An Introduction to Generalized Polygons

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Abstract

We present a survey on generalized polygons, emphasizing the infinite case and mentioning some connection with Model theory. We try to be complementary to the monograph [57]. In an appendix, we overview the classification of Moufang polygons.

1 Some definitions of the notion of a generalized polygon

We start this paper by presenting some definitions of a generalized polygon that can be found in the literature. Most definitions start with a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ (which we will call an *incidence system*), where $\mathcal{P}$ and $\mathcal{L}$ are two sets the elements of which are called points and lines, respectively, and where $\mathtt{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric relation, called the incidence relation. We use common terminology such as *a point lies on a line, a line goes through a point, a line contains a point, a line passes through a point*, etc., to denote the incidence between a point and a line.

A *path* in Γ is a sequence $(x_0, x_1, \ldots, x_k)$ of points and lines such that $x_{i-1}\mathtt{I}x_i$, for all $i \in \{1, 2, \ldots, k\}$, and $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$ (if the latter is not satisfied, then we call it a *sequence*). We say that the path (or sequence) *joins the elements x_0 and x_k* (and we do not distinguish between a path and its "inverse" here). The positive natural number k is called the *length* of the path. If $x_0 = x_k$, then the path is called *closed.* Note that a closed path of Γ always has even length. A closed path $(x_0, x_1, \ldots, x_{2n} = x_0)$ of length $2n > 2$ is called an *ordinary n-gon* if $x_1 \neq x_{2n-1}$. The *girth* of Γ is the length of a closed path of minimal length, if such a path exists. If not, then the girth is said to be ∞. The *distance* $\delta(x, y)$ between two elements x, y of Γ is the length of a path of minimal length joining x and y, if such a path exists. If not, then the distance between x and y is by definition ∞. The *diameter* of Γ is the maximal value that can occur for the distance between two elements of Γ (where ∞ is by definition bigger than any natural number). Two elements at distance the diameter of Γ are called *opposite.* A *flag* of Γ is a pair $\{x, L\}$ consisting of a point x and a line L which are incident with each other. We say that a flag $\{x, L\}$ is contained in a path γ if x and L occur in γ at adjacent places. The set of flags of Γ will be denoted by $\mathcal{F}$.

Finally we call Γ *thick* (*firm*) if each element is incident with at least three (two) elements.

In the following, let $n \geq 3$ be a positive integer, and let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be an incidence system.

(I) The first definition is the original one by Tits [41]. We call Γ a *generalized n-gon of type* (I) if

  (Ia) every two elements of $\mathcal{P} \cup \mathcal{L}$ can be joined by at most one path of length $< n$,

  (Ib) every two elements of $\mathcal{P} \cup \mathcal{L}$ can be joined by at least one sequence of length $\leq n$.

(II) The second definition is also due to Tits [45]. We call Γ a *generalized n-gon of type* (II) if

  (IIa) the diameter of Γ is equal to n,

  (IIb) the girth of Γ is equal to $2n$.

(III) The third definition is also due to Tits and emerges from his more general definition of "buildings". We call Γ a *generalized n-gon of type* (III) if

  (IIIa) every two elements of $\mathcal{P} \cup \mathcal{L} \cup \mathcal{F}$ are contained in a common ordinary n-gon of Γ,

  (IIIb) if $x, y \in \mathcal{P} \cup \mathcal{L} \cup \mathcal{F}$ are both contained in two ordinary n-gons $(x_0, x_1, \ldots, x_{2n})$ and $(y_0, y_1, \ldots, y_{2n})$, where $x_0, y_0 \in \mathcal{P}$, then there exists an integer k such that the map $\theta : x_i \mapsto y_{i+2k}$ (reading subscripts modulo $2n$) has the property $\theta(x) = x$ and $\theta(y) = y$ (it is clear how to define $\theta(x)$ if $x \in \mathcal{F}$).

(IV) The fourth definition was first explicitly mentioned by Schroth [29], who attributes it to folklore. We call Γ a *generalized n-gon of type* (IV) if

  (IVa) there are no ordinary k-gons in Γ, with $k < n$,

  (IVb) every two elements of $\mathcal{P} \cup \mathcal{L}$ of Γ are contained in an ordinary n-gon.

(V) The last definition is due to Abramenko and Van Maldeghem [2]. We call Γ a *generalized n-gon of type* (V) if

  (Va) the diameter of Γ is equal to n, (Vb) for every element $x \in \mathcal{P} \cup \mathcal{L}$ and every element $y \in \mathcal{P} \cup \mathcal{L}$ at distance $n-1$ from x, there exists a unique element $x' \mathtt{I} x$ at distance $n-2$ from y.

The most general definitions of this lot are undoubtedly the first one and the last one. For instance, if $\mathcal{P} = \{x, y, z\}$, $\mathcal{L} = \{K, L, M, N\}$ and $\mathtt{I}$ is defined by $K \mathtt{I} x \mathtt{I} L \mathtt{I} y \mathtt{I} M \mathtt{I} z \mathtt{I} N$, then Γ is a generalized 6-gon of both type (I) and (V), and also a generalized k-gon of type (I) for all $k \geq 6$. On the other hand, Γ is not a generalized n-gon of any type (II), (III), (IV) for any value of n. Notice that generalized polygons of type (II), (III) and (IV) are automatically firm. For types (III) and (IV) this is trivial, and for type (II) this follows from Lemma 1.5.10 of [57]. Also, the results in Chapter 1 of [57] and in Section 4 of [2] show the following theorem.

Theorem 1.1 *If* Γ *is firm, then the definitions* (I), (II), (III), (IV), (V) *are all equivalent.*

If Γ is firm, then we call a generalized polygon of any of the types above a *weak generalized polygon*. If Γ is moreover thick, then we call it a generalized polygon. This is the terminology used in [57], which is conform the more general standard terminology of (weak) buildings.

For $n = 2$, one finds that a weak generalized 2-gon (digon) is a rather trivial geometry in which every point is incident with every line. For $n = 3$, a generalized 3-gon is nothing other than a projective plane. For higher values, one sometimes uses the terminology generalized *quadrangle* (4-gon), *pentagon* (5-gon), *hexagon* (6-gon), *heptagon* (7-gon) and *octagon* (8-gon). Grammatically, this is not very consistent, but it is borrowed from the usual terminology of regular n-gons in the Euclidean plane.

Let us end this section by explaining briefly why Tits defined generalized polygons in [41] originally as the ones of type (I) in that generality.

The reason is that in [41], Tits classifies the trialities of the triality quadrics (the buildings of type D_4) having at least one absolute point. The possible configurations of absolute points and lines precisely form a generalized hexagon (6-gon) of type (I), and some cases are not firm, just like some polarities in projective 3-space over a field of characteristic 2 have as set of absolute lines a plane line pencil. Moreover, the diameter of this absolute geometry can be strictly smaller than 6, when there are no closed paths. All these cases are covered by definition (I). Of course, the non-trivial examples are the firm ones, and people have restricted their attention only to those. On top of that, we will see below that also the firm non-thick ones can be reduced to thick ones; therefore usually people restrict themselves to the thick case.

Finally, note that, if $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a (weak) generalized polygon, then so is $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathtt{I})$, which is called the *dual* of Γ.

2 Isomorphisms

Let $\Gamma_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathtt{I}_1)$ and $\Gamma_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathtt{I}_2)$ be two weak generalized polygons. A *morphism* φ from Γ_1 to Γ_2 is a pair of mappings $\varphi_1 : \mathcal{P}_1 \to \mathcal{P}_2$ and $\varphi_2 : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\varphi_1(p)\mathtt{I}_2\varphi_2(L)$, for all $p \in \mathcal{P}_1$ and all $L \in \mathcal{L}_1$, whenever $p\mathtt{I}_1 L$. If the diameters of Γ_1 and Γ_2 are different, then there is some theory and there are interesting examples; we direct the reader to [16, 17]. From now on, we assume that Γ_1 and Γ_2 are both generalized n-gons, for some $n \geq 2$. If both φ_1 and φ_2 are bijections, and if also the inverses φ_1^{-1} and φ_2^{-1} define a morphism, then we say that φ is an *isomorphism*, and we call Γ_1 and Γ_2 isomorphic.

We have the following characterization of isomorphisms.

Theorem 2.1 *With the above notation, a morphism φ is an isomorphism whenever $n \geq 3$ and one of φ_1, φ_2 is bijective.*

Proof. Without loss of generality we may assume that φ_1 is bijective. First we show that φ_2 is surjective.

Let M be any line of Γ_2, and let q, q' be two arbitrary but distinct point of Γ_2 both incident with M. Let $p, p' \in \mathcal{P}_1$ be such that $\varphi_1(p) = q$ and $\varphi_1(p') = q'$. Let $(p = x_0, x_1, \ldots, x_k = p')$ be a path of length $k \leq n$ (see definition (I)). If the sequence $\gamma = (M, \varphi_1(x_0), \varphi_2(x_1), \varphi_1(x_2), \ldots, \varphi_2(x_{k-1}), \varphi_1(x_k), M)$ is a closed path, then $k + 2 \geq 2n$, implying $n + 2 \geq 2n$, hence $n = 2$, a contradiction. Consequently γ is not a path. We show by induction on $k \geq 2$ that all lines of γ coincide with M. For $k = 2$, this follows from the assumption $n \neq 2$. So suppose now $k > 2$. Since γ is not a path, and since we may assume that $\varphi_2(x_1) \neq M \neq \varphi_2(x_{k-1})$, the bijectivity of φ_1 implies that there exists $j \in \{2, 4, \ldots, k-4, k-2\}$ such that $\varphi_2(x_{j-1}) = \varphi_2(x_{j+1})$. So we may formally remove $\varphi_1(x_j)$ and $\varphi_2(x_{j+1})$ from γ, and obtain a sequence with the same properties as γ (except that it does not arise from a path in Γ_1, but that is not essential for the argument). The induction hypothesis implies $M = \varphi_2(x_1)$ after all. So φ_2 is surjective.

Next, we prove that φ_2 is injective. Suppose by way of contradiction that $\varphi_2(L) = \varphi_2(L')$, for two distinct lines $L, L' \in \mathcal{L}$. Let $(L = x_0, x_1, \ldots, x_k = L')$ be a path of length $k \leq n$. The image is a closed sequence, but cannot be a path, as before. Hence we again conclude that $\varphi_2(L) = \varphi_2(x_2)$. This means that we may assume that L and L' meet in a point p. Now consider any element x opposite p in Γ. By considering minimal paths joining x and L respectively L', we see that x, L, L' are contained in a common closed path γ of length $2n$ (joining p with itself). The image of γ is again a closed sequence, but can be shortened by removing $\varphi_1(p)$ in the beginning and a the end, leaving a closed sequence of length $2n - 2$. This cannot be a path, and a similar argument as before shows that all lines of that sequence coincide. We have shown that $\varphi_1(x)\mathtt{I}_2\varphi_2(L)$, if n is even, and each point on x is mapped on $\varphi_2(L)$, if n is odd. In other words, every point p' at distance $n-1$ or n from p is mapped onto $\varphi_2(L)$. Similarly, one shows that every point at distance $n-1$ or n from such a point p' is mapped onto $\varphi_2(L)$. But it is easy to see that every point p'' lies at distance $n-1$ or n from some such point p'. This of course contradicts the bijectivity of φ_1.

Hence φ_2 is bijective. Left to show is that the inverse is also a morphism. Suppose by way of contradiction that $p \in \mathcal{P}_1$ and $L \in \mathcal{L}_1$ are not incident, but $\varphi_1(p)\mathtt{I}_2\varphi_2(L)$. The image of a minimal path joining p and L is a closed sequence which cannot be a path, contradicting the injectivity of both φ_1 and φ_2.

The theorem is proved. $\square$

There is another characterization of isomorphisms, which is proved in [15]. We denote the distance function in Γ_i by δ_i, $i = 1, 2$.

Theorem 2.2 *Let Γ_i, $i = 1, 2$ be a generalized n-gon as above, $n \geq 4$. Let $i \in \{1, 2, \ldots, n-1\}$ arbitrary.*

(i) If i is even, then let $\varphi_1 : \mathcal{P}_1 \to \mathcal{P}_2$ be a bijection such that, for all $p, p' \in \mathcal{P}_1$, we have $\delta_1(p, p') = i$ if and only if $\delta(\varphi_1(p), \varphi_1(p')) = i$.

(ii) If i is odd, then let $\varphi_1 : \mathcal{P}_1 \to \mathcal{P}_2$ and $\varphi_2 : \mathcal{L}_1 \to \mathcal{L}_2$ be bijections such that, for all $p \in \mathcal{P}_1$ and $L \in \mathcal{L}_1$, we have $\delta_1(p, L) = i$ if and only if $\delta(\varphi_1(p), \varphi_2(L)) = i$.

Then in case (i), there is a unique bijection $\varphi_2 : \mathcal{L}_1 \to \mathcal{L}_2$ such that (φ_1, φ_2) is a morphism. In both cases (i) and (ii), the pair (φ_1, φ_2) is an isomorphism.

For the case $i = n$, there do exist counterexamples, see [15] again. Also, a similar characterization using a map $\theta : \mathcal{F}_1 \to \mathcal{F}_2$ exists; see [1, 14].

The foregoing characterizations of isomorphisms are global characterizations. A beautiful local characterization is proved by Bödi and Kramer [4].

Theorem 2.3 *With the above notation, a morphism φ is an isomorphism whenever both φ_1 and φ_2 are surjective and for at least one element $x \in \mathcal{P}_1 \cup \mathcal{L}_1$, the restriction of φ_i to the set of elements of Γ_1 incident with x is bijective onto the set of elements of Γ_2 incident with $\varphi_i(x)$, $i = 1$ if x is a point, $i = 2$ if x is a line.*

3 The structure Theorem and subpolygons

As already mentioned before, the theory of weak generalized polygons can be reduced to the theory of (thick) generalized polygons. This is due to Tits [43], see also [67] and [57].

In the following, a *thick* element is a point or a line incident with at least three elements. The structure theorem roughly says that every weak n-gon arises from a generalized m-gon, where the dihedral group of order $2m$ is contained in the dihedral group of order $2n$. In this formulation, the result can be seen as a special case of the more general result for spherical buildings (the dihedral groups are the Weyl groups, see the paper of Grundhöfer in these proceedings). However, a more geometric formulation is the following.

Theorem 3.1 *Let Γ be a weak generalized ngon. Then either there are no thick elements and Γ is basically an ordinary n-gon, or there are exactly two thick elements which are opposite, or the minimal distance k between two thick elements is not bigger than $n/2$ and the set of thick elements can be partitioned into two subsets $\mathcal{P}'$ and $\mathcal{L}'$ such that, if we define an incidence relation $\mathtt{I}'$ by $p'\mathtt{I}'L'$ if $\delta(p', L') = k$, then the incidence system $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ is a (thick) generalized n/k-gon. In the latter case, every element of $\Gamma_{ik}(x)$, with $x \in \mathcal{P}' \cup \mathcal{L}'$ and $1 \leq i \leq n/k$, belongs to $\mathcal{P}' \cup \mathcal{L}'$.*

In the last case of the previous theorem we say that Γ is a *multiple* of Γ' (if $k = 2$, sometimes called the *double*). It is up to duality uniquely determined by Γ' and k.

Note that the elements p' and L' in Theorem 3.1, though regarded as a point and a line of Γ', respectively, are not necessarily a point and a line of Γ, respectively. Indeed, it can very well happen that all thick elements of Γ are points, for instance.

The above structure theorem is a very useful tool in the study of weak subpolygons of a given generalized polygon. Let us give an example. First we need some preliminaries about subpolygons.

A weak subpolygon $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ of a weak generalized polygon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consists of the sets $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L}' \subseteq \mathcal{L}$ such that, with $\mathtt{I}'$ defined as the restriction of $\mathtt{I}$ to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$, the incidence system $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I})$ is a weak generalized polygon. Not much is known in case the diameters of Γ and Γ' are different, so from now on we assume that Γ and Γ' always have the same diameter.

The weak subpolygon Γ' of Γ is called a *full* (respectively *ideal*) weak subpolygon if, for every line L' (respectively point p') of Γ', every element of Γ incident with L' (respectively p') in Γ also belongs to Γ'.

For instance, Γ itself is a full and ideal weak subpolygon if itself. Conversely, it is not so hard to see that every full and ideal weak subpolygon of the weak generalized polygon Γ coincides with Γ (cp. Proposition 1.8.2 of [57]).

Another type of weak subpolygons Γ' of a generalized polygon Γ are the ones with the property that for every thick element x of Γ' all elements of Γ incident with x belong to Γ'. We call such a weak subpolygon containing at least one thick element a *solid* weak subpolygon.

An important class of sub-n-gons in the case of $n = 3$ (the projective planes) are the Baer subplanes. Let us recall the definition (also valid in the infinite case). A *Baer subplane* of a projective plane Γ is a subplane Γ' with the property that every element (point or line) of Γ is incident with at least one element of Γ'. Here is the generalization of this notion to higher n.

Let Γ be a weak generalized n-gon. A *Baer (weak) sub-n-gon* of Γ is a weak sub-$n$gon Γ' of Γ with the property that every element (point or line) of Γ is at distance $\leq n/2$ from at least one element of Γ'.

For odd n we find examples through the following theorem. For any generalized polygon Γ, any point or line x of Γ, and any positive integer i, we denote by $\Gamma_i(x)$, $\Gamma_{\leq i}(x)$ the set of elements of Γ at distance i, at distance at most i, respectively.

Theorem 3.2 *Let Γ be a generalized $(2m+1)$-gon, and suppose that σ is an involution of Γ (an isomorphism of order 2 from Γ to itself). Let $\mathcal{P}'$ (respectively $\mathcal{L}'$) be the set of fixed points (respectively fixed lines) of σ. Denote by $\mathtt{I}'$ the induced incidence relation. Then either the incidence system $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ is a Baer weak subpolygon of Γ, or there exist a unique point p and a unique line $L\mathtt{I}p$ of Γ such that $\mathcal{P}' = \Gamma_{\leq n}(p)$ and $\mathcal{L}' = \Gamma_{\leq n}(L)$.*

Proof. Suppose first that there is closed path γ of length $4m + 2$ pointwise fixed by σ. Then clearly the girth of Γ' is equal to $4m + 2$, and the diameter is not smaller than $2m + 1$. But if x, y are two elements fixed by σ, and $\delta(x, y) \leq 2m$, then the unique path joining x and y is also fixed pointwise by σ, and so $\delta'(x, y) = \delta(x, y) \leq 2m$ (denoting the distance in Γ' with δ'). Suppose now that x and y are fixed by σ and are opposite each other. Applying the previous argument on x and an element of γ not opposite x, we see that there exists an element $x' \in \Gamma_1(x)$ fixed by σ. Now $\delta(x', y) = 2m$ and the previous argument shows $\delta'(x', y) = 2m$, implying $\delta'(x, y) = 2m + 1 = \delta(x, y)$. Hence the diameter of Γ' is equal to $2m + 1$ and by definition (II), Γ' is a generalized $(2m + 1)$-gon. Now let z be any element of γ. Then, since $2m + 1$ is odd, z^σ is not opposite z. Hence there is a unique path $(z = z_0, z_1, \ldots, z_{2k} = z^\sigma)$ of even length $2k \leq 2m$ joining z and z^σ. Clearly, this path is mapped onto $(z^\sigma = z_{2k}, z_{2k-1}, \ldots, z_1, z_0 = z)$. So we see that z_k is fixed by σ, with $\delta(z, z_k) = k \leq m$. We have shown that Γ' is a Baer weak subpolygon.

Hence we may assume that σ does not fix any closed path of length $4m + 2$ pointwise. Let $x \in \mathcal{P} \setminus \mathcal{P}'$ be arbitrary. There is a unique path $(x = x_0, x_1, \ldots, x_{2k} = x^\sigma)$ joining x and x^σ, with $\delta(x, x^\sigma) = 2k$. By choosing an arbitrary element $x_{-1} \in \Gamma_1(x) \setminus \{x_1\}$, we see that $\delta(x_{-1}, x_{-1}^\sigma) = 2k + 2$. Going on like this, we may assume that $\delta(x, x^\sigma) = 2m$, and the unique element z' of $\Gamma_m(x) \cap \Gamma_m(x^\sigma)$ is fixed by σ. Now consider an element y incident with x, and opposite x^σ. Then again $\delta(y, y^\sigma) = 2m$ and the unique element z'' of $\Gamma_m(y) \cap \Gamma_m(y^\sigma)$ is fixed by σ. It is easy to see that $z', x, y, z'', x^\sigma, y^\sigma$ belong to a common ordinary $(2m+1)$-gon, and so z' is opposite z''. By varying y, we see that we obtain different elements z'' opposite z'. All these elements z'' are all either points or lines. Consequently, these elements are not mutually opposite, hence the unique paths joining them are also pointwise fixed by σ. In particular there is some element $u' \mathtt{I} z'$ fixed by σ. We conclude that there is a path $(z' = z'_0, z'_1, \ldots, z'_{2m+1} = z'')$ of length $2m+1$ joining z' and z'' fixed pointwise by Γ. We now show that σ fixes $\Gamma_{\leq m}(z'_m) \cup \Gamma_{\leq m}(z'_{m+1})$ pointwise.

It suffices to show that, if $w' \in \Gamma_m(z'_m) \cap \Gamma_{m+1}(z'_{m+1})$, then w' is fixed by σ. Choose any element $w \in \Gamma_m(w') \cap \Gamma_{2m}(z'_m)$. Then clearly $\delta(w, z'_{m+1}) = 2m+1$. As before, the middle element v' (for which holds $\delta(w, v') \leq m$) of the unique minimal path joining w and w^σ is fixed by σ. By the triangle inequality we have $\delta(z'_m, v') \geq m$ and $\delta(z'_{m+1}, v') \geq m+1$. If $\delta(z'_{m+1}, v') > m+1$, then some path of length $> 2m+1$ joining v' and either z'_0 or z'_{2m+1} is fixed pointwise by σ, implying that there is some ordinary $(2m+1)$-gon fixed by σ, a contradiction. Hence $\delta(z'_{m+1}, v') = m+1$ and so z'_m and v' are both points or lines, or one is a point and the other a line according whether m is even or odd. Hence, if $\delta(z'_m, v') > m$, then $\delta(z'_m, v') > m+1$, and this leads similarly as above to a contradiction. All this implies $\delta(z'_m, v') = m = \delta(w, v')$. But there is only one element at distance m from both z'_m and w, and it is w'.

The theorem is proved. $\square$

So the previous theorem shows that Baer weak subpolygons arise naturally with automorphisms of order 2. We now show the following restriction in the non-thick case.

Theorem 3.3 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized $(2m+1)$-gon, and suppose that $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ is a Baer weak non-thick subpolygon. Then Γ' is solid.*

Proof. Since Γ' is non-thick, either it has no thick elements or there exists a unique maximal odd positive integer $k \neq 1$ dividing $2m+1$ such that the distance between any two thick elements of Γ' is a multiple of k. So, if there exist thick elements in Γ', let $p \in \mathcal{P}'$ and $L \in \mathcal{L}'$ be two thick opposite elements, otherwise let them just be two opposite elements of Γ'.

Suppose by way of contradiction that Γ' is not solid. Hence we may assume without loss of generality that there is some point $a \mathtt{I} L$ of Γ that does not belong to Γ'. Let M be an arbitrary line of Γ' incident with p and let b be any point of Γ not belonging to Γ' and incident with M. The point b exists since M is not a thick line of Γ' (otherwise $k = 1$). Since both L, M belong to Γ', every element of the unique path of length $2n$ joining L and M belongs to Γ', hence $\delta(a, b)$ cannot be equal to $2m-2$ and so $\delta(a, b) = 2m$. Consequently there is a unique element x of Γ at distance m from both a and b. Since Γ' is

Baer weak subpolygon, there is an element z of Γ' at distance $\leq m$ from x. If $\delta(x, z) < m$, then all elements of the unique paths of length $\delta(z, L) < 2m + 1$ and $\delta(z, M) < 2m + 1$ joining z with L and M, respectively, belong to Γ', a contradiction since at least one of these paths contains x. So $\delta(x, z) = m$ and it follows that z is a point. Note that this argument also shows that $\delta(z, b) = \delta(z, a) = 2m$. If z were not opposite M, then x would belong to the unique path of length $2m - 1$ joining z and M, a contradiction. Hence z is opposite M, and, similarly, z is also opposite L.

Suppose first that Γ' contains thick elements, hence that k is well defined. Since k divides $2m+1$, this implies that z is a thick point in Γ' (as its distance to L is a multiple of k). But for the same reason it now follows that also M is thick in Γ', and this is a contradiction.

So we may assume that Γ' is an ordinary $(2m + 1)$-gon. But in such a polygon, every element is opposite precisely one other element, again a contradiction since z is opposite both L and M.

The theorem is proved. $\square$

It is actually not difficult to see that in the previous theorem k cannot be equal to $2m+1$, if $m > 1$. Hence non-thick weak Baer subpolygons of generalized $(2m + 1)$-gons, with $m > 1$, are always multiples of other generalized polygons.

For generalized n-gons with $n = 2m$ even, the previous theorem is not valid. Indeed, for generalized digons, every ordinary sub-2-gon is a weak non-thick Baer subpolygon. But one can do better. Let $\mathbb{K}$ be any field and let Γ be the generalized quadrangle arising from a nonsingular quadric in $\mathbf{PG}(4, \mathbb{K})$ of maximal Witt index (see below). Consider a hyperplane section that corresponds with a nonsingular quadric of maximal Witt index in that hyperplane (a ruled nonsingular quadric). This defined a weak non-thick full subquadrangle Γ' of Γ. Now delete one line and all points incident with it from Γ' and denote the weak non-thick subquadrangle that remains by Γ''. It is clear that Γ' is a Baer weak subquadrangle which is not solid.

Actually, in the even case the definition of a Baer subpolygon is not optimal. One obtains better results by defining in this case a Baer (weak) subpolygon of a generalized $2m$-gon Γ as a weak subpolygon Γ' such that either every point or every line of Γ lies at distance $\leq m - 1$ from some element of Γ'. In that case it is easy to see that every weak non-thick Baer subpolygon of a generalized $2m$-gon is either full or ideal, in particular solid. But also every thick Baer subpolygon must be full or ideal. Hence, in the even case, there is no completely satisfactory definition for what a Baer subpolygon should be.

4 Some examples — BN-pairs and the Tits property

In this section, we give some examples of generalized polygons. For a detailed list we once again refer to [57].

4.1 Some general constructions

First, let us show that we are not talking about the empty set in this paper. In fact, there exist generalized n-gons for any $n \geq 2$. These can be constructed by some free construction process introduced by Tits [45]. Briefly, this goes as follows. One starts with an incidence system with girth $\geq 2n$ and joins every pair of elements at distance $n+1$ by a path of length $n-1$ using new elements. This is repeatedly performed and in the limit one obtains a (usually thick, if the original incidence system is not too poor) generalized n-gon.

The generalized n-gons thus obtained have some symmetry. Indeed, at least the automorphisms of the original incidence system can be extended to automorphisms of the generalized n-gon, in the obvious way. This way it possible to obtain involutions of generalized $(2m+1)$-gons, for all $m \geq 1$, fixing Baer subpolygons. Also, by modifying the construction, one can construct for each n generalized n-gons Γ with an automorphism group G acting transitively on the set of pairs (f, Σ), where f is a flag and Σ is an ordinary sub-n-gon of Γ containing f. This is called the *Tits property* for (Γ, G) in [57]. These generalized polygons correspond precisely to groups with a (saturated) BN-pairs of rank 2. We will not give a precise definition of this popular notion. We content ourselves by mentioning that such a group is basically equivalent with an automorphism group G of a generalized polygon Γ such that the pair (Γ, G) has the Tits property. The notion "BN" refers to the fact that the stabilizer of a flag is denoted by B (and called the *Borel subgroup*) and the stabilizer of an ordinary n-gon containing the flag stabilized by B is denoted by N. Then, putting $T := B \cap N$, the quotient N/T is well defined and is the *Weyl group* of the BN-pair. For a generalized n-gon, it is isomorphic to the dihedral group of order $2n$.

Now we note that a weak generalized n-gon is thick precisely if it contains an ordinary $(n+1)$-gon as a subgeometry, see Lemma 1.3.2 of [57]. A natural question is whether there exist generalized n-gons Γ with an automorphism group acting transitively on the set of pairs (f, Ω) with f a flag and Ω an ordinary $(n+1)$-gon containing f. Let us call this the *Joswig property* for (Γ, G), since it was Michael Joswig who first asked me if one could classify such pairs in the finite case (and this gave rise to the papers [40] and [55], where this classification is completed).

In the infinite case, Katrin Tent [30] constructs for each $n \geq 2$ infinitely many examples of pairs (Γ, G), with Γ a generalized n-gon, possessing the Joswig property. Her method uses model theory and we refer to the paper of Poizat in these proceedings. Hence a classification of all such pairs seems out of reach, unlike the restriction to the finite case. However, if one requires a regular action on the pairs (f, Ω) as above, then using involutions and the theory of weak Baer subplanes introduced above, one can show that in the odd case (by which we mean generalized n-gons with n odd) we necessarily have a Pappian projective plane i.e., a projective plane arising naturally from a vector space of dimension 3 over a commutative field, see below. In the even case this problem is still open, although we hope to finish the cases $n = 4, 6$ under the additional assumption of self-duality soon.

The above discussion shows that, whenever we have a group with a BN-pair of rank 2, then we have a generalized polygon with a lot of automorphisms. Standard examples of such BN-pairs are given by semi-simple and almost simple algebraic groups of relative rank 2, by certain classical groups and groups of mixed type, and by the Ree groups of characteristic 2, as defined by Tits [47] over arbitrary (not necessarily perfect) fields of characteristic 2 admitting a so-called *Tits endomorphism*, which is a square root of the Frobenius.

Most examples of the previous paragraph can also be constructed in a more geometric way. The so-called Desarguesian projective planes arise from 3-dimensional vector spaces over not necessarily commutative fields by considering the vector lines and vector planes, with natural incidence relation. Also, any polarity of projective 3-space having fixed lines defines a (weak) generalized quadrangle by considering the fixed lines and the fixed incident point-plane pairs. Similarly, any triality of the triality quadric having fixed lines gives rise to a (weak) generalized hexagon. In order to construct generalized octagons, one considers polarities of so-called metasymplectic spaces (these are equivalent to buildings of type F_4).

Even more geometrically, one can check that the geometry of points and lines of any quadric Q in any projective space (here necessarily over a commutative field, and of dimension at least 4 — in order to avoid the non-thick case) is a generalized quadrangle, provided that Q contains lines, but no planes (we say that the Witt index is 2), and that no point on Q is collinear on Q with every other point (we say that Q is nonsingular).

Also, nonsingular Hermitian varieties containing lines but no planes define generalized quadrangles. In fact, if the characteristic of the underlying field is equal to 2, then these generalized quadrangles sometimes contain a lot of subquadrangles that are not itself obtained from a Hermitian variety or a quadric. For instance, if Γ is the generalized quadrangle arising from a nonsingular quadric of Witt index 2 in projective 4-space over a field of characteristic 2, or equivalently (but dually), Γ arises from symplectic polarity in projective 3-space over the same field, then the subquadrangles of Γ are precisely the so-called *quadrangles of mixed type*, or briefly the *mixed quadrangles*, because they arise from groups of mixed type (they are called the "indifferent quadrangles" by Jacques Tits and Richard Weiss because of their indifferent behaviour with respect to the Steinberg relations corresponding to a root system).

The examples of the last two paragraphs are special cases of the classical quadrangles, which we review now briefly, but in detail.

4.2 The classical quadrangles

The definitions in this subsection are based on Chapter 10 of [7] and Chapter 8 of [42].

Let $\mathbb{K}$ be a skew field and σ an anti automorphism (that means $(ab)^\sigma = b^\sigma a^\sigma$, for all $a, b \in \mathbb{K}$) of order at most 2. Let V be a — not necessarily finite dimensional — right vector space over $\mathbb{K}$ and let $g : V \times V \to \mathbb{K}$ be a $(\sigma, 1)$-linear form, i.e., for all $v_1, v_2, w_1, w_2 \in V$ and all $a_1, a_2, b_1, b_2 \in \mathbb{K}$, we have

$$\begin{aligned}&g(v_1a_1 + v_2a_2, w_1b_1 + w_2b_2) = \\ &\qquad a_1^\sigma g(v_1, w_1)b_1 + a_1^\sigma g(v_1, w_2)b_2 + a_2^\sigma g(v_2, w_1)b_1 + a_2^\sigma g(v_2, w_2)b_2.\end{aligned}$$

Denote $\mathbb{K}_\sigma := \{t^\sigma - t : t \in \mathbb{K}\}$. We define $q : V \to \mathbb{K}/\mathbb{K}_\sigma$ as

$$q(x) = g(x, x) + \mathbb{K}_\sigma,$$

for all $x \in V$. We call q a *pseudo quadratic*, or more precisely, a *σ-quadratic form (over $\mathbb{K}$)*. Let W be a subspace of V. We say that q is *anisotropic over* W if $q(w) = 0$ if and only if $w = 0$, for all $w \in W$ (where we have written the zero vector as 0, and the element $0 + \mathbb{K}_\sigma$ also as 0). It is *non degenerate* if it is anisotropic over the subspace $\{v \in V \mid g(v, w) + g(y, x)^\sigma = 0, \text{ for all } w \in V\}$. From now on we assume that q is non degenerate.

Noting that, if $q(v) = 0$, then $q(vk) = 0$, for all $k \in \mathbb{K}$, we can define the *Witt index* of q as the dimension of the maximal subspaces of V contained in $q^{-1}(0)$.

For a non degenerate σ-quadratic form q over $\mathbb{K}$ with Witt index 2, we define the following geometry $\Gamma = \mathsf{Q}(V, q)$. The points of Γ are the 1-spaces in $q^{-1}(0)$; the lines are the 2-spaces in $q^{-1}(0)$, and incidence is symmetrized inclusion.

One can now show that $\mathsf{Q}(V, q)$ is a weak generalized quadrangle; it is non thick if and only if the dimension of V is equal to 4 and σ is the identity (and consequently $\mathbb{K}$ is commutative). We call $\mathsf{Q}(V, q)$ and its dual *classical quadrangles*.

The quadrics and Hermitian varieties that we mentioned above are classical quadrangles. For the quadrics, this is easy to see by taking σ the identity. For Hermitian varieties, see Chapter 2 of [57].

For V a 5-dimensional space and σ the identity, there is exactly one non degenerate pseudo quadratic form, up to isomorphism. The dual of the corresponding generalized quadrangle is called a *symplectic quadrangle* $\mathsf{W}(\mathbb{K})$, because $\mathsf{W}(\mathbb{K})$ can be defined as the geometry of points and fixed lines of a 3-dimensional projective space over $\mathbb{K}$ with respect to a symplectic polarity.

By definition, we say that $\mathsf{W}(\mathbb{K})$ is a *Pappian quadrangle*.

4.3 The split Cayley hexagons

We now define the Pappian hexagons, also called the split Cayley hexagons.

We define the generalized hexagon $\Gamma = \mathsf{H}(\mathbb{K})$ as the geometry with as set of points the point set of the quadric $Q(6,\mathbb{K})$ in $\mathbf{PG}(6,\mathbb{K})$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$, and as set of lines the lines of $Q(6,\mathbb{K})$ the Grassmann coordinates of which satisfy the following six linear equations (where the *Grassmann coordinates* of a line containing the two points with coordinates $(x_0,\ldots,x_6)$ and $(y_0,\ldots,y_6)$ are defined, up to a common scalar multiple, as $p_{ij} = x_iy_j - y_ix_j$, $0 \leq i,j, \leq 6$):

$$\begin{array}{lll} p_{12} = p_{34}, & p_{54} = p_{32}, & p_{20} = p_{35}, \\ p_{65} = p_{30}, & p_{01} = p_{36}, & p_{46} = p_{31}. \end{array}$$

Incidence is natural. This description is due to Tits [41]. We call $\mathsf{H}(\mathbb{K})$ a *split Cayley hexagon* and a *Pappian hexagon*. The split Cayley hexagons are precisely the generalized hexagons arising from "linear" trialities (i.e, where no field automorphism in the triality is involved).

All the preceding examples — apart from the free and model theoretic constructions — satisfy the so-called Moufang condition, see the paper of Richard Weiss in the present proceedings. There are also other constructions which give projective planes, generalized quadrangles and hexagons with a fairly large automorphism group, but not Moufang. To mention just one class of examples, consider the building at infinity of any rank 3 affine building (see for instance the paper by Linus Kramer in the present proceedings volume). For other examples, the reader is directed to Chapter 3 of [57].

Let us end by constructing the smallest generalized hexagon $\mathsf{H}(2)$ (i.e., $\mathsf{H}(\mathbb{K})$ for $\mathbb{K}$ the field of two elements) in an alternative way. The construction is taken from [59]. It uses the smallest projective plane $\mathbf{PG}(2,2)$ where each element is incident with exactly three others. Let $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ denote this projective plane. We define a new incidence system $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ as follows. The set of points $\mathcal{P}$ is exactly $\mathcal{P}' \cup \mathcal{L}' \cup (\mathcal{P}' \times \mathcal{L}')$. The set of lines $\mathcal{L}$ is defined as follows. For every point p and every line $L\mathtt{I}'p$ of Γ', let x_1 and x_2 be the two points different from p and incident with L in Γ', and let M_1, M_2 be the two lines different from L incident with p in Γ'. Then the sets $\{p, L, (p, L)\}$ and $\{(p, L), (x_1, M_1), (x_2, M_2)\}$ are members of $\mathcal{L}$; the relation $\mathtt{I}$ is the natural one. We obtain a generalized hexagon where every element is incident with exactly three others. This hexagon and its dual are the only hexagons with that property, see [9].

4.4 Generalized polygons in special categories

We refer to [57] for a discussion of existence and uniqueness of small generalized polygons. We content ourselves here to mention that, if a generalized n-gon is finite, then Feit and

Higman [11] proved that the number $s+1$ of points incident with a line is a constant; likewise the number $t+1$ of lines through a point is a constant; the number n is one of $2, 3, 4, 6, 8$, and if $n = 6$, then st is a perfect square, while if $n = 8$, the number $2st$ is a perfect square. Moreover, some inequalities must hold involving s, t, but we will not mention them explicitly.

The main result of the previous paragraph is that for finite generalized n-gons, the value n is highly restricted! This is a general phenomenon that occurs over and over again. For instance, disregarding the generalized digons, Moufang n-gons only exist for $n \in \{3, 4, 6, 8\}$ ([44, 46, 66]), compact connected n-gons only exist for $n \in \{3, 4, 6\}$ ([21, 22]), generalized n-gons with valuation only exist for $n \in \{3, 4, 6\}$ ([50]), generalized n-gons satisfying a certain geometric regularity condition (namely, for every i, $2 \leq i \leq n/2$, and every point p, the sets $\Gamma_i(p) \cap \Gamma_{n-i}(x)$, with $x \in \Gamma_n(p)$, are determined by any two of their elements) only exist for $n \in \{3, 4, 6\}$ ([54]), generalized n-gons arising from so-called "split" BN-pairs only exist for $n \in \{3, 4, 6, 8\}$ ([33, 35]). Hence it may be clear that the most important values for n are exactly $3, 4, 6$ and *sometimes* 8. It is very tempting to try to prove such restrictions in a model theoretic setting such as finite Morley rank. Here, it seems that whenever one can prove such a restriction, a full classification can be carried out. For instance, groups with a rank 2 BN-pair of finite Morley rank and with Weyl group of order $2n$ (hence corresponding to generalized n-gons), where the panels have Morley rank 1 (the strongly minimal case; a *panel* is the set of elements incident with a given element) have $n \in \{3, 4, 6\}$ and are precisely the Pappian polygons over algebraically closed fields, see [34]. These polygons share a number of interesting and interrelated properties (see throughout [57]).

Finally let us mention that every Moufang polygon together with its automorphism group is a pair that satisfies the Tits property, but not necessarily the Joswig property. It even is a nontrivial exercise to list all Moufang polygons with the Joswig property. So, the Joswig property is not a very natural property to study, in contrast to the Tits property, which translates naturally into the rank 2 BN-pair property for groups.

5 Characterization and classification results

In the previous section we already mentioned some classification results of generalized n-gons satisfying certain conditions. I this section, we do this in a more systematic way, and we emphasize the results that are waiting for an analogy in the theory of geometries and groups of finite Morley rank.

5.1 The Moufang condition

A generalized n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ satisfies the *Moufang condition*, or is said to be *Moufang* if for every path $\gamma = (x_0, x_1, \ldots, x_n)$ of length n, the pointwise stabilizer $G_{[\gamma]}$ of the set $\Gamma_1(x_1) \cup \Gamma_1(x_2) \cup \ldots \cup \Gamma_1(x_{n-1})$ acts transitively on the set of ordinary n-gons containing γ.

It is not difficult to see that $G_{[\gamma]}$ in general actually acts semi regularly on the set of ordinary n-gons containing γ. Hence, for a Moufang polygon, it acts regularly. The group generated by all $G_{[\gamma]}$, for γ ranging over the set of all paths of length n, is called the *little projective group* of the Moufang polygon Γ. The groups $G_{[\gamma]}$ themselves are called the *root groups* of Γ.

There is also a group free definition of a Moufang polygon. Since this has a model theoretic consequence, we will discuss that definition now. It is given by the following theorem, where, for two non opposite elements x, y, the notation $\mathrm{proj}_x y$ stands for the unique element of Γ incident with x and at distance $\delta(x, y) - 1$ from y.

Theorem 5.1 *A generalized n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ satisfies the Moufang condition if and only if for every path $\gamma = (x_1, x_2, \ldots, x_{n-1})$ of length $n-2$, for every closed path $(y_0, y_1, \ldots, y_{2n-1}, y_{2n} = y_0)$ of length $2n$ with $\delta(y_0, x_1) = n-1$, and for every element $w \in \Gamma_1(\mathrm{proj}_{x_1} y_0) \setminus \{x_1\}$, there exists a closed path $(z_0, z_1, \ldots, z_{2n-1}, z_{2n} = z_0)$ of length $2n$ satisfying the following properties.*

(i) $\delta(z_0, x_1) = n - 1$ *and* $\delta(z_0, w) = n - 3$,

(ii) $\delta(x_i, y_j) = \delta(x_i, z_j)$, *for all* $i \in \{1, 2, \ldots, n-1\}$ *and all* $j \in \{1, 2, \ldots, 2n\}$,

(iii) whenever $\delta(x_i, y_j) \neq n$, *for any* $i \in \{1, 2, \ldots, n-1\}$ *and any* $j \in \{1, 2, \ldots, 2n\}$, *then* $\mathrm{proj}_{x_i} y_j = \mathrm{proj}_{x_i} z_j$.

For a proof of this theorem (for $n > 3$), see [39, 51]. Now, Linus Kramer (personal communication during the conference) remarked that a direct consequence of Theorem 5.1 is the following

Corollary 5.2 *The Moufang property is a definable property.*

This answers affirmatively a question asked by Poizat on the conference.

We can now comment on the content of the paper [24]. One of the main results classifies all Moufang polygons of finite Morley rank. They are precisely the Pappian polygons (see above) over algebraically closed fields. The proof of this result heavily uses the classification of Moufang polygons as obtained by Tits and Weiss [48]. The main ingredient of the proof consists of proving that there is some interpretable field around. In the cases where the root groups are the additive groups of commutative fields, this is not so hard, but if this is not the case, various ad hoc arguments have to be find. This holds in particular for the mixed quadrangles where the root groups are vector spaces over fields of characteristic 2. Another example is provided by the class of Moufang quadrangles with non commutative root groups.

As an application, it is shown in [24] that infinite simple groups of finite Morley rank with an irreducible spherical BN-pair of rank ≥ 3 are interpretably isomorphic to a simple algebraic group over an algebraically closed field. This application, however, does not use the classification of Moufang polygons at all, although it uses the classification of irreducible spherical buildings of rank at least 3. Indeed, since all irreducible spherical buildings of rank at least 3 are classified, we have an explicit list of Moufang polygons occurring as a residue in such a building, and that is all that is needed in order to prove the application. Of course, this is only a technical remark since the classification of Moufang polygons is much more elementary and accessible than the classification of irreducible spherical buildings of rank at least 3.

The proof of many characterization theorems consist of showing that the generalized polygon in question satisfies the Moufang condition.

5.2 Variations of the Moufang condition

There are a number of variations on the Moufang condition. Most of them can be find in [57], but we discuss them nevertheless.

Let $k \in \{2, 3, \ldots, n\}$. A generalized n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ satisfies the *k-Moufang condition*, or is said to be *k-Moufang* if for every path $\gamma = (x_0, x_1, \ldots, x_k)$ of length k, the pointwise stabilizer $G_{[\gamma]}$ of the set $\Gamma_1(x_1) \cup \Gamma_1(x_2) \cup \ldots \cup \Gamma_1(x_{k-1})$ acts transitively on the set of ordinary n-gons containing γ.

A generalized n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ satisfies the *half Moufang condition*, or is said to be *half Moufang* if for some $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}\}$ and for every path $\gamma = (x_0, x_1, \ldots, x_n)$ of length n and with $x_1 \in \mathcal{X}$, the pointwise stabilizer $G_{[\gamma]}$ of the set $\Gamma_1(x_1) \cup \Gamma_1(x_2) \cup \ldots \cup \Gamma_1(x_{n-1})$ acts transitively on the set of ordinary n-gons containing γ.

We can combine these two definitions to obtain the following (new) notion.

Let k be an even integer with $2 \leq k \leq n$. A generalized n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ satisfies the *half k-Moufang condition*, or is said to be *half k-Moufang* if for some $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}\}$ and for every path $\gamma = (x_0, x_1, \ldots, x_k)$ of length k and with $x_1 \in \mathcal{X}$, the pointwise stabilizer $G_{[\gamma]}$ of the set $\Gamma_1(x_1) \cup \Gamma_1(x_2) \cup \ldots \cup \Gamma_1(x_{k-1})$ acts transitively on the set of ordinary n-gons containing γ.

Note that the preceding definition only makes sense if k is even, because otherwise every path of length $k-2$ begins or ends with an element of $\mathcal{X}$, and so half-k-Moufang would automatically be equivalent with k-Moufang. Likewise, the half Moufang condition only makes sense for generalized n-gons with n even.

The following theorem is an easy exercise, see Section 6.8 of [57].

Theorem 5.3 *Every Moufang generalized n-gon is a half Moufang polygon, is a k-Moufang polygon for all $k \in \{2, 3, \ldots, n\}$, and is a half k-Moufang polygon for all even $k \in \{2, \ldots, n\}$.*

The ultimate conjecture is that every (half) k-Moufang generalized n-gon, $2 \leq k \leq n$, is a Moufang polygon. In some special cases, this has already been proved. For instance, for projective planes it has been shown in [61] that every half 2-Moufang projective plane is Moufang. For finite generalized polygons, the results in [56, 63] imply that every finite k-Moufang generalized n-gon, $2 \leq k \leq n$, is Moufang.

Hence it is reasonable to state the conjecture that every k-Moufang generalized n-gon of finite Morley rank, $2 \leq k \leq n$, is a Moufang polygon. An additional difficulty here is that we do not have automatically a restriction on n, unlike the finite case.

The first interesting open case concerning half k-Moufang polygons are the half 2-Moufang generalized quadrangles. Even in the finite case, nothing is known without the classification of finite simple groups. Using the classification of finite simple groups one can show that every finite half $2k$-Moufang generalized $2m$-gon, $1 \leq k \leq m$, is a Moufang polygon. This follows basically from [8]. The case $k = m = 2$ can be handled without the classification of finite simple groups, see [38].

An interesting derived conjecture is that every half Moufang generalized polygon of finite Morley rank is Moufang.

We end this subsection by mentioning the following result for half Moufang polygons (see 6.4.9 of [57] and Proposition 5.1 of [33]). A *central elation* in a generalized $2m$-gon Γ is an automorphism fixing all elements of Γ at distance $\leq m$ from some fixed element c (which is then called the *centre* of the central elation).

Theorem 5.4 *If , for $m \geq 3$, a generalized $2m$-gon Γ is half Moufang and all elements of the corresponding root groups are central elations, then Γ is a Moufang hexagon.*

5.3 Split BN-pairs

The proof of the classification of Moufang polygons has a peculiar history. In the thirties, Ruth Moufang proved that Moufang projective planes are coordinatized by "alternative division rings", see [25]. These algebraic structures (the ones that are not skew fields) were classified by Bruck and Kleinfeld [6, 20], and proved to be a Cayley-Dickson division algebra over a field. Already in the late sixties, Tits proved (in some unpublished notes) that the root groups of any Moufang hexagon are coordinatized by a field and by a "quadratic Jordan algebra of degree 3" over that field. It is only during the past thirty years that these structures have been classified. Only in the late seventies, Tits [44, 46] proved that Moufang n-gons do not exist for $n \neq 3, 4, 6, 8$. In [66], Weiss uses some ideas of Tits to compose an ingenious and short proof of the same restriction result. In the meantime a manuscript of Tits (entitled *Quadrangles de Moufang, I.*) circulates classifying the Moufang quadrangles satisfying additionally the geometric regularity condition for all elements mentioned above, and giving rise to all mixed quadrangles. Since the doubles of some of these quadrangles occur in the Moufang octagons, this can be seen as a preparation of the classification of Moufang octagons. Also, around the same time, Faulkner [10] classifies certain classes of Moufang quadrangles and hexagons, starting from stronger assumptions, thus leaving out some cases. The classification of all Moufang octagons is published by Tits in 1987 [47], but Tits made no use of the previously mentioned preprint. So the complete classification of Moufang quadrangles remained conjectural for several decades until Tits and Weiss started the joint project of writing the proof down in a book [48]. In early 1997, Richard Weiss discovered a new class of Moufang quadrangles, that was seemingly unrelated to the theory of semi-simple algebraic groups and groups of mixed type. However, shortly after Weiss discovery, Mühlherr and Van Maldeghem [26] proved that these new quadrangles were of "exceptional type F_4" by constructing them alternatively inside buildings of type F_4, thus relating these quadrangles to groups of mixed type F_4.

Hence, the classification of all Moufang polygons (in particular, Moufang quadrangles) has been completed only recently. Some special cases, however, were already done many years ago. For instance, the compact connected case follows from the classification of simple Lie groups (for an explicit list see [18], see also Theorem 9.6.3 of [57]). The finite case follows from the classification by Fong and Seitz [12, 13] of the finite groups with an irreducible "split" BN-pair of rank 2. This means that, with previously mentioned notation, the subgroup B (which is the stabilizer of some flag f of the generalized polygon Γ) can be written as the (not necessarily semi-direct) product $B = UT$, where U is a normal nilpotent subgroup of B. If we take for U the group generated by all root groups related to paths containing the flag f, then the commutator (Steinberg) relations between the root groups (see the paper of Weiss in the present proceedings) imply that U is nilpotent.

Consequently, in the finite case irreducible rank 2 split BN-pairs are essentially equivalent to Moufang polygons. In the infinite case, the natural BN-pair corresponding with a Moufang polygon is still split, and so one might wonder whether the converse is true in general. In other words, can we use the classification of Moufang polygons to classify the irreducible split BN-pairs of rank 2? If a geometric proof of this existed, this would at the same time serve as a revision of the proof of the result of Fong and Seitz mentioned above. Note that such a result also has to deal with generalized n-gons with $n \neq 3, 4, 6, 8$. This project has almost completely be carried out by Tent and Van Maldeghem [33, 35], resulting in the next theorem.

Theorem 5.5 *The generalized polygons corresponding with groups G with an irreducible rank 2 split* BN*-pair with Weyl group of order $2n \neq 16$ are Moufang, and G contains the little projective group of the Moufang polygon.*

Of course on does not expect that the case $n = 8$ is a true exception. It is still under consideration by Tent and Van Maldeghem.

Needless to say that the previous theorem is important for any kind of classification of classes of simple groups (as in the finite case), for instance the simple groups of finite Morley rank.

5.4 Other conditions and characterizations

Every Moufang generalized $2m$-gon contains central elations, as defined above. This follows from the proofs in [44, 46, 66] (for an explicit argument, see Corollary 5.4.7 of [57]). So, up to duality, every point of a Moufang $2m$-gon is the centre of a non trivial central elation. The converse of this observation for finite generalized hexagons and octagons has been proved by Walker [64, 65].

Theorem 5.6 *If every point of a finite generalized hexagon or octagon Γ is the centre of a non trivial central elation, then Γ satisfies the Moufang condition.*

Note that the proof of this result does not use the classification of finite simple groups. But the proof is hardly geometric and uses non trivial results from permutation group theory, such as Hering's trivial normalizer intersection theorem [19]. In terms of transitivity, the hypothesis of Theorem 5.6 is pretty weak. Indeed, no point or line transitivity can directly be derived. I am not aware of any similar result for other categories of generalized hexagons and octagons, such as compact connected hexagons, or, for instance, generalized $2m$-gons of finite Morley rank. In the latter case, also values for m different from 3 and 4 are conceivable.

Another result characterizes the finite Moufang polygons by means of some kind of relatives of the elements of the root groups, namely, the generalized homologies. Let x, y be two opposite elements of a generalized n-gon Γ. Then we call Γ (x, y)-*transitive* if for some element $z \mathtt{I} x$ the group H fixing $\Gamma_1(x) \cup \Gamma_1(y)$ pointwise acts transitively on the set

$\Gamma_1(z) \setminus \{x, \mathrm{proj}_z y\}$. Note that this action is not necessarily regular (sharply transitive). We call Γ (x,y)-*quasi-transitive* if for some element $z \in \Gamma_2(x) \cap \Gamma_{n-2}(y)$ the group H fixing $\Gamma_1(x) \cup \Gamma_1(y)$ pointwise acts transitively on the set $\Gamma_1(z) \setminus \{\mathrm{proj}_z x, \mathrm{proj}_z y\}$. The elements of H are called *generalized homologies*. The following theorem joins results of different papers and authors [3, 36, 37, 52, 53].

Theorem 5.7 *If a finite generalized polygon is (x,y)-transitive for all pairs of opposite elements x, y, then it is a Moufang polygon (but not all finite Moufang polygons arise, in particular, all finite planes and hexagons do arise, but no octagon arises and the generalized quadrangles obtained from Hermitian varieties in 4-dimensional projective space do not arise either). This remains true for infinite generalized projective planes. If a finite generalized quadrangle is (x,y)-transitive for all pairs of opposite points x, y, then it is a Moufang quadrangle (and up to duality all finite Moufang quadrangles arise). Finally, if a finite generalized octagon is (x,y)-transitive for all pairs of opposite points x, y, and (L,M)-quasi-transitive for all pairs of opposite lines L, M, then it is a Moufang octagon (and all finite Moufang octagons do occur).*

The hypotheses in the previous theorem imply a large transitivity of the automorphism group of the generalized polygon in question. However, for polygons of finite Morley rank this is not enough to classify, hence Theorem 5.7 has not yet an analogy for finite Morley rank polygons.

To end the group theoretic characterizations of (certain classes of) Moufang polygons, we refer to [31] for a characterization of certain Moufang polygons by means of the Tits property and a condition on the groups induced on $\Gamma_1(x)$, for x an element of the polygon.

Finally, we turn to some geometric and combinatorial characterizations. The motivation is here in part that these can be used to prove group theoretical ones as those above. This is in particular the case for generalized hexagons, as in this case the Moufang condition is really equivalent to the geometric condition of regularity (up to duality). This same regularity condition can be used to prove non existence of certain generalized n-gons with $n \neq 3, 4, 6$. For instance, this is how the proof of Theorem 5.4 works. Also, since in the finite Morley rank case the only Moufang polygons are the Pappian ones, we also give a characterization of these (and that characterization can be used as a definition of "Pappian").

We introduce some definitions. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized n-gon, and let i be an integer with $2 \leq i \leq n/2$. An element $x \in \mathcal{P} \cup \mathcal{L}$ is called *distance-i-regular* if the incidence system $\Gamma^i_x = (\Gamma_i(x), \{\Gamma_i(x) \cap \Gamma_{n-i}(y) \mid \delta(x,y) = n\}, \overline{\in})$, where $\overline{\in}$ denotes inclusion made symmetrical, has girth ≥ 6. In other words, every line of Γ^i_x is determined by any two of its points. If an element $x \in \mathcal{P} \cup \mathcal{L}$ is distance-i-regular for all i, $2 \leq i \leq n/2$, then we say that x is *regular*. If an element x is distance-2-regular and for every pair of elements

$y, z \in \Gamma_n(x)$, the intersection $\Gamma_2(x) \cap \Gamma_{n-2}(y) \cap \Gamma_{n-2}(z)$ is non empty, then x is called a *projective element*. The motivation for this name stems from the fact that in this case the incidence system

$$(\Gamma_2(x) \cup \{x\}, \{\Gamma_2(x) \cap \Gamma_{n-2}(y) \,|\, \delta(x,y) = n\} \cup \{\Gamma_1(z) \,|\, z \mathrm{I} x\}, \overline{\in})$$

is a projective plane, called the *derived plane (at x)* .

In the following theorem, we combine results of [27] and [54]. Note that regularity is void for projective planes and generalized digons; therefore we only consider generalized n-gons with $n \geq 4$ when dealing with regularity.

Theorem 5.8 *If a generalized n-gon has a distance-i-regular element, then either $i = n/2$ or $i \leq (n+2)/4$. If all points of a generalized n-gon are distance-i-regular, and if i is even, then either $i = n/2$ or $i \leq n/4$, and i divides n. Hence, if all points of a generalized n-gon, $4 \leq n$, are regular, then $n = 4, 6$. If all points of a generalized hexagon Γ are distance-2-regular, then they are also distance-3-regular, hence regular, and Γ is a Moufang hexagon (and, up to duality, every Moufang hexagon arises in this way).*

So one sees that (Moufang) hexagons behave particularly nice with respect to regularity. For Moufang quadrangles, the regularity of elements translates into the fact that root groups "at distance 2" commute. An important open question is whether we know all finite quadrangles all points of which are regular (the conjecture is we do, and then all such quadrangles would be Moufang). Also in other categories, the regularity of all points of a generalized quadrangle is not enough to yet pin it down. But to the best of my knowledge no generalized quadrangle is known all points of which are regular, but which is not Moufang. Also remark that there are Moufang quadrangles that have no regular elements at all (for instance, all exceptional ones including those of type F_4 mentioned before). Specialized to the case of finite Morley rank, an interesting open problem is whether Tent's [30] construction can be modified in such a way to yield generalized quadrangles of finite Morley rank all points of which are regular. If not, can one classify the generalized quadrangles of finite Morley rank all points of which are regular? More precisely, one could ask whether the symplectic quadrangles over algebraically closed fields are the only generalized quadrangles of finite Morley rank all points of which are regular and all panels of which have the same Morley rank. These are certainly very hard questions.

For generalized octagons, Theorem 5.8 above says that not all points can be distance-3-regular. All elements of the Moufang octagons, however, are distance-4-regular, but not a single element is distance-2-regular. The philosophy *"if a nice property holds for one example, then it should also hold for a standard example with a lot of symmetry"* lead me to prove the following (remarkable to me) result.

Theorem 5.9 *If Γ is a generalized octagon, then not all points, respectively lines, can be distance-2-regular.*

In fact, to the best of my knowledge, no generalized octagon with at least one distance-2-regular element is known.

Now we turn to generalized polygons with projective elements. Recall that a Pappian projective plane is one that arises from a commutative field as described above. The next result combines statements of Schroth [28], Ronan [27] and the author, see Theorems 6.2.1 and 6.3.1 in [57].

Theorem 5.10 *Let Γ be a generalized n-gon, $n \in \{4, 6\}$, and suppose that all points of Γ are (distance-2-)regular. If at least one point is projective, then all points are projective, all derived planes are Pappian and Γ is Pappian itself. All Pappian quadrangles and hexagons arise in this way.*

In fact, using the distance-2-regularity of a certain set of points of some generalized hexagon, one can define another derived structure at a point, which turns out to be a Pappian quadrangle if the hexagon is Pappian. This way, we obtain a short tower of Pappian polygons derived from each other in the above sense. See [62] for more details.

As an application, Theorem 5.10 has been used to classify the algebraic polygons in [23].

Let us also remark that the regularity can be used to prove several other geometric characterizations, such as for example the one with long hyperbolic lines, see [49].

We did not discuss classifications and characterizations of generalized polygons embedded in certain projective spaces. Here, too, the regularity conditions play an important role. Also, many questions are still open in that area, especially if one considers structures of finite Morley rank. The reason for omitting these results here is that they are gathered in a survey paper on embeddings of polygons that will appear around the same time of these proceedings, see [60].

Finally, another area I neglected in the present paper, although new results were found since [57] appeared, is the theory of "projectivities". Non opposition defines a bijection between the sets $\Gamma_1(x)$ and $\Gamma_1(y)$ for opposite elements x, y. Composing such bijection until we are back at x, we obtain a permutation group of $\Gamma_1(x)$, called the *group of projectivities of x*. It is, as a permutation group, independent of x, but depends on the type of x, i.e., whether x is a point of a line. Now there are some results recognizing certain classes of Moufang projective planes and quadrangles by just requiring some properties of the group(s) of projectivities. The new results in that direction can be found in [5] and in [58]. In the latter, the situation of sharply 2-transitive groups of projectivities is analyzed.

For more on the interaction between the theory of generalized polygons and model theory, we refer to the survey paper [32].

The classification of Moufang polygons

In this appendix, we give a short overview of the classification of the Moufang polygons, as obtained by Tits and Weiss [48]. The method of proceeding turned out to be completely different from the one outlined in my monograph [**?**]. In fact, this outline was suggusted by Jacques Tits in his lectures at Collège de France; the co-authorship with Richard Weiss led to a completely different approach, although some basic ideas remain. For instance, in the quadrangle case, a fundamental idea is to first classify the *reduced* Moufang quadrangles, and then extend these (see below). This idea survived.

First, I will state the theorem, then describe some basic ideas of the proof, then I will consider some different cases in more detail. But I will give no proofs; these can be found in the forthcoming book [48].

.1 The classification result

The classification result can best be stated in the way that Tits wrote it down as a conjecture many years ago, see for instance [43]: all Moufang polygons arise from algebraic and classical groups *in one or the other way.* More precisely:

Theorem .11 *All Moufang polygons arise in a canonical way from algebraic groups of relative rank* 2*, from classical groups of rank* 2*, from mixed groups of rank 2, from mixed groups of type* F_4 *and relative rank 2, and from the Ree groups of type* $^2F_4$*.*

In the course of the present proof of this result, it became clear that the quadrangles related to the mixed groups of type F_4 and relative rank 2 were missing in the detailed version of Tits' original conjecture. However, in 1997, the proof of the above theorem was completed by Tits and Weiss and these new Moufang quadrangles were proved to exist via their commutation relations. Soon afterwards, they were recognized by Mühlherr and Van Maldeghem [26] as quadrangles arising as the fixed point sets of certain involutions in certain mixed buildings of type F_4. Hence they can be seen as being from algebraic origin in the sense that they arise from a building associated to an algebraic group by considering first a subbuilding (the mixed F_4-building), and then a subcomplex (the fixed point set under an appropriate involution). This is also the way in which the Moufang octagons arise, but one has to take a polarity rather than an involution as last step. Abstractly, this case is typical: most Moufang polygons arise as fixed point structure of an automorphism of some higher rank building, but in general this building is the building associated to a simple algebraic group and this automorphism is an algebraic one, unlike the cases of Moufang octagons and Moufang quadrangles of type F_4.

We now give a more detailed version of the list in Theorem .11 above.

We now assume that Γ is a Moufang n-gon for some $n \geq 3$. We choose an apartment Σ and label the vertices of Σ by the integers modulo $2n$ so that i is adjacent to $i+1$ and different from $i+2$ for all i. Let U_i be the root group corresponding to the root $(i, i+1, \ldots, i+n)$ for all i and let U_+ denote the subgroup of G generated by the subgroups $U_1, U_2, \ldots, U_n$. The $(n+1)$-tuple $(U_+, U_1, U_2, \ldots, U_n)$ is called the root group sequence associated with Γ; it is unique up to conjugation in G and up to the re-numbering $U_i \mapsto U_{n+1-i}$ of the root groups $U_1, \ldots, U_n$. In [**11**], we show: **Theorem.** *A Moufang n-gon is uniquely determined by the associated root group sequence*

$$(U_+, U_1, U_2, \ldots, U_n).$$

As an example, we consider the case $n = 3$. Let A be an alternative division ring. This is a ring satisfying all the axioms of a skew-field except the law of associativity for multiplication, but with a strengthened law of inverses: For each non-zero $u \in A$, there exists an element u' such that $u' \cdot uv = v$ and $vu \cdot u' = v$ for all $v \in A$. Now let U_1, U_2, U_3 be three groups parametrized by the additive group of A. By this we mean that we can choose isomorphisms $u \mapsto x_i(u)$ from the additive group of A to the (multiplicative) group U_i for all $i \in [1, 3]$. We now impose the relations $[U_1, U_2] = [U_2, U_3] = 1$ and $[x_1(u), x_3(v)] = x_2(uv)$ for all $u, v \in A$. These equations determine the structure of a group $U_+ = \langle U_1, U_2, U_3 \rangle$. It turns out that (U_+, U_1, U_2, U_3) is the root group sequence associated with a generalized triangle which we denote by $\mathcal{T}(A)$. If A is a skew-field, $\mathcal{T}(A)$ is the incidence graph of the projective plane associated with a 3-dimensional right vector space over A. Moufang showed in [**3**] that every Moufang projective plane is parametrized by an alternative division ring. This result can be reformulated as follows: Every Moufang triangle is isomorphic to $\mathcal{T}(A)$ for some alternative division ring A.

Alternative division rings were classified by Bruck and Kleinfeld [**1**,**2**]: An alternative division ring is either a skew-field (possibly commutative) or a kind of 8-dimensional non-associative algebra over a commutative field K called a Cayley-Dickson division algebra.

This case is typical. For each n, we find an algebraic system (in some general sense) with which we can parametrize the groups $U_1, \ldots, U_n$ and give formulas for all the commutators in $[U_i, U_j]$ for all distinct i, j in $[1, n]$ expressed in terms of the parameters. These formulas determine the root group sequence $(U_+, U_1, \ldots, U_n)$ and thus Γ. In each case, there then remains the problem of classifying the relevant algebraic systems. That this strategy has any chance of success rests on the following result [**9**,**12**]: **Theorem.** *Moufang n-gons exist only for $n = 3$, 4, 6 and 8.* We indicate the conclusions in each case beginning with the largest value of n.

Moufang octagons are parametrized (see [**10**]) by pairs (K, σ) where K is a commutative field of characteristic two and σ is an endomorphism of K such that σ^2 is the Frobenius map of K, i.e. $(x^\sigma)^\sigma = x^2$ for all $x \in K$.

Moufang hexagons are parametrized (see [**8**]) by certain triples $(J, F, \#)$, where F is a commutative field, J a vector space over F and $\#$ a map from J to itself satisfying a certain list of properies. These triples are closely related to certain Jordan algebras which have been closely studied by Albert, Jacobson and several of Jacobson's students. They were classified by Petersson and Racine [**5**,**6**]. We give two examples: Let J be a commutative field containing F and suppose either that $J^3 \subseteq F$ or that $[J : F] = 3$ and J/F is separable. We set $x^\# = x^2$ for all $x \in J$ in the first case and $x^\# = x/N(x)$ for all $x \in J^*$ in the second, where N is the norm of the extension J/F. In all the other cases, the dimension of J over F is 9 or 27.

There are three distinct classes of Moufang quadrangles: classical, indifferent and exceptional. The classical quadrangles are parametrized by pseudo-quadratic forms (see 8.2 of [**7**]). The indifferent quadrangles are parametrized by algebraic systems involving certain purely inseparable field extensions in characteristic two. The exceptional quadrangles (of which there are four families) are parametrized by pairs of vector spaces and several maps connecting these vector spaces and the fields over which they are defined. The parameter systems for the first three families involve the even Clifford algebra of a certain type of quadratic form. The parameter systems for the fourth family is still more exotic; these quadrangles (like the indifferent quadrangles and the Moufang octagons) exist only in characteristic two. See [**4**].

The Moufang triangles $\mathcal{T}(A)$ for A a field or a skew-field and the classical quadrangles are the spherical buildings associated with certain classical groups. The remaining Moufang triangles (those parametrized by a Cayley-Dickson division algebra), the remaining quadrangles (except those defined only in characteristic two) and all the Moufang hexagons (except those defined over a purely inseparable field extension in characteristic three) are the spherical buildings associated with k-forms of absolutely simple algebraic groups of k-rank two. All other Moufang polygons, namely those which are defined only in characteristic two or three, are related to groups of mixed type as defined in (10.3.2) of [**7**].

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