ON PROJECTIVE HJELMSLEV PLANES OF LEVEL n

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In this paper, we establish a new (but equivalent) definition of projective Hjelmslev planes of level n. This shows that the nth floor of a triangle building is a projective Hjelmslev plane of level n (a result already announced in **[9]**, but left unproved). This will allow us to characterize Artmann-sequences by means of their inverse limits and to construct new ones. We also deduce a new existence theorem for level n projective Hjelmslev planes. All results hold in the finite as well as in the infinite case.

1. Preliminaries.

DEFINITION 1. An incidence structure $H=(P(H), L(H), I)$ is called a *projective Hjelmslev plane* (or briefly a PH-plane) if it satisfies (H.1), (H.2) and (H.3):

(H.1) there is at least one line joining any two points;

(H.2) there is at least one point on any two lines;

(H.3) there is a canonical epimorphism $\alpha_H : H \rightarrow \mathcal{P}_H$ with $\mathcal{P}_H$ a non-degenerate projective plane, such that $\alpha_H(X)=\alpha_H(Y)$ if and only if either $X, Y \in L(H)$ and X and Y join more than one point, or $X, Y \in P(H)$ and X and Y are on more than one common line, for all $X, Y \in P(H) \cup L(H)$.

DEFINITION 2 (Definition by induction on n). A PH-*plane of level n* is a structure $\mathcal{H}_n=(H_n, H_{n-1}, \ldots, H_1, \alpha^n_{n-1}, \ldots, \alpha^2_1)$ such that

(i) H_1 is a non-degenerate projective plane and H_n is a PH-plane;

(ii) $(H_{n-1}, \ldots, H_1, \alpha^{n-1}_{n-2}, \ldots, \alpha^2_1)$ is a PH-plane of level $n-1$;

(iii) $\alpha^n_{n-1} : H_n \rightarrow H_{n-1}$ is an epimorphism of PH-planes;

(iv) the following conditions (V), (Ma), (Mb), (Mc) and (N) are satisfied.

(V) $\mathcal{P}_{H_n}=\mathcal{P}_{H_{n-1}}$ and $\alpha_{H_{n-1}} \circ \alpha^n_{n-1}=\alpha_{H_n}$.

(Ma) If $P, Q \in P(H_n)$, $L, M \in L(H_n)$, $QILIPIM$, $\alpha^n_{n-1}(P)=\alpha^n_{n-1}(Q)$ and $\alpha_{H_n}(L)=\alpha_{H_n}(M)$, then QIM.

(Mb) The dual statement of (Ma).

(Mc) There exist distinct points $P, Q \in P(H_n)$ such that $\alpha^n_{n-1}(P)=\alpha^n_{n-1}(Q)$ and dually.

The epimorphism $\alpha^n_j : H_n \rightarrow H_j$ is defined by $\alpha^n_j=\alpha^{j+1}_j \circ \alpha^{j+2}_{j+1} \circ \ldots \circ \alpha^n_{n-1}$ for $1 \leq j < n$, and α^n_n is the identity on H_n. Note that $\alpha_{H_n}=\alpha^n_1$.

We define an equivalence relation $(\sim j)$ by $P(\sim j)Q$ if $\alpha^n_j(P)=\alpha^n_j(Q)$, for all $P, Q \in P(H_n)$, $j<n$ and by definition $P(\sim 0)Q$ always. Similarly for lines.

(N) For all $L, M \in L(H_n)$, we have $L(\sim j)M$ if and only if QIM for all $Q \in P(H_n)$ such that QIL and $P(\sim n-j)Q$ for some $P \in P(H_n)$ with $LIPIM$.

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Definitions 1 and 2 are taken from Artmann [**1**] and [**2**].

DEFINITION 3. An *Artmann-sequence* $(H_n, \alpha_n^{n+1})_{n\in\mathbb{N}^*}$ is an infinite sequence of PH-planes together with epimorphisms $\alpha_n^{n+1}: H_{n+1} \to H_n$ such that $(H_n, H_{n-1}, \ldots, H_1, \alpha_{n-1}^n, \ldots, \alpha_1^2)$ is a PH-plane of level n for each n.

B. Artmann showed in [**2**] that there exists an Artmann-sequence $(H_n, \alpha_n^{n+1})_{n\in\mathbb{N}^*}$ for every projective plane H_1.

Besides the notions of projective plane, affine plane and dual affine plane, the following notion will be useful (see [**8**]).

DEFINITION 4. Suppose $\mathcal{P}$ is a projective plane and (P, L) is an incident point-line pair of $\mathcal{P}$. The incidence structure $\mathcal{H}$ obtained from $\mathcal{P}$ by deleting all lines incident with P and all points incident with L is called a *helicopter plane*.

Suppose $\mathcal{H}_n = (H_n, \ldots, H_1, \alpha_{n-1}^n, \ldots, \alpha_1^2)$ is a PH-plane of level n. We remark that (N) implies that every line of H_n is completely determined by the set of points incident with it. Hence we can identify every line with that set. Now let $P \in P(H_n)$; we denote by $\bar{P}^i$, $0 \le i \le n$, the set $\{Q \in P(H_n) \mid P(\sim n - i)Q\}$. We define $\bar{B}_n^i = \{L \cap \bar{P}^i \mid L \in L(H_n),\ P \in P(H_n),\ PIL\}$ for $0 \le i \le n$. Now fix i, $0 \le i \le n-1$, and $b \in \bar{B}_{n-1}^i$. We define an incidence structure $S_b = (P(S_b), L(S_b), I)$ as follows.

$$L(S_b) = \{c \in \bar{B}_n^{i+1} \mid \alpha_{n-1}^n(c) = b\},$$
$$P(S_b) = \{c \cap \bar{P}^i \mid c \in L(S_b),\ P \in c\},$$
$$cIc' \text{ if and only if } c' \subseteq c, \text{ for all } c \in L(S_b) \text{ and } c' \in P(S_b).$$

From Artmann [**1**, Satz 1], it follows that S_b is an affine plane if $b \in \bar{B}_{n-1}^0$ and a dual affine plane if $b \in \bar{B}_{n-1}^{n-1}$.

2. Main Results.

THEOREM. *A series of* PH-*planes* $H_n, H_{n-1}, \ldots, H_1$ *together with epimorphisms* $\alpha_j^{j+1}: H_{j+1} \to H_j$ *for* $j = 1, \ldots, n-1$ *form a* PH-*plane of level* n, $(H_n, H_{n-1}, \ldots, H_1, \alpha_{n-1}^n, \ldots, \alpha_1^2)$, *if and only if they satisfy* $(\mathrm{G.1})_n$, $(\mathrm{G.2})_n$ *and* $(\mathrm{G.3})_n$ *below.*

$(\mathrm{G.1})_n$ $|(\alpha_j^{j+1})^{-1}(X)| > 1$ *for all points and lines* X *in* H_j *and all* j *with* $1 \le j < n$.

Suppose $X, Y \in P(H_n)$ *or* $X, Y \in L(H_n)$ *and let* $\alpha_j^n = \alpha_j^{j+1} \circ \alpha_{j+1}^{j+2} \circ \ldots \circ \alpha_{n-1}^n$, $j < n$ *and* α_n^n *be the identity map in* H_n. *We write* $u(X, Y) = j$ *if* $\alpha_j^n(X) = \alpha_j^n(Y)$ *and* $\alpha_{j+1}^n(X) \ne \alpha_{j+1}^n(Y)$. *Also,* $u(X, Y) = n$ *if* $X = Y$. *If* $P \in P(H_n)$ *and* $L \in L(H_n)$, *then we write* $u(P, L) = j$ *if* $\alpha_j^n(P) I \alpha_j^n(L)$ *and* $\alpha_{j+1}^n(P) \not\!I\, \alpha_{j+1}^n(L)$; $u(P, L) = n$ *if* PIL.

$(\mathrm{G.2})_n$ *If* $P, Q \in P(H_n)$, $L, M \in L(H_n)$ *and* $0 \le k \le \inf\{u(Q, L), u(P, L), u(P, M)\}$, *then*

(i) *there is at least one line joining* P *and* Q *and there is at least one point on both* L *and* M,

(ii) $u(Q, M) \ge k$ *if and only if* $u(Q, P) + u(L, M) \ge k$.

$(\mathrm{G.3})_n$ H_1 *is a non-degenerate projective plane.*

COROLLARY 1. *Suppose $\mathcal{H}_n$ is a* PH-*plane of level n. If $b \in \bar{B}^i_{n-1}$, $0<i<n-1$, then S_b as defined at the end of Section* 1 *is a helicopter plane.*

COROLLARY 2. *Suppose $(H_n, \alpha_n^{n+1})_{n\in\mathbb{N}^*}$ is an Artmann-sequence with inverse limit H_∞. Then H_∞ is a projective plane. Let (R, T) be any coordinatizing* PTR *of H_∞ (see* **[5]** *for the definition); then there exists a surjective map $v: R^2 \to Z \cup \{+\infty\}$ satisfying*

(v.1) *$v(a, b) = +\infty$ if and only if $a = b$, for all $a, b \in R$,*

(v.2) *$v(a, c) \geq \inf\{v(a, b), v(b, c)\}$ and if $v(a, b) \neq v(b, c)$, equality holds, for all $a, b, c \in R$,*

(v.3) *if $T(a_1, b_1, c_1) = T(a_1, b_2, c_2)$ and $T(a_2, b_1, c_1) = T(a_2, b_2, c_3)$, then $v(a_1, a_2) + v(b_1, b_2) = v(c_2, c_3)$.*

Conversely, if $\mathcal{P}$ is a projective plane coordinatized by a PTR(R, T) *admitting a surjective map v as above, then $\mathcal{P}$ is isomorphic to the inverse limit of some Artmann-sequence.*

COROLLARY 3. *Let q be the order of a projective plane, possibly infinite. Let Γ be the set of all projective planes of order q. Then an Artmann-sequence $(H_n, \alpha_n^{n+1})_{n\in\mathbb{N}^*}$ can be constructed step by step which satisfies the following conditions.*

(i) *H_1 is any element of Γ, chosen in advance.*

(ii) *If the level n* PH-*plane $(H_n, \ldots, H_1, \alpha_{n-1}^n, \ldots, \alpha_1^2)$ has already been constructed, then H_{n+1} and the epimorphism α_n^{n+1} can be constructed in such a way that $(H_{n+1}, H_n, \ldots, H_1, \alpha_n^{n+1}, \alpha_{n-1}^n, \ldots, \alpha_1^2)$ becomes a* PH-*plane of level $n+1$ and the following conditions are satisfied. For each $i = 0, 1, \ldots, n$, and each $b \in \bar{B}^i_n$, let $\mathcal{P}_b$ be any prescribed element of Γ. Then S_b is any prescribed helicopter plane, affine plane or dual affine plane arising from $\mathcal{P}_b$ according to whether $0<i<n$, $i = n$ or $i = 0$.*

3. Proofs.

Proof of the theorem. We proceed by induction on $n \in \mathbb{N}^*$. The statement is trivial for $n = 1$. So suppose $n > 1$. We remark that $(G.1)_n$, $(G.2)_n$ and $(G.3)_n$ imply $(G.1)_{n-1}$, $(G.2)_{n-1}$ and $(G.3)_{n-1}$ for $H_{n-1}, \ldots, H_1$ with the epimorphisms α_j^{j+1}.

(I) Assume $H_n, \ldots, H_1$, α_j^{j+1} $(1 \leq j \leq n-1)$ are given satisfying $(G.1)_n$, $(G.2)_n$ and $(G.3)_n$. The conditions (H.1) and (H.2) follow directly from $(G.2)_n$(i). We now show (H.3). Suppose $L, M \in L(H_n)$ and let $\mathcal{P}_{H_n} = H_1$ and $\alpha_{H_n} = \alpha_1^n$. Suppose first $\alpha_{H_n}(L) = \alpha_{H_n}(M)$, so $u(L, M) \geq 1$. Let $P \in P(H_n)$ be incident with both L and M. Let $Q \in P(H_n)$ be such that $u(P, Q) = n-1$ (hence $P \neq Q$) and QIP (Q exists by **[8**, §6.1.1]). Applying $(G.2)_n$(ii) for $k = n$, we obtain $u(Q, M) \geq n$, hence QIM. Suppose now $\alpha_{H_n}(L) \neq \alpha_{H_n}(M)$, so $u(L, M) = 0$. If $P, Q \in P(H_n)$ are incident with both L and M, then applying $(G.2)_n$(ii) for $k = n$, we obtain $u(P, Q) \geq n$, hence $P = Q$. Similarly, one shows the dual. This proves (H.3).

When one remarks that $P(\sim j)Q$ if and only if $u(P, Q) \geq j$ for $P, Q \in P(H_n)$ and similarly for lines, the axioms (V), (Ma), (Mb) and (Mc) become trivial to verify. We now check (N). The "if"-part follows from $(G.2)_n$(ii) for $k = n$. We now show the "only if"-part. Suppose $L, M \in L(H_n)$, $P \in P(H_n)$ with $LIPIM$. Let $P^* \in P(H_n)$ be such that $u(P, P^*) = n - j$ (P^* exists by $(G.1)_n$). Suppose first $u(P^*, L) > n - j$. Let Q^* be a point such that $u(Q^*, L) = 0$ (Q^* is any element in the inverse image under α_{H_n} of any point of H_1 not incident with $\alpha_{H_n}(L)$). Consider any line $L^* \in L(H_n)$ incident with both P^* and Q^*. Since Q^*IL^*, $u(L, L^*) = 0$. Consider the unique point $Q \in P(H_n)$ incident with both L and L^*. Applying $(G.2)_n$(ii) on P^*IL^*IQIL, we obtain $u(P^*, Q) = u(P^*, L) > n - j$. Hence $u(P, Q) = n - j$ and so QIM. By $(G.2)_n$(ii) again, $L(\sim j)M$. Suppose now $u(P^*, L) = n - j$ (it cannot be smaller!). Consider any line M^* incident with both P and P^*. By $(G.2)_n$(ii), $u(L, M^*) = 0$. Let Q^* by any point such that $u(Q^*, L) = u(Q^*, M^*) = 0$ (similar construction to the one above). Choose any line L^* incident with both P^* and Q^*. Let $Q \in P(H_n)$ be incident with both L and L^*. In the same way as before, we obtain $n - j = u(P^*, L) = u(P^*, Q) = u(Q, M^*) = u(Q, P)$ and $u(L, M) \geq j$, hence $L(\sim j)M$ again.

(II) Assume, conversely, $(H_n, \ldots, H_1, \alpha^n_{n-1}, \ldots, \alpha^2_1)$ is a level n PH-plane. We show $(G.1)_n$. The existence of the sequence follows from (V) and (Mc). By Artmann [**1**, Satz 1.a], $|(\alpha^{j+1}_j)^{-1}(X)| > 1$ if $j = n - 1$, and by the induction hypothesis, this is also true for $j < n - 1$. This shows $(G.1)_n$. The condition $(G.2)_n$(i) is equivalent to (H.1) and (H.2). And $(G.2)_n$(ii) is an immediate consequence of (N) if $k = n$, and projecting onto h_k, $k < n$, $(G.2)_n$(ii) follows for all $k < n$. Finally, $(G.3)_n$ follows from (H.3). This completes the proof of the theorem.

By [**8**], this theorem forges a quite unexpected link between two different worlds: the world of affine buildings and the world of PH-planes. It can give a new impulse to the study of the latter. Corollaries 1, 2 and 3 are three first examples of how properties of affine buildings may be translated to properties of level n PH-planes.

Proof of Corollary 1. The axioms $(G.1)_n$, $(G.2)_n$ and $(G.3)_n$ are respectively equivalent to (PS), (RP) and (ND) of [**8**] and [**9**]. The result follows from [**8**, Proposition 6.1.10].

Proof of Corollary 2. The inverse limit H_∞ is a projective plane by Artmann [**2**, Satz über den projektiven Limes]. By [**9**, Theorem (4.4.1)], H_∞ is isomorphic to the geometry at infinity of some triangle building endowed with a maximal set of apartments (see Tits [**7**] for definitions). The result follows from [**9**, Theorem I]. The converse is a direct consequence of [**9**, Main Theorem and §4.4] and the construction of triangle buildings in [**8**].

Proof of Corollary 3. This is a consequence of Ronan's beautiful construction of buildings in [**6**].

Remark. Corollary 3 shows that the structure of level n PH-planes is very "disconnected", in contrast to the impression one might have by considering the constructions of Artmann [**2**], Drake [**4**] and Cronheim [**3**]. In these constructions, wide classes of subgeometries of H_n had to be chosen isomorphic. Note that Corollary 3 generalizes the constructions of Artmann [**2**] and Cronheim [**3**], but not Drake [**4**].

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