c_\epsilon^{\mathcal{O}} &:= \{x_{-\epsilon} \in \mathcal{C}_{-\epsilon} \mid x_{-\epsilon} \mathcal{O} c_\epsilon\}, \\
\ell^*(c_\epsilon, d_{-\epsilon}) &:= \min\{\ell(\delta(x_{-\epsilon}, d_{-\epsilon})) \mid x_{-\epsilon} \in c_\epsilon^{\mathcal{O}}\}, \\
D^*(c_\epsilon, d_{-\epsilon}) &:= \{\delta(x_{-\epsilon}, d_{-\epsilon}) \mid x_{-\epsilon} \in c_\epsilon^{\mathcal{O}} \text{ and } \ell(\delta(x_{-\epsilon}, d_{-\epsilon})) = \ell^*(c_\epsilon, d_{-\epsilon})\}.
\end{aligned}$$

We remark that $c_\epsilon^{\mathcal{O}}$ is necessarily non-empty (cf. [6], Lemma 5.1).

If $|D^*(c_\epsilon, d_{-\epsilon})| = 1$, then we define the *Weyl codistance* $\delta^*(c_\epsilon, d_{-\epsilon})$ as the unique element of $D^*(c_\epsilon, d_{-\epsilon})$. We say in this case that δ^* is well-defined at $(c_\epsilon, d_{-\epsilon})$.

Denoting the *sphere with center a chamber* c_ϵ *and radius* $w \in W$ by

$$\mathcal{S}_w(c_\epsilon) := \{x_\epsilon \in \mathcal{C}_\epsilon \mid \delta(c_\epsilon, x_\epsilon) = w\},$$

we can restate the following lemma due to Mühlherr ([6], Lemmas 5.2 and 5.3).

Lemma 2.1 *Given chambers $x_\epsilon, y_{-\epsilon}$ and an element $w \in D^*(x_\epsilon, y_{-\epsilon})$, we have*

(i) $\ell^*(x_\epsilon, y_{-\epsilon}) = \ell^*(y_{-\epsilon}, x_\epsilon) = \ell(w)$,

(ii) $\mathcal{S}_w(x_\epsilon) \subseteq y_{-\epsilon}^{\mathcal{O}}$ *and* $\mathcal{S}_{w^{-1}}(y_{-\epsilon}) \subseteq x_\epsilon^{\mathcal{O}}$.

This lemma has some interesting consequences.

Corollary 2.2 *Given chambers $x_\epsilon, y_{-\epsilon}, z_{-\epsilon}$ and an element $w \in W$, we have*

(*i*) $w \in D^*(x_\epsilon, y_{-\epsilon})$ *if and only if* $w^{-1} \in D^*(y_{-\epsilon}, x_\epsilon)$,

(*ii*) *if* $w \in D^*(x_\epsilon, y_{-\epsilon})$, $\delta(y_{-\epsilon}, z_{-\epsilon}) = s \in S$ *and* $\ell(ws) < \ell(w)$, *then* $ws \in D^*(x_\epsilon, z_{-\epsilon})$.

Proof.

(*i*) It suffices of course to verify one implication. So assume $w \in D^*(x_\epsilon, y_{-\epsilon})$, and choose $y_\epsilon \in \mathcal{S}_w(x_\epsilon)$. Lemma 2.1 yields $y_\epsilon \mathcal{O} y_{-\epsilon}$ as well as $\ell^*(y_{-\epsilon}, x_\epsilon) = \ell(w) = \ell(w^{-1})$, and hence $w^{-1} \in D^*(y_{-\epsilon}, x_\epsilon)$.

(*ii*) Set $w' := ws$, then $\ell(w') = \ell(w) - 1$ by assumption. Applying Lemma 2.1(*ii*) again, we obtain

$$\mathcal{S}_{w'^{-1}}(z_{-\epsilon}) \subseteq \mathcal{S}_{w^{-1}}(y_{-\epsilon}) \subseteq x_\epsilon^{\mathcal{O}}.$$

Therefore, we have

$$\ell(w) - 1 = \ell^*(x_\epsilon, y_{-\epsilon}) - 1 \leq \ell^*(x_\epsilon, z_{-\epsilon}) \leq \ell(w') = \ell(w) - 1.$$

Hence $w' = ws \in D^*(x_\epsilon, z_{-\epsilon})$. □

Provided that the function $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ is *globally well-defined*, i.e., well-defined at every element of the source of δ^*, Corollary 2.2 shows that δ^* already satisfies Axioms (Tw1) and (Tw2) of the introduction. It is slightly more complicated to deal with (Tw3). We are going to study this axiom locally, i.e., for a fixed chamber $c_\epsilon = x_\epsilon$ and $y_{-\epsilon}$ running through the elements of a panel $P_{-\epsilon} \subseteq \mathcal{C}_{-\epsilon}$.

Lemma 2.3 *Let c_ϵ be a chamber and let $P_{-\epsilon}$ be a panel of type $\{s\}$ such that $|D^*(c_\epsilon, y_{-\epsilon})| = 1$, for all $y_{-\epsilon} \in P_{-\epsilon}$. Then*

$$\delta^*(c_\epsilon, P_{-\epsilon}) := \{\delta^*(c_\epsilon, y_{-\epsilon}) \,|\, y_{-\epsilon} \in P_{-\epsilon}\} = \{w, w'\},$$

for two distinct elements $w, w' \in W$, and either

(*i*) $w' = ws$,

or else

(*ii*) $|P_{-\epsilon}| = 2$ *and* $\ell(w) = \ell(w') < \ell(ws) = \ell(w's)$.

Proof. Without loss of generality, we may put $\epsilon = +$. We choose an element $w \in \delta^*(c_+, P_-)$ of minimal length and an $x_- \in P_-$ with $\delta^*(c_+, x_-) = w$. We first state and show three claims

(1) $\ell(ws) > \ell(w)$.
If on the contrary we had $\ell(ws) < \ell(w)$, then Corollary 2.2(ii) would imply $\ell(\delta^*(c_+, y_-)) = \ell(ws) < \ell(w)$, for all $y_- \in P_- \setminus \{x_-\}$, contradicting our choice of w.

(2) *If $z_- \in P_-$ and $\ell^*(c_+, z_-) > \ell(w)$, then $\delta^*(c_+, z_-) = ws$.*
Choose $c_- \in c_+^{\mathcal{O}}$ with $\delta(c_-, x_-) = w$. Since $\ell(ws) > \ell(w)$ by (1), we have $\delta(c_-, z_-) = ws$. Therefore $\ell^*(c_+, z_-) > \ell(w)$ implies $ws \in D^*(c_+, z_-)$, hence $\delta^*(c_+, z_-) = ws$.

(3) *There is a unique $z_- \in P_-$ such that $\delta^*(c_+, y_-) = w$, for all $y_- \in P_- \setminus \{z_-\}$.*
Choose $x_+ \in \mathcal{S}_w(c_+)$. Then $x_+ \mathcal{O} x_-$ by Lemma 2.1(ii). Let $z_- \in P_-$ be the unique chamber satisfying $z_- \not{\mathcal{O}} x_+$. For any $y_- \in P_- \setminus \{z_-\}$, the fact that $y_- \mathcal{O} x_+$ together with $\ell^*(y_-, c_+) = \ell^*(c_+, y_-) \geq \ell(w)$ implies that w^{-1} belongs to $D^*(y_-, c_+)$. Hence $\delta^*(c_+, y_-) = w$ by Corollary 2.2(i). Now if we had $\delta^*(c_+, z_-) = w$ as well, then Lemma 2.1 would imply that also z_- is opposite x_+, contradicting the choice of z_-.

Set $w' := \delta^*(c_+, z_-)$ with z_- as in (3); in particular $w' \neq w$. If $\ell(w') > \ell(w)$, then $w' = ws$ by (2) and we are in Case (i). So suppose $\ell(w') = \ell(w)$. Then x_- and w can be replaced by z_- and w', respectively, in (1) — yielding $\ell(w's) > \ell(w)$ — and in (3). This shows that $P_- = \{x_-, z_-\}$, for if there were a chamber $y_- \in P_- \setminus \{x_-, z_-\}$, we would obtain $w = \delta^*(c_+, y_-) = w'$, clearly a contradiction. So $\ell(w') = \ell(w)$ implies that we are in Case (ii). □

The counter-example that we discuss below shows that Case (ii) of Lemma 2.3 in fact does occur. However, this is of course impossible for thick buildings, as well as for buildings of rank 2 (since $\ell(w) = \ell(w') < \ell(ws) = \ell(w's)$ already implies $w = w'$ for $|S| = 2$). Therefore, Lemma 2.3 in combination with Corollary 2.2 immediately implies:

Corollary 2.4 *Suppose that δ^* is globally well-defined (in other words, $|D^*(x_\epsilon, y_{-\epsilon})| = 1$ for any chambers x_ϵ and $y_{-\epsilon}$). Assume further that the buildings Δ_+ and Δ_- are thick or of rank 2. Then $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.* □

Counter-example. We want to demonstrate that the single fact of δ^* being well-defined does not necessarily imply that $(\Delta_+, \Delta_-, \delta^*)$ is a twin building. So take an arbitrary Coxeter system (W, S) of rank 3, set $S = \{r, s, t\}$ and consider the standard Coxeter complex $\Sigma = \Sigma(W, S)$ with the elements of W being the chambers of Σ. Take two copies Σ_+, Σ_- of Σ, and write w_+, w_- if $w \in W$ is considered as a chamber of Σ_+, Σ_-, respectively. Now we define a symmetric relation $\mathcal{O}$ as follows:

$$w_+ \mathcal{O} v_- :\Longleftrightarrow wv^{-1} \in \{1, rst\},$$

$$w_-\mathcal{O}v_+ :\Longleftrightarrow wv^{-1} \in \{1, tsr\},$$

for $w, v \in W$. Noting that rst is not a conjugate in W of an element of S, it is easily checked that $\mathcal{O}$ indeed defines a 1-twinning of (Σ_+, Σ_-). Also, δ^* is globally well-defined in this situation because, for instance, $w_+^{\mathcal{O}} = \{w_-, (tsrw)_-\}$, and the gallery distances $\ell(w^{-1}v), \ell(w^{-1}rstv)$ are not congruent modulo 2, so that for any chamber $v_- \in \Sigma_-$, there is a nearest chamber in $w_+^{\mathcal{O}}$. However, $(\Sigma_+, \Sigma_-, \delta^*)$ violates (Tw3) and certainly does not define a twin building.

Next we will show that the question whether δ^* is well-defined or not, as well as the question whether (Tw3) is satisfied can be reduced to an analysis of $\delta^*(c_\epsilon, \cdot)$ for a fixed chamber c_ϵ.

Proposition 2.5 *Suppose there exists a chamber c_ϵ satisfying the following two conditions:*

(a) $|D^*(c_\epsilon, x_{-\epsilon})| = 1$ *for all* $x_{-\epsilon} \in \mathcal{C}_{-\epsilon}$,

(b) for any panel $P_{-\epsilon} \subseteq \mathcal{C}_{-\epsilon}$ of type $\{t\}$ (with $t \in S$), we have $\delta^*(c_\epsilon, P_{-\epsilon}) = \{w, wt\}$ for some $w \in W$.

Then δ^* is globally well-defined and $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.

Proof. Set ϵ equal to $+$. Let d_+ be an s-neighbour of c_+ (with $s \in S$), i.e. $\delta(c_+, d_+) = s$. For an arbitrary chosen chamber x_- we set $v := \delta^*(c_+, x_-)$, and we choose $w \in D^*(d_+, x_-)$. We will show that w is already uniquely determined by v, more precisely,

(1) If $\ell^*(d_+, x_-) \neq \ell^*(c_+, x_-)$ $(= \ell(v))$, then $w = sv$; if $\ell^*(d_+, x_-) = \ell^*(c_+, x_-)$, then $w = v$.

First we note that the identities $\ell(\delta(c_+, d_+)) = 1$, $\ell^*(c_+, x_-) = \ell^*(x_-, c_+)$, $\ell^*(d_+, x_-) = \ell^*(x_-, d_+)$ together with the definition of ℓ^* immediately imply $|\ell^*(c_+, x_-) - \ell^*(d_+, x_-)| \leq 1$.

We first consider the case $\ell^*(c_+, x_-) = \ell^*(d_+, x_-) + 1$. Since $w^{-1} \in D^*(x_-, d_+)$ by assumption and by Corollary 2.2(i), we obtain $w^{-1}s \in D^*(x_-, c_+)$ (and $\ell(w^{-1}s) > \ell(w^{-1})$). Hence $w^{-1}s = \delta^*(x_-, c_+) = v^{-1}$, implying $w = sv$.

Now the case $\ell^*(c_+, x_-) \leq \ell^*(d_+, x_-)$ will be settled by using induction on $\ell^*(d_+, x_-)$ (the claim being trivial for $\ell(v) \leq \ell(w) = 0$). Choose a decomposition $w = w't$ with $t \in S$ and $\ell(w') = \ell(w) - 1$. Using assumption (b), we find a chamber x'_- satisfying $\delta(x_-, x'_-) = t$ and $\delta^*(c_+, x'_-) = vt := v'$. By Corollary 2.2($ii$) we also have $w' = wt \in D^*(d_+, x'_-)$, in particular $\ell^*(d_+, x'_-) = \ell(w') < \ell(w) = \ell^*(d_+, x_-)$. We have to distinguish two cases now.

1. $\ell(v) = \ell(w)$.
   Then $\ell(v') = \ell(v) \pm 1 = \ell(w) \pm 1$. However, the inequality $\ell(v') = \ell^*(c_+, x'_-) \leq \ell^*(d_+, x'_-) + 1 = \ell(w') + 1 = \ell(w)$ excludes the possibility $\ell(v') = \ell(w) + 1$. Therefore $\ell(v') = \ell(w) - 1 = \ell(w')$, and the induction hypothesis yields $w' = v'$, hence $w = v$.

2. $\ell(v) = \ell(w) - 1$.
   Then $\ell(v') = \ell(w) - 1 \pm 1 = \ell(w') \pm 1$ and both possibilities might occur. For $\ell(v') = \ell(w') + 1$, the case already treated implies $w' = sv'$. For $\ell(v') = \ell(w') - 1$, this follows from the induction hypothesis. Hence we have $w = sv$ in both cases.

From (1) immediately follows:

(2) For any chamber x_- we have $|D^*(d_+, x_-)| = 1$ and $\delta^*(d_+, x_-) \in \{v, sv\}$, where $v := \delta^*(c_+, x_-)$.

Next we show:

(3) For any panel $P_- \subseteq \mathcal{C}_-$ of type $\{t\}$ (with $t \in S$), we have $\delta^*(d_+, P_-) = \{w, wt\}$ for some $w \in W$.

Choose $w \in \delta^*(d_+, P_-)$ of minimal length and a chamber $x_- \in P_-$ satisfying $\delta^*(d_+, x_-) = w$. Note that $\ell(wt) > \ell(w)$ (cf. (1) in the proof of Lemma 2.3). Again set $v := \delta^*(c_+, x_-)$ and choose, using Assumption (b), a chamber $z_- \in P_-$ satisfying $\delta^*(c_+, z_-) = vt$. Finally, we set $u := \delta^*(d_+, z_-)$. By (2), we have $w \in \{v, sv\}$ and $u \in \{vt, svt\}$. We will show that $u \neq wt$ implies $u = w$, thus ruling out Case (ii) of Lemma 2.3. Note that $u \neq wt$ already yields $\ell(u) = \ell(w)$ by Lemma 2.3. We again distinguish between two cases.

1. $w = v$ and $u \neq wt$.
   In view of Corollary 2.2, $\delta^*(d_+, x_-) = v$ is only possible if $\ell(sv) > \ell(v)$, hence $\ell(sw) > \ell(w)$. We already deduced $\ell(wt) > \ell(w)$ and $\ell(u) = \ell(w)$ above. Note that $u \in \{vt, svt\} \setminus \{vt\}$, so $u = svt$. Now an easy argument on Coxeter systems gives the implication
   $$\left.\begin{array}{r} \ell(sw) > \ell(w), \\ \ell(wt) > \ell(w), \\ \ell(swt) = \ell(w) \end{array}\right\} \Longrightarrow w = swt = u.$$

2. $w = sv$ and $u \neq wt$.
   We already established $\ell(wt) > \ell(w)$ and $\ell(w) = \ell(u)$. Since $u \in \{vt, svt\}$ and $u \neq wt = svt$, we have $u = vt = swt$, hence also $\ell(w) = \ell(swt)$. Now $\delta^*(d_+, z_-) = vt$ and $\delta^*(d_+, x_-) = sv \neq v$ imply, using Corollary 2.2, that $\ell(v) > \ell(vt)$, consequently $\ell(sw) > \ell(swt) = \ell(w)$. So we are again in the situation $\ell(sw) > \ell(w)$, $\ell(wt) > \ell(w)$ and $\ell(swt) = \ell(w)$, which yields $w = swt = u$.

So we have shown that $wt \in \delta^*(d_+, P_-)$ or that $\delta^*(d_+, x_-) = \delta^*(d_+, z_-)$ for the two distinct elements x_-, z_- of P_-. Therefore $\delta^*(d_+, P_-) = \{w, wt\}$ by Lemma 2.3.

Now we can show that

(4) δ^* is globally well-defined and satisfies (Tw3).

An obvious induction using (2) and (3) yields $|D^*(y_+, x_-)| = 1$ for all chambers y_+, x_-. Also, (3) gives (Tw3) for the sign ϵ being $+$ (i.e., when searching for appropriate neighbours in Δ_-). Finally, (2) and Corollary 2.2(i) imply $\delta^*(x_-, P_+) \subseteq \{w, ws\}$ for some $w \in W$, where $x_- \in \mathcal{C}_-$ and $P_+ \subseteq \mathcal{C}_+$ is an arbitrary panel of type $\{s\}$. Combined with Lemma 2.3, this yields $\delta^*(x_-, P_+) = \{w, ws\}$, thus proving (Tw3) completely.

Now (4) and Corollary 2.2 imply that $(\Delta_+, \Delta_-, \delta^*)$ is a twin building. □

Proposition 2.5 and Lemma 2.3 immediately imply the following

Corollary 2.6 *Suppose Δ_+ and Δ_- are thick or of rank 2. If there exists a chamber c_ϵ satisfying $|D^*(c_\epsilon, x_{-\epsilon})| = 1$ for all $x_{-\epsilon} \in \mathcal{C}_{-\epsilon}$, then δ^* is globally well-defined and $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.* □

Corollary 2.6 for thick buildings was independently proved by Valery Vermeulen (private communication; his — much longer — proof will be included in his doctoral thesis).

The main application of Proposition 2.5 will be given in the next section (cf. Theorem 3.5).

3 1-twinnings and (twin) Apartments

One of the fundamental notions in building theory is the one of an apartment. If 1-twinnings are supposed to lead to something reasonable, meaning twin buildings, then the opposition relation $\mathcal{O}$ must be linked in a natural way to the apartments of the buildings Δ_+, Δ_-. It turns out that the global condition which distinguishes the twin buildings amongst the 1-twinnings is the existence of "sufficiently many twin apartments" (in a sense which will be made precise soon) coming along with $\mathcal{O}$.

In this section we keep all notions and notations introduced in the previous section. In particular, $\mathcal{O}$ will again be a 1-twinning of a pair of buildings (Δ_+, Δ_-) of type (W, S). When we speak of apartments in Δ_ϵ, we always mean members of the maximal system of apartments of Δ_ϵ.

Definition 3.1 (i) A chamber c_ϵ is said to be *twinned* with an apartment $\Sigma_{-\epsilon}$ of $\Delta_{-\epsilon}$ if there is exactly one chamber in $\Sigma_{-\epsilon}$ opposite c_ϵ.

(*ii*) A pair (Σ_+, Σ_-) of apartments is called a *twin apartment* of $(\Delta_+, \Delta_-, \mathcal{O})$ if each chamber of Σ_ϵ is twinned with $\Sigma_{-\epsilon}$ for $\epsilon \in \{+, -\}$.

Remark 3.2 It follows immediately from Lemma 2.1(*ii*) that $|\Sigma_{-\epsilon} \cap c_\epsilon^{\mathcal{O}}| \geq 1$ for any chamber c_ϵ and any apartment $\Sigma_{-\epsilon}$ of $\Delta_{-\epsilon}$.

Remark 3.3 The apparently weaker notion of a chamber twinned with an apartment is in fact equivalent to that of a twin apartment because one can show the following. An apartment Σ_ϵ of Δ_ϵ is part of a twin apartment if and only if $\Sigma_\epsilon^* := \{c_{-\epsilon} \in \mathcal{C}_{-\epsilon} \,|\, |c_{-\epsilon}^{\mathcal{O}} \cap \Sigma_\epsilon| = 1\}$ is non-empty. In that case, Σ_ϵ^* is the set of chambers of an apartment $\Sigma_{-\epsilon}$ of $\Delta_{-\epsilon}$, and $(\Sigma_\epsilon, \Sigma_{-\epsilon})$ is a twin apartment. However, we shall not use this statement in the sequel.

The following easy lemma will prove to be very useful in the rest of this section. It provides a link between the Weyl codistance and twin apartments.

Lemma 3.4 *Suppose the chamber c_ϵ is twinned with the apartment $\Sigma_{-\epsilon} \subseteq \Delta_{-\epsilon}$. Denote by $c_{-\epsilon}$ the unique chamber of $\Sigma_{-\epsilon}$ opposite c_ϵ. Then $|D^*(c_\epsilon, x_{-\epsilon})| = 1$ and $\delta^*(c_\epsilon, x_{-\epsilon}) = \delta(c_{-\epsilon}, x_{-\epsilon})$ for any chamber $x_{-\epsilon}$ of $\Sigma_{-\epsilon}$.*

Proof. Let $x_{-\epsilon}$ be a chamber of $\Sigma_{-\epsilon}$ and $w \in D^*(c_\epsilon, x_{-\epsilon})$. Since $\Sigma_{-\epsilon}$ is an apartment, $\mathcal{S}_{w^{-1}}(x_{-\epsilon}) \cap \Sigma_{-\epsilon} \neq \emptyset$. However, $\mathcal{S}_{w^{-1}}(x_{-\epsilon}) \subseteq c_\epsilon^{\mathcal{O}}$ by Lemma 2.1(*ii*). Therefore necessarily $\mathcal{S}_{w^{-1}}(x_{-\epsilon}) \cap \Sigma_{-\epsilon} = \{c_{-\epsilon}\}$ and hence $w^{-1} = \delta(x_{-\epsilon}, c_{-\epsilon})$. This means that $w = \delta(c_{-\epsilon}, x_{-\epsilon})$ is the only element of $D^*(c_\epsilon, x_{-\epsilon})$. □

Using this lemma and the results of Section 2, twin buildings can now be characterized as follows.

Theorem 3.5 *A* 1*-twinning $\mathcal{O}$ of (Δ_+, Δ_-) induces a twin building $(\Delta_+, \Delta_-, \delta^*)$ if and only if the following condition is satisfied.*

(TA) *There exists a chamber $c_{-\epsilon}$ such that for any $x_\epsilon \in c_{-\epsilon}^{\mathcal{O}}$, there exists an apartment Σ_ϵ of Δ_ϵ satisfying $c_{-\epsilon}^{\mathcal{O}} \cap \Sigma_\epsilon = \{x_\epsilon\}$.*

Proof. It is well known that any twin building satisfies (TA). In fact, if $(\Delta_+, \Delta_-, \delta^*)$ is a twin building, then for any two given chambers x_+, y_-, there exists a twin apartment (Σ_+, Σ_-) with $x_+ \in \Sigma_+$ and $y_- \in \Sigma_-$ (see for instance [1], Lemma 2).

Now let $\mathcal{O}$ be a 1-twinning of (Δ_+, Δ_-). Then we claim the following assertion.

(∗) Let $c_{-\epsilon}$ be any given chamber and let $\mathcal{M}$ be any set of apartments of Δ_ϵ such that $c_{-\epsilon}$ is twinned with each $\Sigma_\epsilon \in \mathcal{M}$. Denote by U_ϵ the union of all members of $\mathcal{M}$. If $c_{-\epsilon}^{\mathcal{O}} \subseteq U_\epsilon$, then $U_\epsilon = \Delta_\epsilon$.

Indeed, we shall show that $y_\epsilon \in U_\epsilon$ for any given chamber $y_\epsilon \in \Delta_\epsilon$ by induction on $\ell^*(c_{-\epsilon}, y_\epsilon)$, the start of the induction being the assumption $c_{-\epsilon}^{\mathcal{O}} \subseteq U_\epsilon$. Choose $w \in D^*(c_{-\epsilon}, y_\epsilon)$, where $\ell(w) = \ell^*(c_{-\epsilon}, y_\epsilon) > 0$. Write $w = w's$ with $s \in S$ and $\ell(w') = \ell(w) - 1$, and let y'_ϵ be such that $\delta(y_\epsilon, y'_\epsilon) = s$. Then $w' \in D^*(c_{-\epsilon}, y'_\epsilon)$ by Corollary 2.2(ii). Applying the induction hypothesis, we find an apartment $\Sigma_\epsilon \in \mathcal{M}$ containing y'_ϵ. Let c_ϵ be the unique chamber of Σ_ϵ opposite $c_{-\epsilon}$, and let z_ϵ be the chamber of Σ_ϵ satisfying $\delta(y'_\epsilon, z_\epsilon) = s$. Now Lemma 3.4 yields $w' = \delta^*(c_{-\epsilon}, y'_\epsilon) = \delta(c_\epsilon, y'_\epsilon)$ and $\delta^*(c_{-\epsilon}, z_\epsilon) = \delta(c_\epsilon, z_\epsilon) = w's = w$. Applying Corollary 2.2(ii) again, we see that $\ell^*(c_{-\epsilon}, x_\epsilon) = \ell(w')$, for all x_ϵ in $R_{\{s\}}(z_\epsilon) \backslash \{z_\epsilon\}$. However, $y_\epsilon \in R_{\{s\}}(z_\epsilon)$ and $\ell^*(c_{-\epsilon}, y_\epsilon) = \ell(w) > \ell(w')$. Hence $y_\epsilon = z_\epsilon \in \Sigma_\epsilon \subseteq U_\epsilon$. This shows our claim.

Now suppose that (TA) is satisfied. Then (∗) shows that for any chamber $x_\epsilon \in \mathcal{C}_\epsilon$, there exists an apartment Σ_ϵ of Δ_ϵ containing x_ϵ such that $c_{-\epsilon}$ is twinned with Σ_ϵ. So Lemma 3.4 first of all implies $|D^*(c_{-\epsilon}, x_\epsilon)| = 1$, for all $x_\epsilon \in \mathcal{C}_\epsilon$. However, Lemma 3.4 secondly implies that $\delta^*(c_{-\epsilon}, z_\epsilon) = wt$, where $w = \delta^*(c_{-\epsilon}, x_\epsilon)$ and z_ϵ is the unique t-neighbour of x_ϵ in Σ_ϵ. This shows that the assumptions of Proposition 2.5 are satisfied here (concerning Assumption (b), see also Lemma 2.3). Hence this proposition implies that $(\Delta_+, \Delta_-, \delta^*)$ is a twin building. □

In the spherical case, Condition (TA) can still be weakened considerably.

Corollary 3.6 *Let Δ_+ and Δ_- be spherical buildings of type (W, S) (so W is finite). Then the following statements are equivalent for a 1-twinning $\mathcal{O}$ of (Δ_+, Δ_-).*

(i) *There exists an apartment $\Sigma_\epsilon \subseteq \Delta_\epsilon$ and a chamber $c_{-\epsilon}$ which is twinned with Σ_ϵ.*

(ii) *There exist chambers $c_{-\epsilon}$ and d_ϵ with $\ell^*(c_{-\epsilon}, d_\epsilon) = \ell(w_0)$, where w_0 denotes the unique elements of maximal length in (W, S).*

(iii) *The codistance δ^* is well-defined and $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.*

Proof. Put ϵ equal to $+$.

(i) ⇒ (ii) : Let c_+ be the chamber in Σ_+ which is opposite c_-, and let d_+ be the chamber in Σ_+ satisfying $\delta(c_+, d_+) = w_0$. Then also $\delta^*(c_-, d_+) = w_0$ by Lemma 3.4.

$(ii) \Rightarrow (iii)$: First of all, because of the uniqueness of w_0, we must have $\delta^*(c_-, d_+) = w_0 = w_0^{-1}$. Then Lemma 2.1($ii$) implies $\mathcal{S}_{w_0}(d_+) \subseteq c_-^{\mathcal{O}}$. However, since $\ell(\delta(y_+, d_+)) < \ell(w_0)$ for any chamber $y_+ \in \mathcal{C}_+ \setminus \mathcal{S}_{w_0}(d_+)$, we also have $c_-^{\mathcal{O}} \subseteq \mathcal{S}_{w_0}(d_+)$, hence $c_-^{\mathcal{O}} = \mathcal{S}_{w_0}(d_+)$. So for any $x_+ \in c_-^{\mathcal{O}}$, there is precisely one apartment Σ_+ of Δ_+ which contains x_+ and d_+ (namely the convex hull of x_+ and d_+ in Δ_+). Since $c_-^{\mathcal{O}} \cap \Sigma_+ = \mathcal{S}_{w_0}(d_+) \cap \Sigma_+ = \{x_+\}$, we see that c_- is twinned with Σ_+. Hence (TA) is satisfied and Theorem 3.5 yields statement (iii).

$(iii) \Rightarrow (i)$: This is clear. □

4 The standard twin building of type $\widetilde{A}_{n-1}$

Twin buildings naturally arise with Kac-Moody groups and more generally with certain group theoretic axiomatic settings like twin BN-pairs and RGD-systems, introduced by Jacques Tits(cf. [14]). Though it is desirable to have at least some concrete descriptions for (certain) twin buildings which do not refer to groups already in the very definition (similar to the well-known standard model for affine buildings of type $\widetilde{A}_{n-1}$ using lattices), almost no examples of this sort have been discussed in the literature so far. The only exception we are aware of is the detailed treatment of the twin tree associated to $\mathrm{SL}_2(k[t, t^{-1}])$ in Section 2 of [10]. In the way it is dealt with there, this example cannot easily be generalized to twin buildings of type $\widetilde{A}_{n-1}$ for $n > 2$ since one would have to consider W-codistances, where W is the affine Weyl group of type $\widetilde{A}_{n-1}$, instead of numerical codistances between vertices which are sufficient in the twin tree case. However, the new characterization of twin buildings derived in Section 3 allows a technically uncomplicated uniform description of the twin buildings associated to $\mathrm{SL}_n(k[t, t^{-1}])$ for arbitrary n without referring to groups or W-codistances. Lattice class models for twin buildings associated to classical groups over $k[t, t^{-1}]$ can probably be deduced similarly, though in a technically more complicated way (involving $k(t)$-vector spaces endowed with forms defined over k).

In order to derive our description of the twin building associated to $\mathrm{SL}_n(k[t, t^{-1}])$, we need some notation.

Let k be a commutative field, $\mathbb{K} = k(t)$ the rational function field over k, let v_+, v_- be the discrete valuations on $\mathbb{K}$ determined by $v_+(k^*) = v_-(k^*) = \{0\}$ and $v_+(t) = v_-(t^{-1}) = 1$, and let $\mho_\epsilon = \{\lambda \in \mathbb{K} \mid v_\epsilon(\lambda) \geq 0\}$, $\epsilon \in \{+, -\}$, be the corresponding discrete valuation rings. Denote by A the Laurent polynomial ring $A = k[t, t^{-1}]$, by V an n-dimensional $\mathbb{K}$-vector space and by M a free A-submodule of V of rank n (hence $M \otimes_A \mathbb{K} \cong V$). In order to use coordinates, we fix an A-basis $b_1, b_2, \ldots, b_n$ of M.

We recall the construction of the affine building Δ_ϵ of type $\widetilde{A}_{n-1}$ naturally associated with (V, v_ϵ), $\epsilon \in \{+, -\}$. Firstly, we set

$$\mathcal{L}_\epsilon := \{L \mid L \text{ is an } \mho_\epsilon - \text{lattice in } V\},$$

where "$\mho_\epsilon$-lattice" means "free $\mho_\epsilon$-submodule of rank n". Secondly, we define the equivalence relation

$$L \sim L' :\Leftrightarrow \exists \lambda \in \mathbb{K}^* : L' = \lambda L, \text{ for } L, L' \in \mathcal{L}_\epsilon$$

and we set $\overline{\mathcal{L}}_\epsilon := \mathcal{L}_\epsilon / \sim$, denoting the class of L in $\overline{\mathcal{L}}_\epsilon$ by $[L]$. Thirdly, we introduce an adjacency (or "incidence") relation in $\overline{\mathcal{L}}_\epsilon$ by calling $\Lambda, \Lambda' \in \overline{\mathcal{L}}_\epsilon$ adjacent if there exist representatives $L \in \Lambda$ and $L' \in \Lambda'$ such that $t^\epsilon L < L' < L$, where $t^+ := t$ and $t^- := t^{-1}$. Now Δ_ϵ is by definition the flag complex with respect to this incidence relation, i.e., the simplices of Δ_ϵ are precisely the (finite) subsets of $\overline{\mathcal{L}}_\epsilon$ of the form $\{\Lambda_1, \dots, \Lambda_r\}$, where Λ_i and Λ_j are adjacent for all $1 \le i < j \le r$. To any $\mathbb{K}$-basis $e_1, \dots, e_n$ of V, we associate a full subcomplex $\Sigma_\epsilon(e_1, \dots, e_n)$ of Δ_ϵ with set of vertices

$$\overline{\mathcal{L}}_\epsilon(e_1, \dots, e_n) := \{[L] \mid L = \bigoplus_{i=1}^{n} t^{m_i} \mho_\epsilon e_i \text{ for some } m_1, \dots, m_n \in \mathbb{Z}\}$$

and with set of simplices those which have all their vertices in $\overline{\mathcal{L}}_\epsilon(e_1, \dots, e_n)$. The following statements are folklore (cf. [11], Chapter II,§1 for $n = 2$ and [5], Chapter 19, for the general case).

Facts 4.1 (*i*) *Each subcomplex $\Sigma_\epsilon(e_1, \dots, e_n)$ of Δ_ϵ is a Coxeter complex of type $\widetilde{A}_{n-1}$, i.e., isomorphic to the Coxeter complex of the affine Weyl group $W = W_{aff}(\widetilde{A}_{n-1})$ associated to a root system of type A_{n-1}.*

(*ii*) *Δ_ϵ is a thick building of rank n and*

$$\mathcal{A}'_\epsilon := \{\Sigma_\epsilon(e_1, \dots, e_n) \mid e_1, \dots, e_n \text{ is a } \mathbb{K}\text{-basis of } V\}$$

is a system of apartments for Δ_ϵ.

(*iii*) *A well-defined numbering* type: $\overline{\mathcal{L}}_\epsilon \to \mathbb{Z}/n\mathbb{Z}$ *of the vertices of Δ_ϵ can be obtained as follows. Given $\Lambda \in \overline{\mathcal{L}}_\epsilon$, choose a representative $L \in \Lambda$ and an element $g \in GL(V)$ such that $L = g(\bigoplus \mho_\epsilon b_i)$. Then the congruence class modulo n of $v_\epsilon(detg)$ is independent of the choice of L and g, and we set* type$(\Lambda) := \epsilon v_\epsilon(detg)$ *mod n. This function canonically extends to a numbering* type: $\Delta_\epsilon \longrightarrow 2^{\mathbb{Z}/n\mathbb{Z}}$ *of the whole building Δ_ϵ.*

The sign ϵ in the definition of type(Λ) is chosen in such a way to make sure that opposite vertices Λ_+, Λ_- (which by definition must have the same type) enjoy a certain natural property (see 4.3(i) below).

To any ordered $\mathbb{K}$-basis $e_1, \dots, e_n$ of V, and any number $0 \le j < n$, we associate

(1) the lattices

$$L^j_\epsilon := L^j_\epsilon(e_1,\ldots,e_n) := \langle te_1,\ldots,te_j,e_{j+1},\ldots,e_n\rangle_{\mho_\epsilon},$$

where $\langle x_1,\ldots,x_n\rangle_{\mho_\epsilon}$ denotes the $\mho_\epsilon$-lattice generated by $x_1,\ldots,x_n$;

(2) the lattice classes

$$\Lambda^j_\epsilon := \Lambda^j_\epsilon(e_1,\ldots,e_n) := [L^j_\epsilon]$$

;

(3) the chambers

$$c_\epsilon(e_1,\ldots,e_n) := \{\Lambda^0_\epsilon,\ldots,\Lambda^{n-1}_\epsilon\} \in \Delta_\epsilon.$$

Definition 4.2 Two chambers $c_+ \in \Delta_+$, $c_- \in \Delta_-$ are called *opposite* — and we again use the notation $c_+\mathcal{O}c_-$ and $c_-\mathcal{O}c_+$ — if there exists an A-basis $e_1,\ldots,e_n$ of M such that $c_+ = c_+(e_1,\ldots,e_n)$ and $c_- = c_-(e_1,\ldots,e_n)$. Two vertices $\Lambda_+ \in \Delta_+$ and $\Lambda_- \in \Delta_-$ are called *opposite* if they are contained in opposite chambers and type(Λ_+) =type(Λ_-).

It is clear that $\mathcal{O}$ is a non-empty symmetric relation. In the rest of this section we shall show that $\mathcal{O}$ is a 1-twinning of (Δ_+,Δ_-) which induces a twin building $(\Delta_+,\Delta_-,\delta^*)$.

Lemma 4.3 *(i) Two vertices Λ_+ and Λ_- are opposite if and only if there exists an A-basis $f_1,\ldots,f_n$ of M such that $\Lambda_\epsilon = [\langle f_1,\ldots,f_n\rangle_{\mho_\epsilon}]$, for $\epsilon \in \{+,-\}$.*

(ii) Let $e_1,\ldots,e_n$ be an A-basis of M and $\Sigma_\epsilon = \Sigma_\epsilon(e_1,\ldots,e_n)$, $\epsilon \in \{+,-\}$. Then for any vertex $\Lambda_\epsilon \in \Sigma_\epsilon$, respectively any chamber $c_\epsilon \in \Sigma_\epsilon$, there is a unique vertex $\Lambda_{-\epsilon} \in \Sigma_{-\epsilon}$ which is opposite Λ_ϵ, respectively a unique chamber $c_{-\epsilon} \in \Sigma_{-\epsilon}$ which is opposite c_ϵ.

Proof. *(i).* Let $f_1,\ldots,f_n$ be an A-basis of M such that $\Lambda_\epsilon = [\langle f_1,\ldots,f_n\rangle_{\mho_\epsilon}]$, for $\epsilon \in \{+,-\}$. Then Λ_+ and Λ_- are contained in the opposite chambers $c_+(f_1,\ldots,f_n)$ and $c_-(f_1,\ldots,f_n)$, respectively. Furthermore, there exists a $g \in$GL(M) such that $gb_i = f_i$, for all $i \in \{1,\ldots,n\}$, hence, in particular, det$g \in A^* = \{\lambda t^m \,|\, \lambda \in k^*, m \in \mathbb{Z}\}$ and $v_+(\mathrm{det}g) = -v_-(\mathrm{det}g)$. So by the definition of the numberings of Δ_+ and Δ_-, we obtain

$$\mathrm{type}(\Lambda_+) = v_+(\mathrm{det}g) \bmod n = -v_-(\mathrm{det}g) \bmod n = \mathrm{type}(\Lambda_-).$$

Now suppose that Λ_+ and Λ_- are opposite vertices, contained in the opposite chambers $c_+(e_1,\ldots,e_n)$ and $c_-(e_1,\ldots,e_n)$, respectively, for an appropriate A-basis $e_1,\ldots,e_n$ of M. Then there are numbers $j,\ell \in \{0,\ldots,n-1\}$ such that $\Lambda_+ = \Lambda^j_+(e_1,\ldots,e_n)$ and $\Lambda_- = \Lambda^\ell_-(e_1,\ldots,e_n)$. By assumption, we also have type(Λ_+) =type(Λ_-). This obviously implies $j = \ell$, and so the A-basis $te_1,\ldots,te_j,e_{j+1},\ldots,e_n$ of M generates an $\mho_\epsilon$-lattice which represents Λ_ϵ for $\epsilon \in \{+,-\}$.

(ii). Suppose that the vertices $\Lambda_+ \in \Sigma_+$ and $\Lambda_- \in \Sigma_-$ are opposite. In view of (i), we can find representatives $L_+ \in \Lambda_+$ and $L_- \in \Lambda_-$ such that $L_+ \cap L_- \cap M$ is an n-dimensional k-vector space (generated by $f_1, \ldots, f_n$ in the notation of (i)) which also generates V as a $\mathbb{K}$-vector space. On the other hand, all representatives $L_+ \in \Lambda_+$ and $L_- \in \Lambda_-$ are of the form $L_+ = \langle t^{m_1}e_1, \ldots, t^{m_n}e_n\rangle_{\mho_+}$ and $L_- = \langle t^{\ell_1}e_1, \ldots, t^{\ell_n}e_n\rangle_{\mho_-}$, respectively, with all $m_i, \ell_i \in \mathbb{Z}$. Hence our requirement concerning $L_+ \cap L_- \cap M$ immediately implies $m_i = \ell_i$ for all i, which shows that $\Lambda_\epsilon \in \Sigma_\epsilon$ uniquely determines its opposite $\Lambda_{-\epsilon} \in \Sigma_{-\epsilon}$. This shows in particular that the set of vertices of a chamber $c_\epsilon \in \Sigma_\epsilon$ uniquely determines the set of vertices of any chamber $c_{-\epsilon} \in \Sigma_{-\epsilon}$ opposite c_ϵ. Hence there can be at most one such $c_{-\epsilon}$, and of course there exists one, namely $c_{-\epsilon} = c_{-\epsilon}(t^{q_1}e_{\sigma(1)}, \ldots, t^{q_n}e_{\sigma(n)})$ if the $q_1, \ldots, q_n \in \mathbb{Z}$ and the permutation σ of $\{1, \ldots, n\}$ are such that $c_\epsilon = c_\epsilon(t^{q_1}e_{\sigma(1)}, \ldots, t^{q_n}e_{\sigma(n)})$ □

Proposition 4.4 *The opposition relation $\mathcal{O}$ introduced in* Definition 4.2 *is a 1-twinning of (Δ_+, Δ_-) which induces a twin building $(\Delta_+, \Delta_-, \delta^*)$. The set of twin apartments of $(\Delta_+, \Delta_-, \delta^*)$ is*

$$\mathcal{A} := \{(\Sigma_+(e_1, \ldots, e_n), \Sigma_-(e_1, \ldots, e_n)) \,|\, e_1, \ldots, e_n \text{ is an } A\text{-basis of } M\}.$$

Proof. Lemma 4.3 implies that any element $(\Sigma_+, \Sigma_-) \in \mathcal{A}$ is a twin apartment with respect to $\mathcal{O}$. Hence by Definition 4.2, any two opposite chambers are contained in a twin apartment, which in particular shows that Condition (TA) of Theorem 3.5 is satisfied. So in order to prove that $(\Delta_+, \Delta_-, \mathcal{O})$ induces a twin building, we just have to verify that $\mathcal{O}$ is a 1-twinning.

Given two opposite chambers c_+, c_-, we choose an A-basis $e_1, \ldots, e_n$ of M such that $c_\epsilon = c_\epsilon(e_1, \ldots, e_n) = \{\Lambda^0_\epsilon, \ldots, \Lambda^{n-1}_\epsilon\}$ and we set $\Sigma_\epsilon = \Sigma_\epsilon(e_1, \ldots, e_n)$ for each $\epsilon \in \{+, -\}$. Each index $0 \leq j < n$ determines a panel P^j_ϵ which consists of all chambers containing $c_\epsilon \setminus \{\Lambda^j_\epsilon\}$. Note that P^j_+ and P^j_- are panels of the same type since type(Λ^j_+) =type(Λ^j_-) (cf. Lemma 4.3(i)). We denote by d^j_ϵ the unique chamber of Σ_ϵ contained in P^j_ϵ and distinct from c_ϵ. We first show

(1) $\{x_\epsilon \in P^j_\epsilon \,|\, x_\epsilon \mathcal{O} c_{-\epsilon}\} = P^j_\epsilon \setminus \{d^j_\epsilon\}$.

We set ϵ equal to $+$ and distinguish two cases.

1. Case $0 < j < n$.
Here we have $d^j_+ = (c_+ \setminus \{\Lambda^j_+\}) \cup \{M^j_+\}$, where $M^j_+ := [\langle te_1, \ldots, te_{j-1}, e_j, te_{j+1}, e_{j+2}, \ldots, e_n\rangle_{\mho_+}]$. By Lemma 4.3 d^j_+ is not opposite c_-. Any chamber $x_+ \in P^j_+ \setminus \{d^j_+\}$ is of the form $x_+ = (c_+ \setminus \{\Lambda^j_+\}) \cup \{\Lambda\}$ with $\Lambda = [\langle te_1, \ldots, te_{j-1}, te_j, e'_{j+1}, e_{j+2}, \ldots, e_n\rangle]$, where $e'_{j+1} = \lambda e_j + e_{j+1}$ for some $\lambda \in k$. For $i \neq j+1$, $1 \leq i \leq n$, we define $e'_i = e_i$. Observe that $\langle e'_j, e'_{j+1}\rangle_{\mho_\epsilon} = \langle e_j, e_{j+1}\rangle_{\mho_\epsilon}$ for any $\epsilon \in \{+, -\}$, and also $\langle te'_j, e'_{j+1}\rangle_{\mho_-} = \langle te_j, e_j\rangle_{\mho_-}$ (since

$t^{-1} \in \mho_-$). This implies $c_+(e'_1, \dots, e'_n) = x_+$ and $c_-(e'_1, \dots, e'_n) = c_-(e_1, \dots, e_n) = c_-$, showing $x_+ \mathcal{O} c_-$.

2. Case $j = 0$.
If we set $M^0_+ := [\langle te_1, e_2, \dots, e_{n-1}, t^{-1}e_n \rangle_{\mho_+}]$, then $d^0_+ = (c_+ \setminus \{\Lambda^0_+\}) \cup \{M^0_+\}$. Again by Lemma 4.3, d^0_+ is not opposite c_-. Any chamber $x_+ \in P^0_+ \setminus \{d^0_+\}$ is of the form $x_+ = (c_+ \setminus \{\Lambda^0_+\}) \cup \{\Lambda\}$ with $\Lambda = [\langle e'_1, e_2, \dots, e_n \rangle]$, where $e'_1 = e_1 + \lambda t^{-1} e_n$ for some $\lambda \in k$. We set $e'_i = e_i$ for $2 \le i \le n$ and observe that $\langle te'_1, e'_n \rangle_{\mho_\epsilon} = \langle te_1, e_n \rangle_{\mho_\epsilon}$ for any $\epsilon \in \{+, -\}$, as well as $\langle e'_1, e'_n \rangle_{\mho_-} = \langle e_1, e_n \rangle_{\mho_-}$. Hence $c_+(e'_1, \dots, e'_n) = x_+$ and $c_-(e'_1, \dots, e'_n) = c_-$, showing $x_+ \mathcal{O} c_-$.

Next we observe

(2) For any $z_\epsilon \in P^j_\epsilon$, there exists a $z_{-\epsilon} \in P^j_{-\epsilon}$ such that $z_\epsilon \mathcal{O} z_{-\epsilon}$.

Indeed, if $z_\epsilon \neq d^j_\epsilon$, we already saw that $z_\epsilon \mathcal{O} c_{-\epsilon}$. So it suffices to verify $d^j_+ \mathcal{O} d^j_-$. However, this follows immediately from $d^j_\epsilon = c_\epsilon(e_1, \dots, e_{j-1}, e_{j+1}, e_j, e_{j+2} \dots, e_n)$ for $0 < j < n$ and from $d^0_\epsilon = c_\epsilon(t^{-1}e_n, e_2, \dots, e_{n-1}, te_1)$, for any $\epsilon \in \{+, -\}$.

Now we apply statement (1) to the pair $z_+ \mathcal{O} z_-$ occurring in (2) as well, thus obtaining

(3) Any chamber $z_\epsilon \in P^j_\epsilon$ is opposite all chambers of $P^j_{-\epsilon}$ but exactly one.

Since the opposite panels P^j_+ and P^j_- were given arbitrarily, we now have verified Condition (1Tw) of Definition 1.2.

Since we now know that $(\Delta_+, \Delta_-, \mathcal{O})$ induces a twin building, we also know that any twin apartment (Σ_+, Σ_-) is uniquely determined by each pair of chambers $c_+ \in \Sigma_+$ and $c_- \in \Sigma_-$ satisfying $c_+ \mathcal{O} c_-$. Therefore the set of all twin apartments cannot be larger than $\mathcal{A}$.

The proposition is proved. □

Remark 4.5 It follows from Proposition 4.4 and general properties of twin buildings (cf. [1], Lemma 2) that the set

$$\mathcal{A}_\epsilon := \{\Sigma_\epsilon(e_1, \dots, e_n) \,|\, e_1, \dots, e_n \text{ is an } A\text{-basis of } M\}$$

is an apartment system of Δ_ϵ. So given any two chambers $x, y \in \Delta_\epsilon$, we can find an A-basis $e_1, \dots, e_n$ of M, integers $m_1, \dots, m_n$ and a permutation σ of the set $\{1, \dots, n\}$ such that $x = c_\epsilon(e_1, \dots, e_n)$ and $y = c_\epsilon(t^{m_1} e_{\sigma(1)}, \dots, t^{m_n} e_{\sigma(n)})$. We have proved this here in passing (almost) without any calculations.

Similarly, we have proved that for any two chambers $x \in \Delta_+$ and $y \in \Delta_-$, there exists an A-basis $e_1, \dots, e_n$ of M, integers $m_1, \dots, m_n$ and a permutation σ of the set $\{1, \dots, n\}$ such that $x = c_+(e_1, ..., e_n)$ and $y = c_-(t^{m_1} e_{\sigma(1)}, \dots, t^{m_n} e_{\sigma(n)})$.

Remark 4.6 It is now clear that the group $G = \mathrm{SL}_n(k[t, t^{-1}])$ — identified with $\mathrm{SL}(M)$ via the basis $b_1, \ldots, b_n$ — acts (type-preservingly) on $(\Delta_+, \Delta_-, \mathcal{O})$ and hence on $(\Delta_+, \Delta_-, \delta^*)$. It is also easily checked that G acts "strongly transitively", i.e., transitively on the set $\{(c_+, c_-) \in \Delta_+ \times \Delta_- \mid c_+, c_- \text{ are opposite chambers}\}$. So again the general theory of twin buildings (cf. [14],§3.2, or [2], Section 2) yields a twin BN-pair (G, B_+, B_-, N, S) in G, and $(\Delta_+, \Delta_-, \delta^*)$ is canonically isomorphic to the twin buildings associated to this twin BN-pair.

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