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A Characterization of $Q(5, q)$ Using One Subquadrangle $Q(4, q)$

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Let Γ be a finite generalized quadrangle of order (q, q^2), and suppose that it has a subquadrangle Δ isomorphic to $Q(4, q)$. We show that Γ is isomorphic to the classical generalized quadrangle $Q(5, q)$ if at least one of the following holds: (1) all linear collineations of Δ extend to Γ; (2) all subtended ovoids are classical (and we present a uniform proof independent of the characteristic). Further, for q odd, we prove that if every triad $\{x, y, z\}$ of Δ is 3-regular in Γ and $\{x, y, z\}^{\perp\perp} \subset \Delta$, then Γ is classical. We also show that, if for every centric triad $\{x, y, z\}$ of an ovoid $\mathcal{O}$ of the quadrangle $\Delta \cong Q(4, q)$, q odd, all points of $\{x, y, z\}^{\perp\perp}$ belong to $\mathcal{O}$, then $\mathcal{O}$ is classical.

1. DEFINITIONS

A *finite generalized quadrangle* Γ *of order* (s, t), with $s \geq 1$ and $t \geq 1$, is an incidence structure of points and lines with $s+1$ points incident with a line and $t+1$ lines incident with a point, such that for every non-incident point–line pair (p, L) there is exactly one incident point–line pair (M, q) such that $p \mathrm{~I~} M \mathrm{~I~} q \mathrm{~I~} L$. The *distance* between two elements x, y is measured on the incidence graph. If two points x, y (respectively lines L, M) are at distance 2, we call them *collinear* (respectively *concurrent*) and write $x \sim y$ (respectively $L \sim M$). If two elements are at distance 4, we call them *opposite*. The set of all elements at distance i from an element u is denoted by $\Gamma_i(u)$. The set of all elements at distance 2 from both elements u and v (u and v both points or both lines) is denoted by $\{u, v\}^{\perp}$. For p and q opposite points, this set is called the *trace*, and may also be denoted by $p^q = q^p$. The set of all elements at distance 2 from all elements of $\{u, v\}^{\perp}$ is denoted by $\{u, v\}^{\perp\perp}$. If p and q are opposite points, $\{p, q\}^{\perp\perp}$ is called the *hyperbolic line* defined by p and q. If two elements u, v are at distance $k < 4$, we denote the unique element at distance 1 from u and at distance $k-1$ from v by $\mathrm{proj}_u v$, and call this the *projection of* v *onto* u.

A *triad* is a set of three points at mutual distance 4. A *center* of a triad is an element at distance 2 from each point of the triad. If a triad has at least one center, it is called *centric*. A triad in a generalized quadrangle of order (q, q^2), $q \neq 1$, has exactly $q+1$ centers [9, 1.2.4]. Such a triad $\{x, y, z\}$ of a generalized quadrangle of order (q, q^2), $q \neq 1$, is called *3-regular* if the set of points collinear with all centers of the triad (i.e., $\{x, y, z\}^{\perp\perp}$), has size $q+1$. Dual notions hold for a triad of lines.

A *subquadrangle* Δ of order (s', t') of a generalized quadrangle Γ of order (s, t) is a subgeometry of Γ which is itself a generalized quadrangle of order (s', t'). If $s' = s$, Δ is called *full*. If $t' = t$, Δ is called *ideal*. A generalized quadrangle of order (s, t) is called *thin*, whenever s or t is equal to 1, and is called *thick* whenever $s, t \geq 2$. The dual of a generalized quadrangle is obtained by interchanging the roles of points and lines.

For a survey on generalized quadrangles, see [9]. For a survey on generalized polygons (the more general notion), see [13] and [15].

An *ovoid* $\mathcal{O}$ *of a generalized quadrangle* Γ of order (s, t) is a set of points of Γ such that each line of Γ is incident with a unique point of $\mathcal{O}$. It follows that $|\mathcal{O}| = st + 1$. Let Γ be a GQ of order (s, t) with a full sub-GQ Δ of order (s, t') and let p be a point of $\Gamma \setminus \Delta$. Then the set of points of Δ which are collinear with p form an ovoid of Δ (see [9, 2.2.1]). Such an ovoid is said to be *subtended by* p.

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An *ovoid of the projective space* $\mathbf{PG}(3, q)$, $q > 2$, is a set of $q^2 + 1$ points of $\mathbf{PG}(3, q)$, no three of which are collinear. An *ovoid of* $\mathbf{PG}(3, 2)$ is a set of five points no four of which are coplanar.

Let Δ be a subquadrangle of the generalized quadrangle Γ. A group G acting on Δ *extends to* Γ, if for all automorphisms $\alpha \in G$, there is at least one automorphism β acting on Γ such that the restriction of β to Δ is exactly α.

A *thick finite classical generalized quadrangle* is, by definition, one of the following:

- the quadrangle arising from a non-singular Hermitian variety in $\mathbf{PG}(4, q^2)$, denoted by $H(4, q^2)$ and of order (q^2, q^3);
- the quadrangle arising from a non-singular Hermitian variety in $\mathbf{PG}(3, q^2)$, denoted by $H(3, q^2)$ and of order (q^2, q);
- the quadrangle arising from a non-singular elliptic quadric in $\mathbf{PG}(5, q)$, denoted by $Q(5, q)$ and of order (q, q^2); it is the dual of $H(3, q^2)$;
- the quadrangle arising from a non-singular (parabolic) quadric in $\mathbf{PG}(4, q)$, denoted by $Q(4, q)$ and of order (q, q);
- the quadrangle arising from a non-singular symplectic polarity in $\mathbf{PG}(3, q)$, denoted by $W(q)$ and of order (q, q); it is the dual of $Q(4, q)$ and it is self-dual if and only if q is even.

In this article, we take a closer look at $Q(5, q)$ and $Q(4, q)$. So the generalized quadrangle $Q(5, q)$ is the incidence geometry consisting of the points and lines on an elliptic quadric Q in the projective space $\mathbf{PG}(5, q)$. If one intersects Q with a non-tangent hyperplane $\mathbf{PG}(4, q)$ of $\mathbf{PG}(5, q)$, then the point–line structure on the resulting parabolic quadric is the finite generalized quadrangle $Q(4, q)$. Hence $Q(4, q)$ is in a natural way a sub-quadrangle of $Q(5, q)$.

We consider a fixed sub-quadrangle $\Delta \cong Q(4, q)$ contained in $\Gamma = Q(5, q)$. The ovoid of the generalized quadrangle Δ subtended by a point p of $Q(5, q)\backslash\Delta$, will be the set of all points of an elliptic quadric in three dimensions. Indeed, all points of Γ collinear with p are inside a hyperplane Π of $\mathbf{PG}(5, q) \supset Q(5, q)$. The intersection of Π and the four-dimensional (4D) space $\mathbf{PG}(4, q)$ that contains Δ, is a three-dimensional (3D) space, containing the elliptic quadric mentioned. The ovoids of Δ which are elliptic quadrics in some 3D space are called *classical*. For other examples of ovoids on $Q(4, q)$ we refer to [14].

2. Main Results

THEOREM 1. *Let Γ be a GQ of order (q, q^2) and let Δ be a sub-GQ of Γ of order (q, q) with the property that every triad $\{x, y, z\}$ of Δ is 3-regular in Γ and $\{x, y, z\}^{\perp\perp} \subset \Delta$. Then Δ is classical and, if q is odd, each subtended ovoid in Δ is classical.*

THEOREM 2. *Let Γ be a GQ of order (q, q^2) and let Δ be a classical sub-GQ of Γ of order (q, q). Then $\Delta \cong Q(4, q)$. If the linear group G acting on Δ extends to Γ, then all subtended ovoids in Δ are classical.*

THEOREM 3. *Let Γ be a GQ of order (q, q^2) and let Δ be a classical sub-GQ of Γ of order (q, q). If all subtended ovoids in Δ are classical, then Γ itself is classical (and hence isomorphic to $Q(5, q)$).*

COROLLARY 4. *Let Γ be a GQ of order (q, q^2) and let Δ be a sub-GQ of Γ of order (q, q) with the property that every triad $\{x, y, z\}$ of Δ is 3-regular in Γ and $\{x, y, z\}^{\perp\perp} \subset \Delta$. If q is odd, then Γ is classical.*

COROLLARY 5. *Let Γ be a GQ of order (q, q^2) and let Δ be a classical sub-GQ of Γ of order (q, q). If the linear group G acting on Δ extends to Γ, then Γ is classical.*

Theorem 1 and Corollary 4 are, for the odd case, the completion of a theorem stated in [12] (see [9, 5.3.12]). In particular, we shall not need to prove that Δ is classical under the hypotheses of Theorem 1 since this is well known. Likewise, Theorem 3 is not new. For q even, this theorem was already stated in [14]. For q odd, a proof using cohomology theory is given in [1]; the same author has recently simplified the necessary calculations and extended his proof to all q in a yet unpublished manuscript [3]. In the present article however, we provide a purely geometrical proof, valid for any q. By doing so, we explain a step in the geometrical proof provided in [14], that was not elaborated in depth.

Remark that we only deal with finite generalized quadrangles in this article, and as $Q(5, 2)$ (respectively $Q(5, 3)$) is the unique generalized quadrangle of order $(2, 4)$ (respectively order $(3, 9)$) (see e.g., [9]), we may assume that $q \geq 4$.

3. PROOF OF THEOREM 1

PROOF. From [12], it follows that Δ is isomorphic to $Q(4, q)$. To prove the assertion for q odd, we proceed as follows. Let $\mathcal{O}$ be an ovoid subtended by a point $p \in \Gamma \setminus \Delta$. We say that a *conic* of Δ is *subtended* by a point $a \in \Gamma$ if all its points are collinear with a.

- Let $x, y \in \mathcal{O}$. First we show that there are at least $\frac{q+1}{2}$ conics on $\mathcal{O}$ through x and y. The trace $\{x, y\}^{\perp}$ has $q + 1$ points in common with Δ. Take a point $a \in \{x, y\}^{\perp} \cap \Delta$. As $\mathcal{O}$ is an ovoid of Δ, each line of Δ through a has a point in common with $\mathcal{O}$. Let z be such a point of $\mathcal{O} \setminus \{x, y\}$ collinear with a. As each triad of $Q(4, q)$, q odd, has exactly zero or two centers in $Q(4, q)$ ([9, 1.3.6.iii]), the triad $\{x, y, z\}$ has a unique second center b in Δ. The trace, in $Q(4, q)$, of two non-collinear points of $Q(4, q)$ is a conic on $Q(4, q)$. We show that the conic $\{a, b\}^{\perp} \cap \Delta = C_{xyz}$ through x, y and z, is completely contained in the ovoid $\mathcal{O}$. As each point of $\{x, y, z\}^{\perp\perp}$ is—by definition—collinear with $a, b \in \{x, y, z\}^{\perp}$ and—by assumption—$\{x, y, z\}^{\perp\perp} \subset \Delta$, each point of $\{x, y, z\}^{\perp\perp}$ is in $\{a, b\}^{\perp} \cap \Delta = C_{xyz}$, with $|C_{xyz}| = |\{x, y, z\}^{\perp\perp}| = q + 1$. Hence $C_{xyz} = \{x, y, z\}^{\perp\perp}$. As each point r of $\{x, y, z\}^{\perp\perp}$ is collinear with $p \in \{x, y, z\}^{\perp}$, $r (\in \Delta)$ will be a point of the ovoid $\mathcal{O}$ subtended by p. Hence the conic C_{xyz} through x, y and z is completely contained in the ovoid $\mathcal{O}$. As we can repeat the same reasoning for all points in $\{x, y\}^{\perp} \cap \Delta$, we obtain exactly $\frac{q+1}{2}$ conics on $\mathcal{O}$ through x and y which are subtended by two points of Δ. A conic on $\mathcal{O}$ subtended by two points of Δ will be called an *s-conic*.
- Now we show that there are $\frac{q(q+1)}{2}$ s-conics on $\mathcal{O}$ through a point $x \in \mathcal{O}$. By the former reasoning, we constructed $\frac{q+1}{2}$ s-conics through each of the $(q^2 + 1)q^2$ pairs of points on $\mathcal{O}$, so there are $\frac{(\frac{q+1}{2})(q^2+1)q^2}{(q+1)q} = \frac{q(q^2+1)}{2}$ such conics on $\mathcal{O}$. Hence there will be $\frac{\frac{q(q^2+1)}{2}(q+1)}{q^2+1} = \frac{q(q+1)}{2}$ s-conics through a single point of $\mathcal{O}$.
- Thirdly, we count the number of s-conics on $\mathcal{O}$ through a point x of $\mathcal{O}$ that share exactly one point (the point x) with a given s-conic $C \subset \mathcal{O}$ through x. As there are q points on C different from x, and as there are $\frac{q-1}{2}$ s-conics different from C through x and a second point of C, there are $q(\frac{q-1}{2})$ s-conics different from C that intersect C in two points. Hence there are $\frac{q(q+1)}{2} - 1 - q(\frac{q-1}{2}) = q - 1$ s-conics that share just the point x with C. We shall denote those s-conics by C_i, $i = 1, \ldots, q - 1$, and put $C = C_0$.

- Now we prove that also those $q-1$ conics C_i, $i>0$, mutually share exactly one point. Suppose C is subtended by the points $a, b \in \Delta$. Take a line L of Δ through x, not through a or b. The projections y', z' on L of the points $y, z \in C \setminus \{x\}$, $y \neq z$, will never be equal, as this would imply that the triad $\{x, y, z\}$ has three centers (i.e., a, b and y'). Hence there is a one-to-one correspondence between the points of C and the points on the line L through x. So every conic on $\mathcal{O}$ subtended by a point of L, will intersect C in at least two points (x included). So none of the points of L can subtend a conic C_i. Hence the subtending points of the $q-1$ conics C_i, $i = 1, 2, \ldots, q-1$, can be found on the lines xa and xb (for each conic, there is one subtending point on xa and one on xb). If two of those conics, say subtended by r respectively s, with $r, s \in xa$, would intersect each other in a point $u \neq x$, there would arise a triangle with vertices u, r and s. So we found q s-conics through x that mutually just have x in common—and hence cover all q^2+1 points of $\mathcal{O}$.
- Now by Gevaert *et al.* [8] all conics C_i, $i = 0, 1, \ldots, q-1$, have a common tangent line T at x. By the same paper, as $\mathcal{O}$ contains conics different from $C_0, C_1, \ldots, C_{q-1}$, the ovoid $\mathcal{O}$ is classical, that is, belongs to a $\mathbf{PG}(3, q)$. □

Remark that we only used the fact that every triad $\{x, y, z\}$ *which is centric in* Δ is 3-regular in Γ and satisfies $\{x, y, z\}^{\perp\perp} \subset \Delta$. Triads without center in Δ are not needed to prove the assertion for q odd.

From the previous proof, we can also deduce the following corollary.

COROLLARY 6. *Let Δ be the classical GQ $Q(4, q)$ of order (q, q), q odd, and let $\mathcal{O}$ be an ovoid of Δ such that for every centric triad $\{x, y, z\}$ of $\mathcal{O}$, the set $\{x, y, z\}^{\perp\perp}$ belongs to $\mathcal{O}$. Then the ovoid $\mathcal{O}$ is classical.*

4. PROOF OF THEOREM 2

PROOF. As each point of Γ will induce an ovoid in Δ, and the classical generalized quadrangle $W(q)$ has no ovoids for q odd (see [9, 3.4.1]), Δ is isomorphic to $Q(4, q)$. This proves the first assertion.

From now on, $\mathcal{O}$ is a subtended ovoid in Δ. The linear group G acting on $\Delta \cong Q(4, q)$ (or, equivalently, acting on the dual $W(q)$), is the group $\mathbf{PGSp}_4(q)$ of all collineations of $W(q)$ induced by $\mathbf{PGL}_4(q)$ (see [15, pp. 152–154]), and has order $q^4(q^4-1)(q^2-1)$.

As every point in $\Gamma \setminus \Delta$ subtends exactly one ovoid, the number of points in $\Gamma \setminus \Delta$ (i.e., $q^2(q^2-1)$) is an upper bound for the size of the orbit $G(\mathcal{O})$ of a subtended ovoid $\mathcal{O}$, and hence we have a lower bound for the size of the stabilizer $G_{\mathcal{O}}$ of a subtended ovoid $\mathcal{O}$ under G.

$$
\begin{array}{rrcl}
 & |G| & = & |G_{\mathcal{O}}| \cdot |G(\mathcal{O})| \\
\Rightarrow & |G_{\mathcal{O}}| & \geq & \frac{|G|}{q^2(q^2-1)} \\
\Rightarrow & |G_{\mathcal{O}}| & \geq & q^2(q^4-1).
\end{array}
$$

Now the proof is split up, according to the characteristic of $\mathbf{GF}(q)$.

For q odd, we proceed as follows. We take a triad in Δ which is centric in Δ, say $\{p_0, p_1, p_2\}$. Let p be a center of the triad in $\Gamma \setminus \Delta$, then p_0, p_1 and p_2 belong to the ovoid $\mathcal{O}_p$ subtended by p. As we know a bound for the size of the group $G_{\mathcal{O}}$ stabilizing $\mathcal{O}_p$, we can deduce that $\{p_0, p_1, p_2\}^{\perp\perp}$ is contained in $\mathcal{O}_p$, hence contained in Δ. By Theorem 1, $\mathcal{O}_p$ is classical.

For q even, we point out that for the (self-) dual generalized quadrangle $W(q)$ in $\mathbf{PG}(3, q)$, the group stabilizing $\mathcal{O}$ is 3-transitive. This allows us to conclude that $\mathcal{O}$ is classical.

q odd

The group $G_{\mathcal{O}}$ has order at least $q^2(q^4-1)$, but cannot act 3-transitively on the point set of $\mathcal{O}$. Indeed, we show that not all triads of $\mathcal{O}$ are centric, and as a centric triad will never be the image of a non-centric triad, $G_{\mathcal{O}}$ is not 3-transitive on $\mathcal{O}$.

Let X be the number of points of Δ that are centers of some triad $\{p_0, p_1, p_2\}$ of $\mathcal{O}$. As a point of $\mathcal{O}$ can never be such a center, and each point not in $\mathcal{O}$ is a center of such a triad, $X = |\Delta \setminus \mathcal{O}| = q^3+q$. So we count $X(q+1)q(q-1)/6 = q^2(q^4-1)/6$ pairs $(c, \{p_0, p_1, p_2\})$ with c a center of the triad $\{p_0, p_1, p_2\}$. If Y is the number of centric triads on $\mathcal{O}$, we count $2Y$ pairs $(c, \{p_0, p_1, p_2\})$ (as any triad has zero or two centers, see [9, 1.3.6iii]). Hence $Y = \frac{q^2(q^4-1)}{12}$, so not all triads of $\mathcal{O}$ (they are $(q^2+1)q^2(q^2-1)/6$ in total) are centric. Similarly, one shows that exactly $\frac{q^2-1}{2}$ triads $\{p_0, p_1, p_2\} \subset \mathcal{O}$, with p_0 and p_1 given, are centric.

Now we concentrate on the stabilizer $G_{\mathcal{O},x_0,x_1,x_2}$ fixing $\mathcal{O}$ and three points $x_0, x_1, x_2 \in \mathcal{O}$. As the orbit for $G_{\mathcal{O}}$ of x_0 has at most q^2+1 elements, the stabilizer $G_{\mathcal{O},x_0}$ of x_0 in $G_{\mathcal{O}}$ has order at least $q^2(q^2-1)$.

As the orbit for $G_{\mathcal{O},x_0}$ of x_1 has size at most q^2, the group $G_{\mathcal{O},x_0,x_1}$ has order at least (q^2-1).

As $G_{\mathcal{O},x_0,x_1}$ is not transitive on the point set of $\mathcal{O} \setminus \{x_0, x_1\}$, the orbit for $G_{\mathcal{O},x_0,x_1}$ of x_2 has less than q^2-1 elements, hence the group $G_{\mathcal{O},x_0,x_1,x_2}$ has order greater than 1. Let $\{p_0, p_1, p_2\} \subset \mathcal{O}$ be a *centric* triad of Δ, with centers x and y.

- Suppose the stabilizer $G_{\mathcal{O},p_0,p_1,p_2}$ has order greater than 2. As the orbit of the center x for $G_{\mathcal{O},p_0,p_1,p_2}$ has size at most 2, the size of the stabilizer of x in $G_{\mathcal{O},p_0,p_1,p_2}$ is greater than 1. Let α be a non-identity collineation of this group $G_{\mathcal{O},p_0,p_1,p_2,x}$. As α fixes the three lines xp_0, xp_1, xp_2, this linear collineation fixes all lines through x. As also y is fixed under α, the trace x^y is pointwise fixed. Let p_3 be a point of $\mathcal{O}$ collinear with x, and suppose $p_3 \notin x^y$. As $p_3 = p_3^\alpha$, the points x, p_3 and $xp_3 \cap x^y$ would be three fixed points on the line xp_3, hence all points on xp_3 are fixed and α must be the identity by [15, 4.4.2 (v)]. Hence $p_3 \in x^y$, and every point of $x^y = \{p_0, p_1, p_2\}^{\perp\perp}$ belongs to the ovoid. So, by Corollary 6, $\mathcal{O}$ is classical.
- Suppose the stabilizer $G_{\mathcal{O},p_0,p_1,p_2}$ has order exactly 2. Hence we can assume that the non-identity collineation of $G_{\mathcal{O},p_0,p_1,p_2}$ interchanges the centers x and y (otherwise, the same reasoning as above holds, to conclude that all points of $\{p_0, p_1, p_2\}^{\perp\perp}$ are inside $\mathcal{O}$).
  Also, the size of the orbit of the (ordered) triple (p_0, p_1, p_2) is at least $\frac{q^2(q^4-1)}{2}$, hence equal to $6Y = \frac{q^2(q^4-1)}{2}$ since exactly $6Y$ *ordered* triples are centric. Hence $G_{\mathcal{O}}$ acts transitively on the set of ordered centric triads. Consequently $G_{\mathcal{O}}$ acts 2-transitively on $\mathcal{O}$. Dually, with $\mathcal{O}$ there corresponds a spread $\mathcal{S}$ of $W(q)$ on which $\mathbf{PGSp}_4(q)$ acts 2-transitively. Now, by [10] and [4], the spread $\mathcal{S}$ is regular, hence $\mathcal{O}$ is classical.

q even

To simplify the argumentation, we consider the symplectic quadrangle $W(q)$ in $\mathbf{PG}(3, q)$ instead of $Q(4, q)$ (which are isomorphic for q even). The group $G_{\mathcal{O}}$ has order at least $q^2(q^4-1)$. Let p_0, p_1 and p_2 be three distinct points of $\mathcal{O}$.

As the orbit for $G_{\mathcal{O}}$ of p_0 has at most q^2+1 elements, the group $G_{\mathcal{O},p_0}$ has order at least $q^2(q^2-1)$.

As the orbit for $G_{\mathcal{O},p_0}$ of p_1 has size at most q^2, the group $G_{\mathcal{O},p_0,p_1}$ has order at least q^2-1.

As the orbit for $G_{\mathcal{O},p_0,p_1}$ of p_2 has at most q^2-1 elements, the group $G_{\mathcal{O},p_0,p_1,p_2}$ is trivial if and only if $G_{\mathcal{O}}$ acts sharply 3-transitively on $\mathcal{O}$, and $G_{\mathcal{O}}$ has order $q^2(q^4-1)$.Note that $\mathcal{O}$ being an ovoid of $W(q)$ is also an ovoid of $\mathbf{PG}(3, q)$; see [11].

- If $G_{\mathcal{O},p_0,p_1,p_2}$ is trivial and so $G_{\mathcal{O}}$ acts 3-transitively on the ovoid $\mathcal{O}$ of $\mathbf{PG}(3,q)$, then $\mathcal{O}$ is an elliptic quadric; see [5, p. 277, 53]. Hence $\mathcal{O}$ is classical.
- So we may assume that $G_{\mathcal{O},p_0,p_1,p_2}$ is not trivial. We show that in this case the order of $G_{\mathcal{O},p_0,p_1,p_2}$ is exactly 2, by pointing out that the non-identity element of $G_{\mathcal{O},p_0,p_1,p_2}$ is unique. First we remark that, since $\mathcal{O}$ is an ovoid of $\mathbf{PG}(3,q)$, the three distinct points $p_0, p_1, p_2 \in \mathcal{O} \subset W(q)$ define a plane in $\mathbf{PG}(3,q)$. If ζ is the symplectic polarity defining $W(q)$ and if π is the plane containing p_0, p_1, p_2, then $\pi^\zeta = x$ is the unique center of $\{p_0, p_1, p_2\}$. As $\{p_0, p_1, p_2\}$ is fixed elementwise by every $\alpha \in G_{\mathcal{O},p_0,p_1,p_2}$, also x is fixed by every such α. As p_0, p_1, p_2 and x are four linearly independent points in the plane $\pi = \langle p_0, p_1, p_2\rangle$, α fixes every point of this plane. Hence π is the axis of the perspectivity α. Let c be the center of α and let a be a point of $\mathcal{O}$ which is not fixed. Then $a, a^\alpha, a^{\alpha^2}$ are three points of $\mathcal{O}$ on the same line ac of $\mathbf{PG}(3,q)$, hence $a = a^{\alpha^2}$. Consequently α is an involution. As there is an odd number of points on a line, the center of the involution α should be in the axis, that is, $c \in \pi$, and hence α is an elation.

  Now we look for the center c of α, somewhere in the plane π. If $c \in \mathcal{O}$, there would be three points of $\mathcal{O}$ on a line of $\mathbf{PG}(3,q)$ (namely, c, a and a^α for all $a \in \mathcal{O} \setminus \pi$). If $c \neq x$, $c \in \pi \setminus \mathcal{O}$, then there are (precisely) q lines of the quadrangle through c, not in π. Let L be such a line, with l the unique point of $\mathcal{O}$ on L. Then l^α also belongs to $\mathcal{O}$, lies on L, and is different from l. Hence there are two points of $\mathcal{O}$ on a line of the quadrangle, a contradiction. So $c = x$ is the center of the elation α. Now we show that α is unique. Suppose α' is different from α and also belongs to $G_{\mathcal{O},p_0,p_1,p_2}$. Let b be a point of $\mathcal{O}$, not in the plane π. Then $b, b^\alpha, b^{\alpha'}$ are three different points of $\mathcal{O}$ on the line xb of $\mathbf{PG}(3,q)$, a contradiction. Hence the order of $G_{\mathcal{O},p_0,p_1,p_2}$ is exactly 2.

  By the formula $|G_{\mathcal{O}}| = |G_{\mathcal{O},p_0,p_1,p_2}||G_{\mathcal{O}}(p_0,p_1,p_2)|$, we know that the orbit of an ordered triple (p_0,p_1,p_2) of $\mathcal{O}$ has length at least $\frac{q^2(q^4-1)}{2}$. Hence $|G_{\mathcal{O}}(p_0,p_1,p_2)|$ is either $\frac{q^2(q^4-1)}{2}$ or $q^2(q^4-1)$. If $|G_{\mathcal{O}}(p_0,p_1,p_2)| = q^2(q^4-1)$, then $G_{\mathcal{O}}$ acts 3-transitively on $\mathcal{O}$ and we are done by [5, p. 277, 53]. So we may assume that $|G_{\mathcal{O}}(p_0,p_1,p_2)| = \frac{q^2(q^4-1)}{2}$. Hence $|G_{\mathcal{O}}| = q^2(q^4-1)$. As $|G_{\mathcal{O}}| = |G_{\mathcal{O},p_0,p_1}||G_{\mathcal{O}}(p_0,p_1)|$ and $|G_{\mathcal{O}}(p_0,p_1)| \leq (q^2+1)q^2$, we have $|G_{\mathcal{O},p_0,p_1}| \geq q^2-1$. Also, $|G_{\mathcal{O},p_0,p_1}| = |G_{\mathcal{O},p_0,p_1,p_2}||G_{\mathcal{O},p_0,p_1}(p_2)|$. We know that $|G_{\mathcal{O},p_0,p_1,p_2}| = 2$. It follows that $|G_{\mathcal{O},p_0,p_1}(p_2)| \geq \frac{q^2-1}{2}$. Hence $|G_{\mathcal{O},p_0,p_1}(p_2)| \in \{q^2-1, \frac{q^2-1}{2}\}$. As q is even, $|G_{\mathcal{O},p_0,p_1}(p_2)| = q^2-1$, and so $|G_{\mathcal{O},p_0,p_1}| = 2(q^2-1)$ and $|G_{\mathcal{O}}(p_0,p_1)| = \frac{(q^2+1)q^2}{2}$. Now let (a,b) and (a',b') be ordered pairs, each consisting of distinct points of $\mathcal{O}$. Let $c_1, c_2 \in \mathcal{O}\setminus\{a,a'\}$, with $c_1 \neq c_2$. As $|G_{\mathcal{O},c_1,c_2}(a)| = q^2-1$, there is an element $\theta \in G_{\mathcal{O},c_1,c_2}$ for which $a^\theta = a'$; let $b^\theta = b''$. Now let $d \in \mathcal{O} \setminus \{a', b'', b'\}$. Then there is an element $\theta' \in G_{\mathcal{O},a',d}$ for which $b''^{\theta'} = b'$. Hence $a^{\theta\theta'} = a'$ and $b^{\theta\theta'} = b'$. It follows that $|G_{\mathcal{O}}(p_0,p_1)| = (q^2+1)q^2$, a contradiction. □

REMARK. Another approach of the proof for q odd goes as follows: one can show that the subgroups of $\mathbf{PGL}_4(q)$ large enough to contain $G_{\mathcal{O}}$ can not contain $G_{\mathcal{O}}$ unless they are isomorphic to the stabilizer of the classical ovoid. The only cases to consider (and exclude) were the stabilizer of a point and the stabilizer of a line, using [6]. This was suggested to us by Penttila.

5. PROOF OF THEOREM 3

5.1. Definitions. Some of the lemmas and most notions used in the following paragraphs can also be found in [1, 2, 14], but we recall them for coherency reasons.

Let Γ be a generalized quadrangle of order (q, q^2), and Δ a generalized subquadrangle of order (q, q), isomorphic to $Q(4, q)$. If L is a line of $\Gamma \setminus \Delta$, then the unique point of L in Δ will be denoted by the corresponding lowercase letter l. An ovoid $\mathcal{O}$ of Δ subtended by a point p of $\Gamma \setminus \Delta$, is denoted by $\mathcal{O}_p$.

An ovoid $\mathcal{O}$ in Δ is called *doubly subtended* if there are exactly two points in $\Gamma \setminus \Delta$ that subtend $\mathcal{O}$.

A *rosette* (*of ovoids*) $\mathcal{R}$ of a $Q(4, q)$ based at a point r of $Q(4, q)$ is a set of ovoids with pairwise intersection $\{r\}$ such that $\{\mathcal{O} \setminus \{r\} | \mathcal{O} \in \mathcal{R}\}$ is a partition of the points of $Q(4, q)$ not collinear with r. The point r is called the *base point* of $\mathcal{R}$. It follows that a rosette has q ovoids.

A *rosette* (*of conics*) R of a $Q^-(3, q)$ based at a point r is a set of plane intersections of size $q + 1$ with pairwise intersection $\{r\}$ such that $\{C \setminus \{r\} | C \in R\}$ is a partition of the points of $Q^-(3, q)$. It follows that a rosette of conics has q elements and that these q conics have the same tangent at r.

A line L of $\Gamma \setminus \Delta$ with $L \cap \Delta = \{l\}$ will *subtend a rosette* as follows: every point of $L \setminus \{l\}$ subtends an ovoid of Δ through l. As there are no triangles in Γ, two ovoids $\mathcal{O}_x, \mathcal{O}_y$ with x, y different points of $L \setminus \{l\}$, will never share a second point. Hence $\mathcal{O}_x, \mathcal{O}_y$ have pairwise intersection l, and $\{\mathcal{O}_x\}_{x \in L \setminus \{l\}}$ is a rosette.

A *flock* $\mathcal{F}$ of an ovoid $\mathcal{O}$ of $\mathbf{PG}(3, q)$ is a partition of all but two points of $\mathcal{O}$ into $q - 1$ disjoint ovals C_i. The remaining points x, y are called the *carriers* of the flock. A flock $\mathcal{F} = \{C_1, \ldots, C_{q-1}\}$ is called *linear* if all planes π_i, with $C_i \subset \pi_i$, contain a common line L. It has been proved that every flock of an ovoid is linear (see [7]).

A linear flock is uniquely defined by its two carriers, or by two of its ovals, or by an oval and a carrier. (Indeed, the line L that is common to all planes π_i of the ovals $C_i \in \mathcal{F}$, is also the intersection of the tangent planes of $\mathcal{O}$ at the carriers of $\mathcal{F}$ (equivalently, if q is odd, L is the polar line of the line xy with respect to the polarity defining $\mathcal{O}$).)

5.2. Lemmas. For the following lemmas, we assume Γ to be a GQ of order (q, q^2) with a classical sub-GQ Δ of order (q, q). We also assume that all subtended ovoids of Δ by points of $\Gamma \setminus \Delta$ are classical.

LEMMA 7. *Each subtended ovoid in Δ is doubly subtended.*

PROOF. For any triad $\{x, y, z\}$ of Γ we have $|\{x, y, z\}^{\perp}| = q + 1$, so an ovoid of Δ is subtended by at most two points of Γ. As there are $\frac{q^2(q^2-1)}{2}$ classical ovoids in $Q(4, q)$ (i.e., the number of elliptic quadrics on $Q(4, q)$), there are at most that many subtended classical ovoids in $Q(4, q)$. As each subtended ovoid in Δ is maximally doubly subtended, there are at most $2\frac{q^2(q^2-1)}{2}$ points in $\Gamma \setminus \Delta$ (as each point of $\Gamma \setminus \Delta$ subtends a classical ovoid). As the number of points of $\Gamma \setminus \Delta$ is equal to $q^2(q^2 - 1)$, each subtended ovoid is exactly doubly subtended. □

If two distinct points $x, y \in \Gamma \setminus \Delta$ subtend the same ovoid, they are called *twins*, and we write $x^{\mathrm{tw}} = y$. Also, we call two ovoids *tangent at a point x* if their intersection is precisely $\{x\}$.

LEMMA 8. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two subtended ovoids in Δ, tangent at a, then there is a unique rosette of classical ovoids through $\mathcal{O}_1$ and $\mathcal{O}_2$, and moreover this rosette is subtended by a line.*

PROOF. Let Π_i be the 3D space containing $\mathcal{O}_i$, with $i = 1, 2$. As $\mathcal{O}_1 \cap \mathcal{O}_2 = \{a\}$, the common plane π of Π_1 and Π_2 contains a. As π contains a unique point of $\mathcal{O}_i$, it is the unique tangent plane of $\mathcal{O}_i$ at a in Π_i, $i = 1, 2$. Let $\mathcal{R}_* = \{\mathcal{O}_i\}_{i=1}^q$ be the rosette we want to construct. If $\langle \mathcal{O}_3 \rangle$ had an intersection plane with $\langle \mathcal{O}_1 \rangle$ different from π, we would have $|\mathcal{O}_1 \cap \mathcal{O}_3| = q + 1$, a contradiction. So all $\langle \mathcal{O}_i \rangle$, with $\mathcal{O}_i$ in $\mathcal{R}_*$, should contain π. Hence taking the intersection of $Q(4, q)$ with the q 3D spaces through π that are not tangent to $Q(4, q)$ at a, we constructed $\mathcal{R}_*$ in a unique way.

Now we show that $\mathcal{R}_*$ is subtended. Let $\mathcal{O}_1$ be subtended by the point k_1. The rosette $\mathcal{R}_L$ subtended by $L := ak_1$ will, of course, contain $\mathcal{O}_1$. Let $\mathcal{O}'_i$ be an ovoid of $\mathcal{R}_L$ subtended by $x_i \in L \setminus \{k_1\}$, x_i collinear with a point of $\mathcal{O}_j \setminus \{a\}$. Let Π'_i be the 3D space containing $\mathcal{O}'_i$. Using the same arguments as above, we conclude that Π_1 and Π'_i intersect in the unique plane π tangent to $\mathcal{O}_1$ at a in Π_1. As this plane is the same as the one constructed above, $\mathcal{O}_j$ coincides with $\mathcal{O}'_i$. Hence $\mathcal{R}_*$ is subtended by the line L. □

From this result, it follows that to each line L of $\Gamma \setminus \Delta$ subtending the rosette $\mathcal{R}_L = \{\mathcal{O}_i\}_{i=1}^q$, one can associate the unique plane π_L being the common plane of all 3D spaces Π_i, with Π_i containing $\mathcal{O}_i$. We shall refer to the plane constructed in this way as *the tangent plane π_L of Δ defined by L*.

LEMMA 9. *If two subtended ovoids $\mathcal{O}_1$ and $\mathcal{O}_2$ of Δ are tangent at some point a, and the point k_i subtends $\mathcal{O}_i$ $(i = 1, 2)$, then either k_1 and k_2 (and hence k_1^{tw} and k_2^{tw}) are collinear, or k_1^{tw} and k_2 (and hence k_1 and k_2^{tw}) are collinear.*

PROOF. By assumption we have $\mathcal{O}_1 \cap \mathcal{O}_2 = \{a\}$. Suppose $k_1^{\text{tw}} \not\sim k_2$, $k_1 \not\sim k_2$. Then the q ovoids subtended by the q points on ak_1 form the unique rosette through $\mathcal{O}_1$ and $\mathcal{O}_2$ (Lemma 8). But the same holds for the points on ak_1^{tw} and ak_2. Hence there are $3q$ different points defining q ovoids. This is impossible, as we know that each ovoid is doubly subtended (Lemma 7). □

LEMMA 10. *Let $\mathcal{R}$ be a rosette of classical ovoids with base point r, and let $\mathcal{O}$ be a classical ovoid not belonging to this rosette. If $r \notin \mathcal{O}$, then the intersection of $\mathcal{R} \cup \{\pi_r\}$, with π_r the tangent hyperplane of $Q(4, q)$ at r, and $\mathcal{O}$ consists of a flock $\mathcal{F}$ and its carriers a, b. If $r \in \mathcal{O}$, then the intersection of $\mathcal{R}$ and $\mathcal{O}$ is a rosette of q conics on $\mathcal{O}$ through r.*

PROOF. Obvious. □

5.3. Sketch of the proof of Theorem 3. In order to prove the result, we use the concept of a regular pair of lines. A pair of lines of a generalized quadrangle of order (s, t) is called *regular* if it is contained in a (necessarily unique) subquadrangle of order $(s, 1)$.

In the first part of the proof, we show that all pairs of lines of Γ are regular if they contain twins. Secondly, we show the same for lines not containing twins. These results make sure that we can use a lot of grids for constructing a lot of classical subquadrangles, as shown in the third part. In the fourth part, we show that we constructed enough classical subquadrangles (i.e., one through every dual window of Γ), so that we must conclude that Γ is classical too.

5.4. Part 1*: regularity for line pairs containing twins.*

THEOREM 11. *Let Γ and Δ be as above. Let the points l' and k' of $\Gamma \setminus \Delta$ be twins, and consider a line L through l', and a line K through k', with $L \cap K = \phi$. Then (L, K) is a regular pair of lines.*

PROOF. The subtended ovoid $\mathcal{O} = \mathcal{O}_{l'} = \mathcal{O}_{k'}$ intersects L in l and K in k. The flock of $\mathcal{O}$ with carriers l and k is denoted by $\mathcal{F}$.

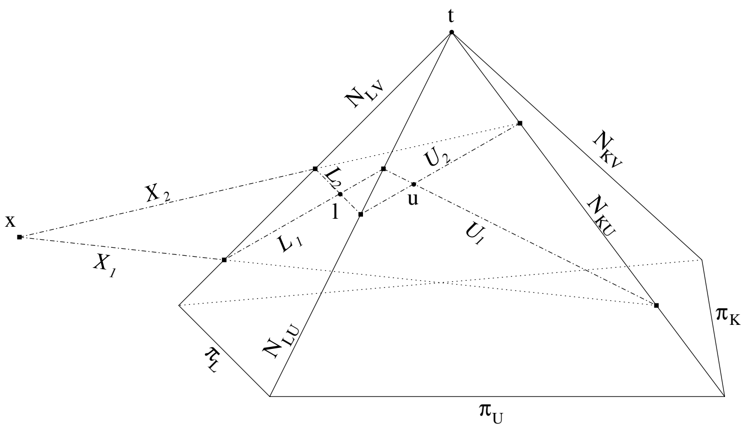
1. First we show that every line of $\{L,K\}^{\perp} \setminus \{l'k, lk'\}$ corresponds to the flock $\mathcal{F}$ of $\mathcal{O}$.
   Consider a line U of $\{L,K\}^{\perp}$, different from lk' and $l'k$. We put $U \cap \Delta = \{u\}$, $U \cap L = \{l''\}$, $U \cap K = \{k''\}$. Let $\mathcal{R}$ be the rosette of ovoids with base point u subtended by the line U. As $u \notin \mathcal{O}$ (avoiding triangles), $\mathcal{R}$ intersects $\mathcal{O}$ in a flock together with its two carriers (Lemma 10). As $l'' \in U \cap L$ subtends an ovoid $\mathcal{O}_{l''}$ touching $\mathcal{O}$ in l, l'' defines the single point l on $\mathcal{O}$. Similarly for k defined by $k'' \in U \cap K$. Hence every line $U \in \{L,K\}^{\perp} \setminus \{l'k, lk'\}$ defines on $\mathcal{O}$ the flock $\mathcal{F}$ of $\mathcal{O}$ with carriers l and k.
2. Now we can show the regularity of L and K.
   Put $U_0 := lk'$, $U_1 := l'k$ and $\{L,K\}^{\perp} := \{U_i\}_{i:0\to q}$. Let N be any line of Γ distinct from L and from K. We claim that, if N intersects U_2 and U_3, then it will also intersect U_0 and U_1. Using this result, we shall show that N also intersects U_i for $i \geq 4$.
   The intersection points of N with U_2 and U_3 are respectively n_2 and n_3. As n_2 and n_3 are on lines of $\{L,K\}^{\perp}$, both conics $C_{n_2} := \mathcal{O} \cap \mathcal{O}_{n_2}$ and $C_{n_3} := \mathcal{O} \cap \mathcal{O}_{n_3}$ belong to the flock $\mathcal{F}$ of $\mathcal{O}$. Hence, by Lemma 10, every point n_i of N will define an element $\mathcal{O}_{n_i}$ of $\mathcal{F} \cup \{l,k\}$. So one of the points of N, say n_0, will define the carrier l, or, equivalently, subtend an ovoid tangent to $\mathcal{O}$ at the point l. Hence $n_0 \sim l$. But $\mathcal{O}_{n_0}$ tangent to $\mathcal{O}$ implies $n_0 \sim l'$ or $n_0 \sim k'$ (see Lemma 9). The first case ($n_0 \sim l'$) yields a triangle, so n_0 is collinear with k'. This implies $n_0 \in lk' = U_0$, so N and U_0 intersect.
   The same argument holds for the point $n_1 \in N$ that defines the carrier k of $\mathcal{F}$: the point n_1 belongs to $l'k = U_1$, so N and U_1 intersect. This shows our claim.
   Now we show that, if N intersects U_2 and U_3 (and hence U_0 and U_1), N also intersects U_i for $i \geq 4$. To avoid too many indices, we show this for $i = 4$. Put $\mathrm{proj}_{U_4} n_2 = p$. By our claim, the line $n_2 p$ intersects $k'l$, inducing a triangle if $n_2 p \neq N$. Hence $p \mathrm{\ I\ } N$. This concludes the proof. □

5.5. *Part 2: regularity for line pairs not containing twins.*

THEOREM 12. *Let Γ and Δ be as above. Let L, K be two opposite lines of $\Gamma \setminus \Delta$, such that no pair of points (l', k'), with $l', k' \notin \Delta$, can be found such that $l' \in L$, $k' \in K$ and $l'^{\mathrm{tw}} = k'$. Then (L,K) is a regular pair of lines.*

PROOF. Consider two lines U, V of $\Gamma \setminus \Delta$ in $\{L,K\}^{\perp}$. Again, corresponding uppercase and lowercase letters are used for a line of $\Gamma \setminus \Delta$, respectively the unique point of Δ on that line. So we can consider the four points l, k, u and v in Δ, and we assume that they are all different. By Theorem 11 we may suppose that $\{U,V\}^{\perp}$, respectively $\{L,K\}^{\perp}$, does not contain two lines A and B for which there exist points a', b' with $a' \in A$, $b' \in B$ and $a'^{\mathrm{tw}} = b'$.

1. In the first part of this proof, we show that l, k, u and v belong to a common plane.
   Consider the tangent planes π_L, π_K, π_U and π_V at Δ defined by respectively L, K, U and V (see definition following Lemma 8).
   - Let a be the common point of U and L. As a subtends the ovoid $\mathcal{O}_a$ that belongs to the rosette $\mathcal{R}_L$ as well as to the rosette $\mathcal{R}_U$, the planes π_L and π_U both belong to the 3D space Π_a defined by $\Pi_a \cap Q(4,q) = \mathcal{O}_a$. Hence π_L and π_U share a common line (as $l \neq u$, π_L and π_U are not equal). The same result holds for each of the pairs (π_L, π_V), (π_K, π_U) and (π_K, π_V). Let $\pi_L \cap \pi_U = N_{LU}$—with similar notation for all other above pairs of planes.
   - Now we show that π_L and π_K only have a point in common. Indeed, if $\pi_L \cap \pi_K$ were a line and $l \sim k$, then $\langle \pi_L, \pi_K \rangle$ would be a 3D space intersecting $Q(4,q)$ in the cone $Q(4,q) \cap \langle l^{\perp} \rangle$ respectively $Q(4,q) \cap \langle k^{\perp} \rangle$, yielding a contradiction. If $\pi_L \cap \pi_K$ were a line and $l \not\sim k$, then $\langle \pi_L, \pi_K \rangle$ is a 3D space intersecting $Q(4,q)$



in an ovoid touching both π_L and π_K, which hence is subtended by a point of L and by a point of K. As $L \cap K = \phi$, this would imply that L and K contain a twin pair (l', k'), in contradiction with the assumptions.

If π_U and π_V intersected in a line, then U and V would contain a twin pair (u', v') $(u' \in U, v' \in V)$, a contradiction. So $\pi_U \cap \pi_V$ is a point. This also implies that the four lines N_{LU}, N_{LV}, N_{KU} and N_{KV} are all distinct. Since both π_K and π_L contain $N_{LU} \cap N_{KU}$ and $N_{LV} \cap N_{KV}$, these points coincide. Hence all lines contain a common point t.

- Now we are ready to show that l, k, u and v belong to a common plane.† (We refer to the picture.) From now on, throughout the whole argument and unless stated otherwise, we work in the standard quadratic extension $\mathbf{PG}(4, q^2)$ of the ambient projective space $\mathbf{PG}(4, q)$ of $Q(4, q)$. Hence, for instance, the plane π_L will be viewed as a plane over $\mathbf{GF}(q^2)$ and contains $q^4 + q^2 + 1$ points. Also, the quadric $Q(4, q)$ extends uniquely to a quadric $Q(4, q^2)$ in $\mathbf{PG}(4, q^2)$.

  First we consider π_L and π_U. In $\mathbf{PG}(4, q)$, the 3D space $\langle \pi_L, \pi_U \rangle$ intersects $Q(4, q)$ in an ovoid tangent to π_L at l and tangent to π_U at u. In $\mathbf{PG}(4, q^2)$, however, the intersection of $Q(4, q^2)$ with π_L is the union of two lines through l, say L_1 and L_2. The same holds for $Q(4, q^2) \cap \pi_U$: this is the union of two lines U_1, U_2 through u. Up to choice of indices, L_1 and U_1 will intersect in a point of $N_{LU} = \pi_L \cap \pi_V$—as L_2 and U_2 will do. The line through the points $L_1 \cap \pi_V$ and $U_1 \cap \pi_K$ is denoted by X_1; the line through the points $L_2 \cap \pi_V$ and $U_2 \cap \pi_K$ is denoted by X_2. Hence we obtain two triangles with lines respectively $\{L_1, U_1, X_1\}$ and $\{L_2, U_2, X_2\}$, that are in perspective from the point t (indeed, the vertices of both triangles are on N_{LU}, N_{KU} and N_{LV}). Hence we can apply the theorem of Desargues to conclude that l, u and x, with $\{x\} = X_1 \cap X_2$, are collinear.

  Using the same arguments in the 3D space $\langle \pi_K, \pi_V \rangle$, we can conclude that k, v and x (indeed the same point x) are collinear.

  Hence l, k, u and v are in the same plane $\pi_{lkuv} := \langle l, k, u, v \rangle$, and this plane clearly also defines a plane of $\mathbf{PG}(4, q)$, since it contains the non-collinear set of

†This is the point where the proof of Theorem 7.1 of [14] is incomplete. At p. 250 (a), two planes (in particular π_l and lmu, with m renamed k in our version) are supposed to intersect in a line, whereas this is not the case in the general 4D setting.

points $\{l, k, u, v\}$. We conclude that l, k, u and v are either on an irreducible conic or on two different lines (lk and uv) of $Q(4, q)$.

2. In the second part of this proof, we show that (L, K) is a regular pair of lines.
   - Suppose the conic $\pi_{lkuv} \cap Q(4, q) = C$ defined by L, K, U, V is irreducible. Put $\{L, K\}^{\perp} = \{U, V, W_1, \ldots, W_{q-1}\}$ where $l \in W_1$, $k \in W_2$. Let w_i be the common point of W_i and Δ ($i \geq 3$). Then L, K, U, W_i ($i \geq 3$) also define the conic C (as a plane is defined by three non-collinear points), implying $w_i \in C$. Hence $C = \{l, k, u, v, w_3, \ldots, w_{q-1}\}$.

     To prove that (L, K) is regular, we have to check the following: if Y intersects $U, V \in \{L, K\}^{\perp}$, then Y will also intersect W_i, $i \in \{1, \ldots, q-1\}$. And indeed, interchanging the roles of L, K and U, V in the first part of this section, it follows that $y \in C$. Now again by this reasoning (substituting Y for K), every line containing a point of L and a point of Y, should meet $Q(4, q)$ in a point of C. Hence W_i and Y are concurrent for all i. Hence $Y \in \{L, K\}^{\perp\perp}$. It follows that the pair (L, K) is regular.
   - Secondly, consider the case where $\pi_{lkuv} \cap Q(4, q) = C$ is reducible. So lk and uv are distinct lines, and the conic $C = lk \cup uv$ is uniquely defined by any three of the points l, k, u and v. Let $\{L, K\}^{\perp} = \{U, V, W_1, \ldots, W_{q-1}\}$ with $W_1 = lk$. Let w_i be the common point of W_i and $Q(4, q)$ for $i > 1$ and let w_1 be the common point of lk and uv. Then U, W_i, L and K, $i > 1$, also define the conic C, so $w_i \in C$. Clearly $w_i \in uv$, $i > 1$. Hence $uv = \{u, v, w_1, \ldots, w_{q-1}\}$. Let $Y \in \{U, V\}^{\perp} \setminus \{L, K, uv\}$. Then, if y is the common point of Y and $Q(4, q)$, we have $y \in lk$. Now, interchanging roles of L and Y, we see that every line containing a point of uv and a point of L must contain a point of Y. Hence for $i \geq 1$, W_i and Y are concurrent. Hence $Y \in \{L, K\}^{\perp\perp}$. It follows that the pair (L, K) is regular. □

COROLLARY 13. *All lines of Γ are regular.*

PROOF. This follows from Theorems 11 and 12. □

COROLLARY 14. *The intersection of Δ and a grid not contained in Δ is a conic (either irreducible or consisting of two distinct lines).*

PROOF. This follows from the proof of previous theorems. □

5.6. Part 3: construction of sub-GQs. As all lines of Γ are regular, two opposite lines U, V define a $(q + 1) \times (q + 1)$-grid $\mathcal{G}$ in Γ. We shall say $\mathcal{G}$ is the *grid based on* U, V and denote it by $\mathcal{G}(U, V)$.

In this part, we give the construction of a lot of new sub-GQs of order (q, q) in Γ. Starting from an elliptic quadric (respectively a quadratic cone, a hyperbolic quadric) inside Δ, we choose an additional line of $\Gamma \setminus \Delta$ containing a point of the elliptic quadric (respectively quadratic cone, hyperbolic quadric) and construct a sub-GQ Δ' of order (q, q) containing this structure.

THEOREM 15. *Let Γ and Δ be as above. Given an elliptic quadric $\mathcal{O}$ in Δ and a line L of $\Gamma \setminus \Delta$ intersecting this ovoid, with L a line not containing a point subtending $\mathcal{O}$, there exists a sub-GQ Δ' of order (q, q) of Γ through $\mathcal{O}$ and L.*

PROOF. *Construction of Δ'.*
Let $\mathcal{O}$ be an elliptic quadric in Δ, L a line of $\Gamma \setminus \Delta$ intersecting $\mathcal{O}$ in l, and L not through a point subtending $\mathcal{O}$. We construct Δ' as follows.

- The *basic line* of Δ' is—by definition—the line L itself.
- As the ovoid $\mathcal{O}$ is not subtended by any point of L, and the base point l of the rosette $\mathcal{R}_L$ belongs to $\mathcal{O}$, the rosette $\mathcal{R}_L$ will intersect $\mathcal{O}$ in a rosette of conics (see Lemma 10). This means that every point x of $L \setminus \{l\}$ is collinear with $q+1$ points of $\mathcal{O}$, constituting a conic C_x through l. The q lines joining this point x to the set $C_x \setminus \{l\}$, are also lines of Δ', and are said to be of the *first generation*. Hence there are q^2 lines of the first generation in Δ'. Every point of such a line will be a point of Δ', so we have already defined q^3+q+1 points of Δ'. These points, including the point l, are the points of the *first generation*.
- The third set of lines belonging to Δ' is constructed as follows: take two opposite lines U, V of the first generation. As all lines of Γ are regular, we can construct the $(q+1)\times(q+1)$-grid $\mathcal{G}(U, V)$ based on these lines U, V. This grid contains L, and intersects $\mathcal{O}$ in a conic C through l, but this conic is not one of the conics in the rosette $\mathcal{R}_L \cap \mathcal{O}$. All (new) lines of the grid $\mathcal{G}(U, V)$ that are opposite L belong to the *second generation* of lines of Δ'.
- Every line that is the projection of a line of the second generation onto l, belongs to the *third generation*. These are precisely the lines through l belonging to the above grids. In total, there will be q such lines (this will be proved by showing that Δ' is indeed a GQ; see the last part of the proof for more explanation), and the q^2 new points on these lines are the points of the *third generation*.

Note that through each conic C of $\mathcal{O}$ through l, not belonging to the rosette $\mathcal{R}_L \cap \mathcal{O}$ (i.e., not defined by one of the q points of $L \setminus \{l\}$), one can construct a unique grid $\mathcal{G}(U, V)$ based on two lines of the first generation. Indeed, choose $u, v \in C \setminus \{l\}$ and put $U := \mathrm{proj}_u L$ (so $U \cap L$ is the unique point of L collinear with u) and $V := \mathrm{proj}_v L$. Then, as C does not belong to the rosette $\mathcal{R}_l \cap \mathcal{O}$, U, V will be at distance 4 and of the first generation. By Corollary 14, the grid $\mathcal{G}(U, V)$ intersects $\mathcal{O}$ in a conic which must necessarily coincide with C because it shares three points u, v, l with C.

(∗) We now claim that if a line K of Γ through a point p of the first generation with $p \notin \mathcal{O}$, $p \notin L$, intersects the ovoid $\mathcal{O}$, then K is of the first or second generation.
Indeed, suppose K is not of the first generation and $K \cap \mathcal{O} = \{k\}$. If we project L onto k and put $\mathrm{proj}_k L = V$, then V is a line of the first generation. As $p \in K$ is a point of the first generation, it belongs to a line U of the first generation. As K intersects both U and V, K belongs to the grid $\mathcal{G}(U, V)$ and hence K is of the second generation. The claim is proved.

Δ' is indeed a GQ

We show that for p a point and K a line of Δ', $p \notin K$, the line $M := \mathrm{proj}_p K$ belongs to Δ'. This is obvious if K is the basic line. We now consider all other cases.

(1, 1) If p and K both belong to the first generation, $\mathrm{proj}_p K = M$ belongs—by definition of the second generation of lines—to Δ'.

(1, 2) Let p be of the first, and let K be of the second generation. If $p \in L$, then clearly M belongs to Δ'. So assume $p \notin L$. Hence p belongs to a unique line S of the first generation, and K belongs to some grid $\mathcal{G}(U, V)$ with S, U, V three lines of the first generation (i.e., intersecting L and $\mathcal{O}$ in two different points). We may assume $U \neq S \neq V$. If we can show that the line $M = \mathrm{proj}_p K$ intersects $\mathcal{O}$, then by (∗) the line M belongs to Δ'. We put $S \cap L = \{s'\}$. The line $W := \mathrm{proj}_{s'} K$ belongs to the grid $\mathcal{G}(U, V)$, so W intersects $\mathcal{O}$ in a point w. We may assume $S \cap K = \phi$, otherwise we are done. The line W also belongs to the grid $\mathcal{G}(S, K)$, so this grid intersects $\mathcal{O}$ in the conic C_{skw} through s, k and w. As M belongs on its turn to the grid $\mathcal{G}(S, K)$, the point $\{m\} = M \cap \Delta$ belongs to the conic C_{skw} by Corollary 14. Hence $m \in \mathcal{O}$, and this part of the proof is finished.

(3, 1) Let p be of the third, and let K be of the first generation. Then p is on a line L' through l, with L' through a point u' of a line U of the second generation. So the line U intersects $\mathcal{O}$ in the point u. The point $k'' := \mathrm{proj}_K u'$ is of the first generation as $k'' \in K$. As $u'k''$ is a line of the second generation taking account of case (1, 2), the line $u'k''$ meets $\mathcal{O}$ in a point x. So the grid $\mathcal{G}(L', K)$ meets $\mathcal{O}$ in the conic C_{kxl}. As $M := \mathrm{proj}_p K$ belongs to the same grid $\mathcal{G}(L', K)$, the line M meets $\mathcal{O}$ in the same conic. Hence, by $(*)$, M is of the second generation and so it belongs to Δ'.

(1, 3) Let p be of the first, and let K be of the third generation. Clearly we may assume that $p \notin L$. The line $U := \mathrm{proj}_p L$ is of the first generation and intersects $\mathcal{O}$ in the point u. As K is of the third generation, K contains l and a point k' on a line N of the second generation. If $p \in U$ we are done, so assume $p \notin U$. The line $J := \mathrm{proj}_{k'} U$ is of the second generation, as it is the projection of a line of the first generation on a point of the third generation (see case (3, 1)); so J intersects $\mathcal{O}$ in the point j. Hence the grid $\mathcal{G}(K, U)$ intersects $\mathcal{O}$ in at least l, j and u, so $M = \mathrm{proj}_p K$, belonging to $\mathcal{G}(K, U)$, will also intersect $\mathcal{O}$. By $(*)$, the line M is of the second generation, and so it belongs to Δ'.

(3, 2) Let p be of the third, and let K be of the second generation. Then p is on a line L' through l, with L' through a point u' of a line U of the second generation. We may assume that $u' = p$. So U intersects $\mathcal{O}$ in the point u. As K is of the second generation, K intersects $\mathcal{O}$ in a point k. Take a point $u'' \in U \setminus \{p\}$, which is necessarily of the first generation. We may assume that $K \cap U = \phi$, otherwise we are done. The line $V := \mathrm{proj}_{u''} K$ belongs to either the first or the second generation (by case (1, 2)), so V intersects $\mathcal{O}$ in the point v. Hence $\mathcal{G}(U, K)$ intersects $\mathcal{O}$ in a conic C_{uvk}. As $M = \mathrm{proj}_p K$ also belongs to $\mathcal{G}(U, K)$, the line M meets $\mathcal{O}$ in a point of C_{uvk}. If this point is l, M is of the third generation, so the proof is done. If this point is different from l, the point $M \cap K$ is of the first generation. Indeed, K is of the second generation, so it has one point in $\mathcal{O}$, $q - 1$ points of the first generation not in $\mathcal{O}$, and one point of the third generation; if $M \cap K$ were of the third generation, the points $M \cap K$, l and u' would constitute a triangle. Hence, relying on $(*)$, M is of the second generation.

(3, 3) Let p as well as K be of the third generation. This case is trivial.

Hence Δ' is a generalized quadrangle. Clearly it is thick. As each line of Δ' contains $q + 1$ points of Δ', and as any point of $L \setminus \{l\}$ is incident with $q + 1$ lines of Δ', the quadrangle Δ' has order (q, q). □

THEOREM 16. *Let Γ and Δ be as above. Given a quadratic cone $\mathcal{C}$ in Δ, i.e., a set of $q + 1$ lines through a point p, and a line L of $\Gamma \setminus \Delta$ intersecting this cone in a point different from p, there exists a sub-GQ Δ' of order (q, q) of Γ through $\mathcal{C}$ and L.*

PROOF. The proof is completely similar to the previous case. Let us just indicate how Δ' is defined.

Let $\mathcal{C}$ be a quadratic cone in Δ with vertex p, L a line of $\Gamma \setminus \Delta$ intersecting $\mathcal{C} \setminus \{p\}$. Put $L \cap \mathcal{C} = \{l\}$. We construct a sub-GQ Δ' as follows.

- The *basic lines* of Δ' are the $q + 1$ lines of the cone $\mathcal{C}$ and the line L.
- The lines of the *first generation* are the q^2 lines joining a point $x \in L \setminus \{l\}$ and a point $y \in \mathcal{C} \setminus \{pl\}$. (For every point $x \in L \setminus \{l\}$, the $q + 1$ points on $\mathcal{C}$ collinear with x constitute a conic C_x through l.) In this way, one obtains $q^2(q - 1)$ new points of Δ'.

Those points, together with the $(q+1)^2$ points on $\mathcal{C} \cup L$, constitute the *first generation of points*.

- The lines of the *second generation* are the $q^3 - q$ new lines opposite L of the q^2 grids $\mathcal{G}(U, V)$ with U, V lines of the first generation.
- The lines of the *third generation* are the lines through l intersecting a line of the second generation. The proof will imply that there are $q-1$ such lines. On these lines, we find $q(q-1)$ new points of Δ', said to be of the *third generation*. (Again, no points of the second generation are defined.) □

THEOREM 17. *Let Γ and Δ be as above. Given a hyperbolic quadric $\mathcal{G}$ in Δ and a line L of $\Gamma \setminus \Delta$ intersecting this hyperbolic quadric, there exists a sub-GQ Δ' of order (q, q) of Γ through $\mathcal{G}$ and L.*

PROOF. Again similar to the proof of Theorem 15. The construction of Δ' is now as follows. Put $L \cap \mathcal{G} = \{l\}$.

- The *basic lines* of Δ' are the $2q+2$ lines of $\mathcal{G}$ and the line L.
- The lines of the *first generation* are the q^2 lines joining a point $x \in L \setminus \{l\}$ and a point $y \in \mathcal{G}$, with y not on a line of Δ containing l. (For every such point x the $q+1$ points of $\mathcal{G}$ collinear with x constitute a conic C_x through l.) Including all points of $\mathcal{G}$ we obtain in this way q^3+3q+1 points of Δ', said to be of the *first generation*.
- The lines of the *second generation* are the new lines in the grids $\mathcal{G}(U, V)$ with U, V opposite lines of the first generation. There are $q^3 - 2q$ lines of the second generation.
- The lines of the *third generation* are the lines containing l and concurrent with any line of the second generation. The points of the *third generation* are the new points incident with lines of the third generation. As the structure Δ' defined in this way turns out to be a GQ, there are $q-2$ lines of the third generation and $q^2 - 2q$ points of the third generation. □

5.7. *Part* 4*: sub-GQs through every dual window.* A *dual window* of a generalized quadrangle is a set of five points, two of which, say a and b, are at distance 4, while the other three are in a^b, together with the six lines through the pairs of collinear points.

LEMMA 18. *Let Γ be a GQ of order (q, q^2). Through every dual window of Γ, there is at most one sub-GQ of order (q, q).*

PROOF. Let Γ_1 and Γ_2 be two subquadrangles of order (q, q) of Γ. As each line of Γ_1 intersects Γ_2 ([9, 2.2.1]), the intersection $\Gamma_1 \cap \Gamma_2$ of these subquadrangles is a grid of Γ_1, or an ovoid of Γ_1, or the set of all points of Γ_1 collinear with a fixed point of Γ_1. As a dual window is never contained in $\Gamma_1 \cap \Gamma_2$, we have a contradiction. □

THEOREM 19. *Let Γ be a GQ of order (q, q^2) and let Δ be a classical sub-GQ of order (q, q) of Γ, such that every subtended ovoid of Δ is classical. Then there exists a sub-GQ Δ' of order (q, q) through every dual window of Γ. Hence Γ is classical.*

PROOF. We perform a double counting on the pairs $(\mathcal{W}, \mathcal{D})$ with $\mathcal{W}$ a dual window of Γ, and $\mathcal{D}$ a subquadrangle constructed as explained in Theorems 15, 16 or 17, such that $\mathcal{W} \subset \mathcal{D}$. By Lemma 18, there is at most one subquadrangle of order (q, q) through every dual window. The number of dual windows in Γ is $W = \frac{1}{12}(q^3+1)(q^2+1)(q+1)^2 q^6 (q-1)$. Given a fixed subquadrangle $\mathcal{D}$ of order (q, q), one counts $x = \frac{1}{12}(q^2+1)(q+1)^2 q^4 (q-1)$ dual windows in $\mathcal{D}$. We count the number S of subquadrangles of order (q, q) constructed so far as follows. There are $\frac{q^2(q^2-1)}{2}$ classical ovoids in Δ. Through every such ovoid, one constructed $q-2$

subquadrangles Δ' different from Δ (through every point p of the ovoid, there are $q^2 - q - 2$ lines to choose for starting the construction of Δ', but there are $q + 1$ lines of Δ' through p). There are $\frac{q^2(q^2+1)}{2}$ grids in Δ. Through every grid, one constructed q new subquadrangles Δ'. There are $(q^2 + 1)(q + 1)$ cones in Δ. Through every cone, one constructed $q - 1$ new subquadrangles Δ'. This gives us a total of $S = q^5 + q^2$ subquadrangles (Δ included). We conclude that $W = xS$, and hence we constructed exactly one subquadrangle through every dual window. Hence Γ is classical by [9, 5.3.5(ii)]. □

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