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Ten Exceptional Geometries from Trivalent Distance Regular Graphs

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Dedicated to H.S.M. Coxeter

Abstract. The ten distance regular graphs of valency 3 and girth > 4 define ten non-isomorphic neighborhood geometries, amongst which a projective plane, a generalized quadrangle, two generalized hexagons, the tilde geometry, the Desargues configuration and the Pappus configuration. All these geometries are bislim, i.e., they have three points on each line and three lines through each point. We study properties of these geometries such as embedding rank, generating rank, representation in real spaces, alternative constructions. Our main result is a general construction method for homogeneous embeddings of flag transitive self-polar bislim geometries in real projective space.

Keywords: Coxeter graph, Biggs-Smith graph, real embedding, universal embedding

1. Introduction

In 1971, Biggs and Smith [1] classified the trivalent distance regular graphs. There are exactly thirteen of those, eight of which are bipartite. Hence these eight can be regarded as the incidence graph of a geometry. For the other five, one can consider the so-called *neighborhood geometry*, obtained from the graph by taking as points the vertices and as lines a second copy of the vertices with the incidence relation induced by adjacency. In this way, we obtain 14 geometries, because there is one graph which has no vertex transitive automorphism group, and so this graph gives rise to two non-isomorphic geometries (generalized hexagons). Three graphs contain cycles of length 4, and so they give rise to non-linear geometries. In view of the embedding problem we will discuss, these are trivial and so we do not consider them (they are also trivial as geometries: two of them are isomorphic to the geometry of all 3-sets of a 4-set, the other has three points and three lines, and all points are incident with all lines). Hence there remain 11 more interesting geometries, two of which are mutual isomorphic (see below). So we end up with a rather exceptional set of 10 *bislim* highly transitive and regular geometries (bi-slim here means that each object is incident with three others; see below). Some of these are well known and studied extensively. We mention the Desargues and the Pappus configurations, the smallest (thick) projective plane, generalized quadrangle and generalized hexagons, and the tilde geometry. In this paper, we are mainly interested in real embeddings of these geometries, which can be regarded as n_3 configurations. There has been some interest from configurational point of view of realizations of such structures, but most of the known results are obtained by computer, or are enumeration results for rather small n, see for instance [6]. Therefore, they can not be applied to most of our geometries. For instance, one of the problems we will solve is the existence of a real embedding of the tilde geometry. This geometry has 45 points and 45 lines, and it not captured by any previously developed theory. It is also too big to handle with a computer in that the final result (whether it has a real embedding or not) does not deserve a huge computer search, but it should rather be the case that the proof is more worth mentioning (because of its beauty or the idea behind it) than the result itself. In the present paper, we will develop some general theory about real embeddings of bislim self-polar flag transitive geometries, and show how to apply it to all our self dual geometries. Moreover, we will also provide some direct geometric constructions of the universal embeddings in projective spaces over the field $\mathbf{GF}(2)$ of two elements.

We are also interested in some alternative constructions of some of our exceptional geometries, sometimes giving rise to a new construction of the trivalent distance regular graph. In particular, we present a geometric construction of the Biggs-Smith graph, which makes the full automorphism group apparent, and from which the classical construction can be derived. This will yield a short new (computer free) proof of the existence of that graph as distance transitive graph.

The motivation of this work is twofold. First, we hope that the geometric look will tell us more about the graphs themselves. Both the geometries and the graphs are amongst the most symmetrical of their kind and so they reflect exceptional properties of small groups. Some of these properties may now be explained geometrically, or at least illustrated in a geometric way. Secondly, it has been a basic question to determine whether a given geometry can be drawn in the real plane using points and straight lines. Our work answers this question for many highly symmetric and thus beautiful but still small geometries, so that one could really draw such a picture and admire its many surprising and beautiful features.

2. Definitions

A point-line geometry is a system $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consisting of a point set $\mathcal{P}$, a line set $\mathcal{L}$ and a symmetric incidence relation $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$ expressing precisely when a point is incident with a line. Usually we think of a line as the set of points incident with it and we accordingly use phrases like "a point is on a line", "a line goes through a point", etc. The *dual* of Γ is the point-line system Γ^D obtained from Γ by interchanging the point set with the line set. If all lines of Γ carry the same number $s+1$ of points and all points are incident with the same number $t+1$ of lines, then we say that Γ has *order* (s,t). If $s=1$, then the geometry is usually called *thin*, while if both s and t are at least 2, the geometry is *thick*. If $s = 2$, then we call Γ *slim*. If the dual of Γ is also slim, i.e., if also $t = 2$, then we call Γ *bislim*. The *incidence graph* $I(\Gamma)$ or *Levi graph* of Γ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and adjacency relation $\mathtt{I}$. The *gonality* of Γ is half of the girth of $I(\Gamma)$ (the girth being the length of the smallest cycle in $I(\Gamma)$ is indeed an even number since $I(\Gamma)$ is obviously bipartite). If the gonality is at least 3, then lines are determined by their point sets. A *flag* is an incident point-line pair. The geometry Γ is called *connected* if its incidence graph is connected. An *incidence matrix* of a finite point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a matrix $\mathbf{A}$ the columns of which are indexed by $\mathcal{P}$, the rows of which are indexed by $\mathcal{L}$, and the (L, x)-entry of which is equal to 1 or 0 according whether $x \mathtt{I} L$ or not, for all $x \in \mathcal{P}$, $L \in \mathcal{L}$. Clearly, an incidence matrix completely determines the geometry.

An *monomorphism* (*isomorphism*) from the point-line geometry Γ to the point-line geometry Γ' is an injective map (bijection) from the point set of Γ to the point set of Γ' together with an injective map (bijection) from the line set of Γ to the line set of Γ' such that, for any two elements of Γ, their images in Γ' are incident (precisely) when the elements themselves are incident. A *collineation* of a point-line geometry Γ is an isomorphism from Γ to itself. The *full collineation group* is the group of all collineations (and every subgroup is called a *collineation group*) of Γ and is denoted $\mathrm{Aut}(\Gamma)$. If $\mathrm{Aut}(\Gamma)$ acts primitively on the point set of Γ, then we say that Γ *is primitive*. A *duality* is an isomorphism from Γ to its dual. If a duality exists, then we say that Γ is *self-dual*. A *polarity* is a duality of order 2; if it exists, then we say that Γ is *self-polar*. It is easy to see that self-polar point-line geometries have some symmetric incidence matrices.

Let $\mathcal{G} = (V, E)$ be a graph, where V is the set of vertices, and E is the set of edges. Denote the (usual) distance between two vertices v, w by $\delta(v, w)$. Then $\mathcal{G}$ is called *distance regular* if for all positive integers i, j, k, and for all pairs of vertices $(v, w) \in V^2$ such that $\delta(v, w) = k$, the number of vertices of $\mathcal{G}$ at distance i from v and at the same time at distance j from w is constant (which does not depend on the pair (v, w), but only on the numbers i, j, k). Let G be a group of automorphisms of a connected graph $\mathcal{G}$. Then $\mathcal{G}$ is called *distance transitive with respect to G* if G acts transitively on V and if for every positive integer k and every vertex v of $\mathcal{G}$, the stabilizer G_v acts transitively on the set of vertices at distance k from v.

Now let $\mathcal{G}$ be a connected graph which is not bipartite. Then the adjacency matrix $\mathbf{A}$ of $\mathcal{G}$ is the incidence matrix of a unique point-line geometry $\Gamma(\mathcal{G})$, which we call the *neighborhood geometry of $\mathcal{G}$* (we borrow this terminology from [7] where an equivalent definition of these geometries arising from graphs is given in the spirit of the one in the introduction). Note that $\Gamma(\mathcal{G})$ is indeed unique since it is isomorphic to its dual. It is easy to see that $\Gamma(\mathcal{G})$ is connected (precisely because Γ is not bipartite). If $\mathcal{G}$ is a connected bipartite graph, then it is clearly the incidence graph of two mutually dual connected point-line geometries. In this way, we have attached to every connected graph a pair of mutually dual (but not necessarily non-isomorphic) connected point-line geometries.

We will study the point-line geometries arising in this way from the distance regular trivalent graphs. But before we start our investigation, we give some definitions and prove some general results about embeddings.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a point-line geometry and $\mathbf{PG}(d, \mathbb{K})$ the d-dimensional projective space over the field $\mathbb{K}$. An *embedding of* Γ *in* $\mathbf{PG}(d,\mathbb{K})$ is a monomorphism of Γ into the point-line geometry of $\mathbf{PG}(d,\mathbb{K})$ such that the image of $\mathcal{P}$ is not contained in any hyperplane of $\mathbf{PG}(d,\mathbb{K})$. Usually, one identifies a point with its image in $\mathbf{PG}(d,\mathbb{K})$. If Γ is slim, then we call an embedding of Γ in $\mathbf{PG}(d,\mathbb{K})$ *barycentric* if fixed projective coordinates can be chosen for each point of Γ in such a way that, if three distinct points p_1, p_2, p_3 of Γ are on one common line of Γ, then the sum of their coordinates is equal to the zero $(d+1)$-tuple. An embedding is *semi-barycentric* provided fixed projective coordinates can be chosen for each point of Γ in such a way that, if three distinct points p_1, p_2, p_3 of Γ are on one common line of Γ, then either the sum of their coordinates is equal to the zero $(d+1)$-tuple, or the sum of the coordinates of two of them equals the third coordinate tuple. An embedding is *G-homogeneous*, for $G \leq \mathrm{Aut}\,\Gamma$, if every collineation of Γ belonging to G is induced by the semi-linear projective group $\mathbf{P\Gamma L}_{d+1}(\mathbb{K})$. An $\mathrm{Aut}\,\Gamma$-homogeneous embedding is also simply called *homogeneous*.

With the previous definitions, it is not forbidden that in some embedding of a geometry Γ a point p of Γ is not incident with a line L of Γ, but that these objects are incident in the projective space in which Γ is embedded. Embeddings for which this does not occur will be called *exclusive embeddings*. It is shown in [10] that every bislim geometry admits a representation in $\mathbf{PG}(2,\mathbb{R})$ such that for at most one line of the geometry the three points on that line are not collinear in $\mathbf{PG}(2,\mathbb{R})$. For exclusive embeddings, there are counterexamples for any number of such lines (see [13]).

Let Γ be a slim geometry. If the positive integer d is maximal with respect to the property that Γ admits an embedding in $\mathbf{PG}(d,2)$ (respectively a barycentric embedding in $\mathbf{PG}(d,\mathbb{R})$), then $d+1$ is called the *universal embedding rank of* Γ (respectively the *real* (*barycentric*) *embedding rank of* Γ). If $d>0$ does not exist, then we say that the universal embedding rank (respectively real (barycentric) embedding rank) is zero. Also, we say that a subset of points of Γ *generates* Γ if the smallest slim subgeometry of Γ with the property that, whenever two points of that subgeometry are collinear in Γ, they are also collinear in the subgeometry, coincides with Γ. The *generating rank of* Γ is the minimal number of points needed to generate Γ. Obviously the generating rank of Γ cannot be smaller than the universal and real embedding ranks of Γ.

Finally, we need the notion of a cover. Let $\Gamma=(\mathcal{P},\mathcal{L},\mathtt{I})$ and $\Gamma'=(\mathcal{P}',\mathcal{L}',\mathtt{I})$ be two point-line geometries, and let there be given a surjective map from $\mathcal{P}$ onto $\mathcal{P}'$ and a surjective map from $\mathcal{L}$ onto $\mathcal{L}'$ such that incident elements are mapped onto incident elements. Then we call Γ (together with this pair of maps) a *cover* (sometimes also a *local isomorphism*) if for each element $v \in \mathcal{P}\cup\mathcal{L}$, the set of elements of Γ incident with v is mapped bijectively onto the set of elements incident with the image of v. It is easy to show that the size of the inverse images (*fibers*) of two collinear points is equal to the size of the inverse image of the line they determine. In particular, the size of fibers is constant if Γ is connected. If that size is k, then we speak of a *k-fold cover of* Γ'. A 2-fold (3-fold) cover is also called a *double* (*triple*) *cover*. Every collineation of Γ preserving the fibers *induces* a collineation in Γ; we also say that some collineation group of Γ' *is induced by* some collineation group of Γ. It is clear what is meant. If the fibers are equivalence classes in $\mathcal{P}$ and in $\mathcal{L}$ with respect to the relation *"is at maximal distance of"*, then we say that Γ *is an antipodal cover of* Γ'.

Finally, we mention the following well known result (see [8]).

Lemma 2.1. *Let Γ be a thick geometry embedded in $\mathbf{PG}(d,\mathbb{K})$, for some field $\mathbb{K}$. Then the only linear collineation of $\mathbf{PG}(d,\mathbb{K})$ fixing every point of Γ is the identity.*

3. Construction of Homogeneous Embeddings

Our first result states a necessary condition for the existence of barycentric embeddings of slim point-line geometries.

Lemma 3.1. *Suppose the slim point-line geometry Γ with n points and m lines has some barycentric embedding in $\mathbf{PG}(d,\mathbb{R})$, for some $d \geq 2$. Then the rank of any incidence matrix of Γ is at most $n-d-1$.*

Proof. Let $\mathbf{A}$ be an incidence matrix, with the rows indexed by the lines and the columns indexed by the points of Γ. Build the $n\times(d+1)$ matrix B, where the rows are labelled by the points of Γ, by defining the row corresponding to the point $p \in \mathcal{P}$ as the coordinate tuple of p with respect to the barycentric embedding in $\mathbf{PG}(d,\mathbb{R})$ under consideration. Then clearly $\mathbf{A}B$ is the zero matrix and the result follows from the fact that the rank of B is equal to $d+1$. ∎

The second result is connected with homogeneous and real semi-barycentric embeddings. We define $\mathbf{Sc}_n(\mathbb{R})$ as the (multiplicative) group of real scalar $n\times n$ matrices.

Lemma 3.2. *Let Γ be a flag transitive (with respect to a collineation group G) slim point-line geometry G-homogeneously embedded in $\mathbf{PG}(d,\mathbb{R})$, $d \geq 2$, for some $G \leq$ $\mathrm{Aut}(\Gamma)\cap\mathbf{PGL}_{d+1}(\mathbb{R})$. Then the group G lifts to a subgroup of $\mathbf{GL}_{d+1}(\mathbb{R})$ if and only if the embedding is barycentric. Also, some nontrivial central extension $2\cdot G$ of G lifts to a subgroup of $\mathbf{GL}_{d+1}(\mathbb{R})$, with $2\cdot G/(2\cdot G\cap\mathbf{Sc}_{d+1}(\mathbb{R})) = G$, if and only if the embedding is semi-barycentric, but not barycentric. In the latter case there is a connected double cover Γ' of Γ.*

Proof. Suppose first the embedding is barycentric and identify each point of Γ with a fixed coordinate tuple such that $p_1+p_2+p_3=0$ if and only if p_1,p_2,p_3 are collinear, for each triple of distinct points (where we write 0 for the 0-coordinate tuple). Let $\theta\in G$, and choose an arbitrary point p of Γ. We choose the matrix M of θ such that pM (obvious notation; M is unique by Lemma 2.1) is the fixed coordinate tuple of p^θ. It follows easily that for all points p' collinear with p, the coordinate tuple $p'M$ is the fixed chosen one (use $p_1+p_2+p_3=0$ if and only if $p_1M+p_2M+p_3M=0$). By connectivity, this is the case for all points of Γ. Now let n be the order of θ, then M^n is proportional to the identity (because the points of Γ generate $\mathbf{PG}(d,\mathbb{R})$); since for each point p of Γ we have $pM^n = p$, we see that M^n is equal to the identity matrix and so $\det M = \pm 1$. The mapping $\theta\mapsto M$ defines a lifting of G to $\mathbf{GL}_{d+1}(\mathbb{R})$.

Now suppose that the embedding is semi-barycentric, but not barycentric. For each point p of Γ, denote by p^+ and p^- (with $p^+=-p^-$) two fixed coordinate tuples corresponding to p such that, for each triple $\{p_1,p_2,p_3\}$ of distinct points, p_1,p_2,p_3 are collinear if and only if either $p_1^+ + p_2^+ + p_3^+ = 0$, or $p_1^+ + p_2^+ + p_3^- = 0$, or $p_1^+ + p_2^- + p_3^+ = 0$, or $p_1^- + p_2^+ + p_3^+ = 0$. Since we assume that the embedding is not barycentric, there is no way to choose p^+ in such a way that $p_1^+ + p_2^+ + p_3^+ = 0$ whenever p_1,p_2,p_3 are three distinct collinear points of Γ. For $\theta\in G$, we choose matrices $\pm M_\theta$ such that, for some fixed point p of Γ, the coordinate tuples $p(\pm M_\theta)$ are the two fixed chosen ones. Similarly as in the previous paragraph, this is well defined and independent of the point p. Also similarly one proves that $\det M_\theta \in \{1,-1\}$. The mapping $\pm M_\theta \mapsto \theta$ is clearly an epimorphism with kernel $\{I,-I\}$ (I is the identity matrix) from the group $G^\dagger = \{\pm M_\theta \mid \theta \in G\} \le \mathbf{GL}_{d+1}(\mathbb{R})$ to G. Clearly the kernel is in the center of $G^\dagger$, and so $G^\dagger$ is a central extension of G, with $G^\dagger/(G^\dagger \cap \mathbf{Sc}_{d+1}(\mathbb{R})) = G$. If it were trivial, then $G^\dagger$ could be written as $2 \times G'$, with $G \cong G'$. Since the size of the stabilizer of any point of Γ in G' is equal to that in $2\times G'$, we see that G' would have two orbits on the fixed chosen coordinate tuples of points of Γ. We claim that this defines a barycentric embedding, which leads to a contradiction. Indeed, for three arbitrary but fixed collinear points p_1, p_2, p_3 of Γ, we may choose p_1^+, p_2^+, p_3^+ as the coordinate tuples in one orbit of G'. Suppose by way of contradiction that $p_1^+ = p_2^+ + p_3^+$. By flag transitivity of G', there exists an element $M' \in G'$ mapping p_1^+ onto p_2^+ while the set $\{p_1^+, p_2^+, p_3^+\}$ is preserved. Hence we obtain $p_2^+ = p_1^+ + p_3^+$, implying $2p_3^+ = 0$, a contradiction. So $p_1^+ + p_2^+ + p_3^+ = 0$, and the claim follows.

Now suppose that G lifts to a subgroup G' of $\mathbf{GL}_{d+1}(\mathbb{R})$. Let p be an arbitrary point of Γ, and let p^+ and p^- be two fixed coordinate tuples for p, with $p^+ = -p^-$. Define for each point x of Γ the coordinate tuples x^+, x^- as the image of p^+, p^- under some suitable element of G', where an element is said to be suitable to G if it maps p to x. This is well defined since we claim that, if θ stabilizes p, then the corresponding element M in G' stabilizes $\{p^+, p^-\}$. Indeed, M maps p^+ to some multiple $r \cdot p^+$. If n is the order of θ, then M^n is the identity and maps p^+ to $r^n \cdot p^+$. Hence $r^n = 1$ and the claim follows. Now the group generated by $-I$ and G' is isomorphic to the direct product $2 \times G'$, and similar arguments as those in the previous paragraph show that G' has two orbits on the set $\{p^+, p^- \mid p \text{ is a point of } \Gamma\}$, each of which defines a barycentric embedding (if p_1, p_2, p_3 are three distinct collinear points, and p_1^+, p_2^+, p_3^+ are in the same orbit with respect to G', then, since every transitive group on three letters contains a 3-cycle, from the flag transitivity it follows that $p_1^+ + r_2 p_2^+ + r_3 p_3^+ = 0$ implies $p_2^+ + r_2 p_3^+ + r_3 p_1^+ = 0$, which easily leads to $r_2 = r_3 = 1$).

Finally, suppose that some nontrivial central extension $2 \cdot G$ of G lifts to a subgroup $G^\dagger$ of $\mathbf{GL}_{d+1}(\mathbb{R})$, with $G^\dagger/(G^\dagger \cap \mathbf{Sc}_{d+1}(\mathbb{R})) = G$. As before, one easily shows that for each point p there can be chosen two fixed coordinate tuples p^+ and p^-, with $p^+ = -p^-$ such that $G^\dagger$ acts transitively on the set $\{p^+, p^- \mid p \text{ is a point of } \Gamma\}$. The conditions imply that $-I \in G^\dagger$. Let p_1, p_2, p_3 be three distinct collinear points, and suppose $p_1^+ + r_2 p_2^+ + r_3 p_3^+ = 0$. Using flag transitivity, this implies $p_2^+ \pm r_2 p_3^+ \pm r_3 p_1^+ = 0$, which yields $r_2 = \pm 1$ and $r_3 = \pm 1$. Hence the embedding is semi-barycentric, and not barycentric by the second paragraph of our proof. Moreover, it is clear that the point-line geometry $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I})$, with $\mathcal{P}' = \{p^+, p^- \mid p \text{ is a point of } \Gamma\}$, $\mathcal{L}' = \{\{p_1^{\varepsilon_1}, p_2^{\varepsilon_2}, p_3^{\varepsilon_3}\} \mid p_1, p_2, p_3 \text{ are points of } \Gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\}, p_1^{\varepsilon_1} + p_2^{\varepsilon_2} + p_3^{\varepsilon_3} = 0\}$, and natural incidence relation $\mathtt{I}$, is a double cover of Γ. If it would not be connected, then one could show that each connected component defines a barycentric embedding and the stabilizer of one component is isomorphic to G, leading to the fact that the extension is trivial after all, a contradiction.

The lemma is proved. ∎

The previous lemma says that, if we want to construct real G-homogeneous semi-barycentric embeddings for flag transitive collineation groups G, then we ought to look at real representations of G. One possibility to find a real embedding this way is to see whether a point stabilizer G_x, for x a point of Γ, fixes a 1-dimensional subspace in the representation. This way, one finds the points of Γ in the associated real projective space, and then one has to check whether collinear points of Γ are also collinear in this real projective space. This approach is clearly very individual and does not guarantee any success. However, if one considers the real representation arising from the permutation representation of G on the points of Γ, if Γ is self-polar and G acts primitively on the point set of Γ, then some general results can be obtained.

Theorem 3.3. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a connected self-polar point-line geometry with n points and let G a flag transitive collineation group acting primitively on $\mathcal{P}$. Let $\mathbf{A}$ be a symmetric incidence matrix of Γ, and consider it as the matrix of a linear endomorphism of real n-space. If $\det\mathbf{A} = 0$, then the projection of the basis vectors from $\mathrm{Im}\mathbf{A}$ into $\mathrm{Ker}\mathbf{A}$ yields a G-homogeneous barycentric embedding of Γ. If $\det\mathbf{A} \neq 0$, and if Γ admits a self-polar double cover Γ' with corresponding symmetric incidence matrix $\mathbf{A}'$ such that $\det\mathbf{A}' = 0$ and such that Γ' admits a collineation group G' which induces G in Γ, then, considering $\mathbf{A}'$ as the matrix of a linear endomorphism of real $2n$-space, the projection of the basis vectors from $\mathrm{Im}\mathbf{A}'$ into $\mathrm{Ker}\mathbf{A}'$ yields a G-homogeneous semi-barycentric (non barycentric) embedding of Γ (basis vectors corresponding to points of Γ' in the same fiber are indeed projected onto opposite vectors).*

Proof. First suppose that $\det\mathbf{A} = 0$. Let V be a real n-space. From linear algebra it follows that $V = \mathrm{Im}\mathbf{A} \oplus \mathrm{Ker}\mathbf{A}$, and we consider the action of $\mathbf{A}$ on V on the left (so on column vectors). We denote the polarity corresponding to $\mathbf{A}$ by ρ. We may identify each point p of Γ with a basis vector $\overline{p}$ of V in such a way that, if p corresponds to the ith row of $\mathbf{A}$, then $\overline{p}$ is the ith basis vector $(0,\dots,0,1,0,\dots,0)$, with the 1 on the ith entry. So we can unambiguously write $\overline{p} = \overline{p}_{\mathrm{Im}} + \overline{p}_{\mathrm{Ker}}$, with $\overline{p}_{\mathrm{Im}} \in \mathrm{Im}\mathbf{A}$ and $\overline{p}_{\mathrm{Ker}} \in \mathrm{Ker}\mathbf{A}$. It is clear that G is isomorphic with a subgroup of $\mathbf{GL}(V)$ stabilizing $\mathrm{Ker}\mathbf{A}$. Moreover, G acts on the set $\overline{\mathcal{P}} := \{\overline{p}_{\mathrm{Ker}} \,|\, p \in \mathcal{P}\}$ and this action is permutation equivalent to the action of G on $\mathcal{P}$ whenever $|\overline{\mathcal{P}}| > 1$ (using the primitivity of G on $\mathcal{P}$). We now show that $\overline{p}_{\mathrm{Ker}} + \overline{p}'_{\mathrm{Ker}} + \overline{p}''_{\mathrm{Ker}} = 0$, for distinct collinear points p, p', p'' of Γ. This is clearly equivalent to showing that $\overline{p}_{\mathrm{Ker}} + \overline{p}'_{\mathrm{Ker}} + \overline{p}''_{\mathrm{Ker}} \in \mathrm{Im}\mathbf{A}$, for p, p', p'' as above, which is on its turn equivalent with $\overline{p} + \overline{p}' + \overline{p}'' \in \mathrm{Im}\mathbf{A}$. But this is clear, as $\overline{p} + \overline{p}' + \overline{p}''$ is the image of the basis vector $\overline{x}$, with x^ρ equal to the line $pp'p''$. Now remark that $\overline{\mathcal{P}}$ generates $\mathrm{Ker}\mathbf{A}$ and so, if $|\overline{\mathcal{P}}| = 1$, then $\overline{p}_{\mathrm{Ker}} = 0$, for every $p \in \mathcal{P}$, contradicting the assumption $\det\mathbf{A} = 0$. So distinct points of Γ define distinct elements of $\mathrm{Ker}\mathbf{A}$. Hence there remains to show that distinct lines of Γ define distinct lines of $\mathbf{PG}(\mathrm{Ker}\mathbf{A})$ (implying in particular that $\dim\mathrm{Ker}\mathbf{A} \geq 3$ and so, in view of the primitivity of G on $\mathcal{P}$ again, distinct points of $\mathcal{P}$ define distinct points of $\mathbf{PG}(\mathrm{Ker}\mathbf{A})$).

First we remark that, if two different collinear points p, p' define proportional vectors $\overline{p}_{\mathrm{Ker}}, \overline{p}'_{\mathrm{Ker}}$, then by flag transitivity, every point x collinear with p defines a vector $\overline{x}_{\mathrm{Ker}}$ proportional to $\overline{p}_{\mathrm{Ker}}$. By connectivity, all elements of $\overline{\mathcal{P}}$ are proportional to each other and so $\dim\mathrm{Ker}\mathbf{A} = 1$. Since Γ contains at least 3 points, there is some element of G whose restriction to $\mathrm{Ker}\mathbf{A}$ acts as a linear isomorphism with determinant different from 1 and -1, contradicting the finiteness of both G and Γ.

So we may assume that lines of Γ really define lines of $\mathbf{PG}(\mathrm{Ker}\mathbf{A})$. Assume, by way of contradiction, that two concurrent lines L_1, L_2 of Γ define the same line L of $\mathbf{PG}(\mathrm{Ker}\mathbf{A})$. Let p be the intersection of L_1 with L_2. Since G is flag transitive, there is a collineation in G fixing p and inducing a 3-cycle on the set of lines of Γ through p. This implies easily that all lines of Γ through p define the same line L of $\mathbf{PG}(\mathrm{Ker}\mathbf{A})$. Mapping p to any point x collinear with p, thereby preserving the line px, we see that all points at distance 4 from p also correspond with points of $\mathbf{PG}(\mathrm{Ker}\mathbf{A})$ on L. By connectivity, and going on like this, all elements of $\overline{\mathcal{V}}$ lie on L and hence $\dim \mathrm{Ker}\mathbf{A} = 2$.

We recoordinatize $\mathrm{Ker}\mathbf{A}$ such that $\overline{p}_{\mathrm{Ker}}$ has coordinates $(1,0)$, and the two other points of some line through p in Γ correspond to the vectors $(0,1)$ and $(-1,-1)$. By flag transitivity, we can fix $(1,0)$ and map $(0,1)$ to some vector (a,b) corresponding to a point of Γ collinear with p, but distinct from both $(0,1)$ and $(-1,-1)$, and we can do this using a collineation whose order is divisible by 3. The 2×2 matrix M corresponding to that transformation has finite multiplicative order if and only if either $b \in \{1,-1\}$ and $a = 0$, or $b = -1$. In the latter case

$$M = \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}$$

and consequently we see that in both cases $M^2 = I$, contradicting the fact that the order of M must be divisible by 3.

So we have shown the theorem in the case that $\mathbf{A}$ is singular.

Now suppose that $\det \mathbf{A} \neq 0$ and that Γ admits a self-polar double cover $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ with corresponding symmetric incidence matrix $\mathbf{A}'$ such that $\det \mathbf{A}' = 0$, and such that Γ' admits a collineation group G' inducing G in Γ. Note that $|G'| = 2|G|$, since the unique involution interchanging the elements of each fiber is a collineation of Γ'. Denote by ξ the natural epimorphism from Γ' down to Γ.

Let V' be a real $2n$-space. We again have $V' = \mathrm{Im}\mathbf{A}' \oplus \mathrm{Ker}\mathbf{A}'$ and consider the action of $\mathbf{A}'$ on V' on the left. We again identify each point p of Γ' with a basis vector $\overline{p}$ of V' in such a way that, if p corresponds to the ith row of $\mathbf{A}'$, then $\overline{p}$ is the ith basis vector $(0,\ldots,0,1,0,\ldots,0)$, with the 1 on the ith entry. We again write $\overline{p} = \overline{p}_{\mathrm{Im}} + \overline{p}_{\mathrm{Ker}}$, with $\overline{p}_{\mathrm{Im}} \in \mathrm{Im}\mathbf{A}'$ and $\overline{p}_{\mathrm{Ker}} \in \mathrm{Ker}\mathbf{A}'$. The group G' is isomorphic with a subgroup of $\mathbf{GL}(V')$ stabilizing $\mathrm{Ker}\mathbf{A}'$ and acting on the set $\overline{\mathcal{P}}' := \{\overline{p}_{\mathrm{Ker}} \mid p \in \mathcal{P}'\}$ permutation equivalently to the action of G' on $\mathcal{P}'$ if $|\overline{\mathcal{P}}'| > 1$. We now claim that $\overline{p}_{\mathrm{Ker}} + \overline{p}'_{\mathrm{Ker}} = 0$, for p, p' two distinct points of the same fiber of Γ'. As before, it suffices to show that $\overline{p} + \overline{p}' \in \mathrm{Im}\mathbf{A}'$. First we note that there is a natural injective morphism $\alpha : V \leftarrow V'$ which maps the basis vector $\overline{x}$, $x \in \mathcal{P}$, to the vector $\overline{x}_1 + \overline{x}_2$, where $x_1, x_2 \in \mathcal{P}'$ are the distinct points in the fiber of x. It is easily verified on the basis that for every vector $\overline{w} \in V$ we have $\alpha(\mathbf{A}\overline{w}) = \mathbf{A}'\alpha(\overline{w})$. As $\det \mathbf{A} \neq 0$, there is a (unique) vector $\overline{v} \in V$ with $\mathbf{A}\overline{v} = \overline{p^{\xi}}$, so $\mathbf{A}'\alpha(\overline{v}) = \overline{p} + \overline{p}'$ and the claim is proved.

Similar to the first part of the proof, one shows that $\overline{p}_{\mathrm{Ker}} + \overline{p}'_{\mathrm{Ker}} + \overline{p}''_{\mathrm{Ker}} = 0$, for distinct collinear points p, p', p'' of Γ' and that $\overline{\mathcal{P}}'$ generates $\mathrm{Ker}\mathbf{A}'$. Hence there remains to show that points (lines) in different fibers of Γ' define distinct points (lines) of $\mathbf{PG}(\mathrm{Ker}\mathbf{A}')$. But this is similar to the first part of the proof.

Hence we do obtain a G-homogeneous semi-barycentric embedding of Γ in $\mathbf{PG}(\mathrm{Ker}\mathbf{A})$. It cannot be barycentric by Lemma 3.1.

The theorem is proved. ∎

This theorem has an interesting corollary. For a given slim point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, we let $\mathbb{R}_p$ be a copy of the additive group of real numbers, where we denote the 1 by p. We define the group $G_{\mathcal{P}}$ as the direct product of these copies $\mathbb{R}_p$, for $p \in \mathcal{P}$. We then define the quotient group G_Γ by requiring the relations $p_1 + p_2 + p_3 = 0$, for each triple of distinct collinear points of Γ. Now let there be given any barycentric embedding of Γ in $\mathbf{PG}(d, \mathbb{R})$, then we may fix the coordinates of every point of Γ and obtain a representation of Γ in the vector space $V = \mathbb{R}^{d+1}$, where the sum of each three distinct collinear points is zero. It is clear that there is a canonical epimorphism from G_Γ onto the additive group of V such that, for each point p of Γ, the point of G_Γ corresponding to p is mapped onto the point of V corresponding to p. Hence, in this case, the representation of Γ in G_Γ can be considered as a universal barycentric real embedding of Γ. It follows that, if a slim point-line geometry Γ has some barycentric real embedding, then it has a unique universal one. In view of Lemma 3.1, Theorem 3.3 now implies:

Corollary 3.4. *Suppose a slim self-polar flag transitive point-line geometry* Γ *admits at least one real barycentric embedding. Let* $\mathbf{A}$ *be a symmetric incidence matrix of* Γ*, and consider it as the matrix of a linear endomorphism of real n-space, with n the number of points of* Γ*. Then* $\det \mathbf{A} = 0$*, and the projection of the basis vectors from* $\mathrm{Im}\mathbf{A}$ *into* $\mathrm{Ker}\mathbf{A}$ *yields the* (*homogeneous*) *universal real barycentric embedding of* Γ*. Hence the real barycentric embedding rank of* Γ *is equal to the corank of the matrix* $\mathbf{A}$*.*

Proof. This is clear if Γ admits a flag transitive collineation group acting primitively on the point set of Γ. If not, and if $\det \mathbf{A} = 0$, then we must show that, if the construction of the Corollary does not yield an embedding, then Γ does not admit any real barycentric embedding at all.

The proof of Theorem 3.3 shows that, if the given algorithm does not lead to an embedding, then it leads to an embedding of some quotient geometry, and so two points p, p' of Γ give rise to either opposite vectors $\overline{p}_{\mathrm{Ker}} = -\overline{p}'_{\mathrm{Ker}}$ or equal vectors $\overline{p}_{\mathrm{Ker}} = \overline{p}'_{\mathrm{Ker}}$ (using the notation of the proof of Theorem 3.3). It follows that, with the right choice of sign, the vector $\overline{p} \pm \overline{p}'$ belongs to $\mathrm{Im}\mathbf{A}$. But $\mathrm{Im}\mathbf{A}$ is generated by vectors of the form $\overline{x}_1 + \overline{x}_2 + \overline{x}_3$, with x_1, x_2, x_3 distinct collinear points of Γ (using the symmetry of $\mathbf{A}$). It follows that the relation $p \pm p' = 0$ in $G_{\mathcal{P}}$ follows from the relations $x_1 + x_2 + x_3 = 0$, with x_1, x_2, x_3 distinct and collinear. This means that there is no universal barycentric embedding, and hence no real barycentric embedding at all. ∎

This yields in principal a complete classification of all barycentric embeddings of slim self-polar flag transitive geometries. Indeed, if some incidence matrix is non-singular, then there is no real barycentric embedding; otherwise we apply the construction given in Theorem 3.3 or Corollary 3.4 and see whether it gives rise to an embedding (in the primitive case it always does).

One can compare the previous results with the situation over $\mathbf{GF}(2)$. Indeed, if a slim geometry admits any embedding in some projective space $\mathbf{PG}(d, 2)$ over the field $\mathbf{GF}(2)$ of two elements, then it admits a universal such embedding and every other embedding over $\mathbf{GF}(2)$ arises from the universal embedding by projection. The universal embedding is always homogeneous (so is the universal real barycentric one). But there is no concrete general geometric construction algorithm known in this case.

We will provide some ad hoc constructions for our exceptional geometries in the next section. Notice that the dimensions of the universal embedding and the real universal barycentric embedding may be different.

4. Application to Ten Exceptional Bislim Geometries

We now apply the previous results to the ten exceptional bislim geometries arising from the ten trivalent distance regular graphs of girth ≥ 5. We also mention some other properties of these geometries, in casu, some alternative descriptions. Note that all the geometries we consider admit flag transitive collineation groups.

We arrange the geometries by their underlying graphs, sometimes considering small classes of two or three related graphs.

4.1. The Petersen Graph, the Desargues Graph and the Dodecahedron

These graphs give rise to the Desargues configuration **Dc** and a certain unique double cover $\widetilde{\mathbf{Dc}}$ of it. We present some known, but still worthwhile mentioning, constructions.

First of all, there is the classical construction of the Desargues configuration in the real projective plane with two triangles in perspective from a point, implying that the corresponding sides of the triangles meet on a common line (and hence the triangle are also in perspective from that line). It is easy to check that this can never give rise to a homogeneous embedding. However, the *axiom of Desargues*, from which this construction emerges, is valid in every Desarguesian projective plane. In particular, it holds in finite Pappian planes. In some small planes, there are alternative ways to describe it, and they yield G-homogeneous embeddings for rather large collineation groups G of **Dc**.

Consider a non-degenerate conic in the projective plane $\mathbf{PG}(2,5)$. Then the points of **Dc** are the 10 *internal points* of the conic (i.e., the points not incident with any tangent line to the conic) and the lines are the 10 *external lines* (i.e., the lines not meeting the conic), while incidence is natural. From this construction it is clear that $\mathbf{PGL}_2(5)$ is a collineation group, and that **Dc** is a self-polar bislim geometry. Hence, this construction yields a homogeneous (exclusive) embedding into $\mathbf{PG}(2,5)$ (it is indeed well known that the full collineation group is isomorphic to $\mathbf{PGL}_2(5) \cong \mathbf{S}_5$).

In fact, it is rather straightforward to calculate using coordinates that:

Proposition 4.1. *If a Desargues configuration in any Desarguesian projective plane is homogeneously embedded, then the characteristic of the underlying skew field is equal to* 5 *and there is a subplane over* $\mathbf{GF}(5)$ *such that* **Dc** *is embedded in that subplane as in the previous paragraph.*

Consider a triangle T in the projective plane $\mathbf{PG}(2,3)$ (this is a set of three non-collinear points together with the three joining lines). The points of **Dc** are the points of $\mathbf{PG}(2,3)$ different from those of T and the lines are the lines of $\mathbf{PG}(2,3)$ different from those of T. A point p and a line L are incident in **Dc** if they are incident in $\mathbf{PG}(2,3)$ and if either p lies on a line of T or L contains a point of T, or both conditions hold. One checks that this indeed yields **Dc**. Viewing $\mathbf{S}_4$ as a subgroup of Aut**Dc**, this yields a $\mathbf{S}_4$-homogeneous (non-exclusive) embedding of **Dc** in $\mathbf{PG}(2,3)$ (it can be seen as a "little Desargues configuration" from each of the four points not on any of the lines of T).

In fact, the way to prove that projective spaces of dimension ≥ 3 are always Desarguesian is to consider a Desargues configuration in projective 3-space, i.e., a point together with two triangles in perspective from that point, but lying in different planes, and together with the intersection points of the corresponding sides and the intersection line of the two planes. For the smallest projective 3-space (the one over $\mathbf{GF}(2)$), there is a more elegant description as follows. Consider an elliptic quadric in $\mathbf{PG}(3,2)$. The points of **Dc** are the points off that quadric, and the lines are the lines disjoint from the quadric. Incidence is natural. All embeddings in projective 3-space of **Dc** are homogeneous and exclusive. Over the real numbers, it is never barycentric, for otherwise any incidence matrix must be singular. But such a matrix is an adjacency matrix of the Petersen graph, and this matrix is non-singular. However, such an embedding is homogeneous over any field.

A combinatorial construction goes as follows. The points are the pairs of the 5-set $\{1,2,3,4,5\}$; the lines are the triples of that 5-set and incidence is natural (and the action of the symmetric group $\mathbf{S}_5$ is apparent here). We now recognize the Desargues configuration as the neighborhood geometry of the Petersen graph. Indeed, the vertices of the Petersen graph can also be taken as the pairs of the 5-set $\{1,2,3,4,5\}$, with two vertices adjacent if the corresponding pairs are disjoint. Hence the lines of the neighborhood geometry are triples of pairs disjoint from a fixed pair, or in other words, contained in the complementary 3-set.

It is an elementary exercise to show that the generating rank of **Dc** is equal to 4. Since there are embeddings of **Dc** in $\mathbf{PG}(3,2)$ the universal embedding rank is equal to 4. Since any incidence matrix is non-singular, the real embedding rank is 0.

We now consider the question whether there is a semi-barycentric embedding of **Dc** in $\mathbf{PG}(3,\mathbb{R})$ and $\mathbf{PG}(2,\mathbb{R})$. Therefore, we consider the neighborhood geometry of the dodecahedron. Since the dodecahedron graph is a double cover of the Petersen graph, this geometry, which we will denote by $\widetilde{\mathbf{Dc}}$, is an antipodal double cover of **Dc**. It is straightforward to show the following.

Proposition 4.2. *The geometry* **Dc** *admits, up to isomorphism, a unique double cover which does not contain any triangle as a subgeometry, and this double cover is isomorphic to* $\widetilde{\mathbf{Dc}}$.

It is a peculiar fact that $\mathrm{Aut}\,\widetilde{\mathbf{Dc}}$ is not entirely induced by the automorphism group of the dodecahedron graph. Indeed, every collineation of **Dc** lifts to some collineation of $\widetilde{\mathbf{Dc}}$, hence $|Aut\,\widetilde{\mathbf{Dc}}| = 240$, while the automorphism group of the dodecahedron graph has size 120. In fact, the full correlation group of $\widetilde{\mathbf{Dc}}$ is isomorphic to the group $\mathbf{GL}_2(5)$. Note that these observations provide a counterexample to Lemma 3.5 of [7] (which is stated without proof, and which claims that the automorphism group of a graph and its neighborhood geometry are the same).

Clearly an adjacency matrix of the dodecahedron graph is an incidence matrix of $\widetilde{\mathbf{Dc}}$. Now it is known that such a matrix is singular, hence we can apply Theorem 3.3 and we obtain a semi-barycentric embedding of **Dc** in real projective 3-space. We also conclude with Corollary 3.4 that $\widetilde{\mathbf{Dc}}$ does not admit any real barycentric embedding.

With some tedious calculations, one can prove that $\widetilde{\mathbf{Dc}}$ does admit a real embedding in $\mathbf{PG}(2,\mathbb{R})$, but I was not able (yet) to find a (nice) geometric construction of such an embedding or a geometric argument about why such an embedding should exist.

4.2. The Pappus Graph

The Pappus graph is the incidence graph of the Pappus configuration **Pc**. The points of **Pc** are the 9 vectors of a 2-dimensional vector space over the field $\mathbf{GF}(3)$; the lines are the translates of all vector lines but one (arbitrarily chosen). In other words, it is the biaffine plane of order 3, or, with still different terminology, the net of order 3 and degree 3.

The Pappus configuration originates from the following property in *Pappian* projective planes (planes coordinatized by commutative fields). Consider three collinear points a_1, a_2, a_3, and three collinear points b_1, b_2, b_3 such that the intersection point of the corresponding lines is not amongst $a_1, \dots, b_3$. Then the lines a_ib_j and a_jb_i meet at some point c_k, with $\{i, j, k\} = \{1, 2, 3\}$, and c_1, c_2, c_3 are automatically collinear. The points and lines just mentioned define an embedding of the Pappus geometry. This embedding cannot be barycentric since it is easy to check that any incidence matrix of **Pc** has corank 2. But a simple calculation shows that it can neither be semi-barycentric. Also easy to see is that **Pc** is generated by 3 points, hence the universal embedding rank can be at most 3. But clearly **Pc** cannot be embedded in $\mathbf{PG}(2,2)$ since the later only has 7 points. So **Pc** does not admit any embedding at all in any projective space over $\mathbf{GF}(2)$.

Note that **Pc** is not a primitive geometry; it is the unique (antipodal) 3-fold cover of the trivial bislim geometry (trivial in the sense that all points are incident with all lines) which does not contain digons itself (i.e. which does not contain two different lines incident with two different points).

4.3. The Heawood Graph, the Coxeter Graph, Tutte's 12-Cage

The Fano geometry $\mathbf{PG}(2,2)$ is the smallest projective plane the incidence graph of which is the Heawood graph. The point set of $\mathbf{PG}(2,2)$ is the set of 7 nonzero vectors of a 3-dimensional vector space over the field $\mathbf{GF}(2)$ of 2 elements, the lines are the 7 vector planes in this vector space, with natural incidence relation.

Another well known construction of $\mathbf{PG}(2,2)$ is to take as point set the integers modulo 7, and the lines are the translates of the 3-set $\{0, 1, 3\}$. We refer to this as the *cyclic construction*.

From this geometry, one defines the Coxeter graph as follows. The vertices are the *antiflags* (non-incident point-line pairs) and two vertices $\{x_1, L_1\}, \{x_2, L_2\}$, with x_1, x_2 points of $\mathbf{PG}(2,2)$ and L_1, L_2 lines of $\mathbf{PG}(2,2)$ such that $x_i \not I L_i$, $i = 1, 2$, are adjacent whenever $|\{x_1, x_2, L_1, L_2\}| = 4$ and $\{x_1, L_2\}, \{x_2, L_1\}$ are also antiflags. The neighborhood geometry **Cox** of the Coxeter graph has not been studied yet individually in the literature, although it is mentioned as an example of neighborhood geometry in Table 1 of [7]. In view of the definition of the Coxeter graph, it can be described as follows (other constructions follow from other constructions of the Coxeter graph, see [5]). The points are the antiflags of the Fano geometry, three antiflags form a line of **Cox** if their union forms a triangle in $\mathbf{PG}(2,2)$. In fact, this definition may be generalized to arbitrary projective planes to yield many slim geometries with many symmetries. Each line of **Cox** is corresponding to a unique point of **Cox** (the line viewed as a triangle in $\mathbf{PG}(2,2)$, this point is the antiflag none of whose elements are incident with any element of the triangle) and this induces a (unique) polarity without absolute points (a point being *absolute* with respect to a polarity if it is incident with its image under the polarity). Clearly, **Cox** is primitive, and every adjacency matrix $\mathbf{A}$ of the Coxeter graph is an incidence matrix of **Cox**. It is well known that the rank of $\mathbf{A}$ is 28, hence by Lemma 3.1 **Cox** does not admit a real barycentric embedding. But it can be shown (unpublished) that **Cox** admits an antipodal double cover with collineation group $\mathbf{GL}_2(7)$; in fact this geometry is the neighborhood geometry of an (antipodal) non-bipartite graph with 56 vertices and automorphism group $\mathbf{SL}_2(7)$. So, as is the case with the neighborhood geometry of the dodecahedron, the collineation group of the geometry is not completely induced by the automorphism group of the graph.

We now show that the generating rank is equal to 8. We first exhibit a set of 8 points that generates **Cox**. Afterward, we construct an embedding of **Cox** in $\mathbf{PG}(2,7)$ proving that the generating rank is 8 and that this embedding is the universal one.

First we recall an old notation, namely, $\{n\}+\{n/m\}$ refers to the graph with vertex set $\{1,2,\ldots,n,1',2',\ldots,n'\}$ and adjacency relation $\sim$ defined by $i\sim j$ (respectively $i'\sim j'$ and $i\sim j'$) if and only if $i-j\equiv \pm 1 \bmod n$ (respectively $i-j\equiv \pm m \bmod n$ and $i=j$). For instance, the Petersen graph is $\{5\}+\{5/2\}$ in this notation and the incidence graph of the Desargues configuration is $\{10\}+\{10/3\}$.

Now, in the cyclic construction of $\mathbf{PG}(2,2)$, consider the antiflags $(i,\{i-2, i-1,i+1\})$, $i\in\mathbb{Z} \bmod 7$. They form a 7-gon in **Cox** with as additional points on the sides the antiflags $(i,\{i+1,i+2,i+4\})$, $i\in\mathbb{Z} \bmod 7$. These 14 points form a geometric hyperplane the complement of which contains two 7-gons with respective point sets $\mathcal{C}_1=\{(i,\{i+2,i+3,i+5\})\,|\,i\in\mathbb{Z} \bmod 7\}$ and $\mathcal{C}_2=\{(i,\{i-4,i-3,i-1\})\,|\,i\in\mathbb{Z} \bmod 7\}$, so organized that, together, they form the connected graph $\{7\}+\{7/2\}$. Hence the points $(i,\{i-2,i-1,i+1\})$, $i\in\mathbb{Z} \bmod 7$, together with any element of $\mathcal{C}_1\cup\mathcal{C}_2$, generate **Cox**. Hence the generating rank of **Cox** is at most 8.

We now give an explicit construction of the universal embedding of **Cox**. All 3×3 matrices over $\mathbf{GF}(2)$ form a 9-dimensional vector space V with standard basis $\{E_{ij}\,|\,i,j=1,2,3\}$. We consider the quotient space $W=V/T$, where T is the vector line generated by the identity matrix $I=E_{11}+E_{22}+E_{33}$. Note that the points of $\mathbf{PG}(2,2)$ can be identified with triples $(x_1,x_2,x_3)\in\mathbf{GF}(2)^3$ and the lines with triples $[a_1,a_2,a_3]\in\mathbf{GF}(2)^3$ such that a point (x_1,x_2,x_3) is incident with a line $[a_1,a_2,a_3]$ if and only if $a_1x_1+a_2x_2+a_3x_3=0$. We identify an antiflag $\{(x_1,x_2,x_3),[a_1,a_2,a_3]\}$ with the point $\sum_{i,j}x_ia_jE_{ij}+T$. We claim that this defines an embedding in $\mathbf{PG}(W)$. Since the collineation group of $\mathbf{PG}(2,2)$ is transitive on triangles, we can represent every line of **Cox** as a triple $(\{(1,0,0)M,[1,0,0]M^{-t}\},\{(0,1,0)M,[0,1,0]M^{-t}\},\{(0,0,1)M,[0,0,1]M^{-t}\})$. The sum of the coordinates of points of this line of **Cox** in V is equal to $M^tE_{11}M^{-t}+M^tE_{22}M^{-t}+M^tE_{33}M^{-t}=M^tIM^{-t}=I$; hence they represent collinear points of $\mathbf{PG}(W)$. It remains to show that these points generate $\mathbf{PG}(W)$. But this follows by considering the antiflags (denoting the antiflag $\{(x_1,x_2,x_3),[a_1,a_2,a_3]\}$ briefly by $\langle x_1x_2x_3,a_1a_2a_3\rangle$) $\langle 100,100\rangle$, $\langle 100,110\rangle$, $\langle 100,101\rangle$, $\langle 010,010\rangle$, $\langle 010,110\rangle$, $\langle 010,011\rangle$, $\langle 001,101\rangle$, $\langle 001,011\rangle$. So we have the satisfying situation that the universal embedding rank is equal to the generating rank of **Cox**.

Tutte's 12-cage can also be constructed from $\mathbf{PG}(2,2)$, see for instance [12]. It is convenient to use the elements of the Heawood graph and the Coxeter graph in terms of the points, lines, flags and antiflags of $\mathbf{PG}(2,2)$. Then the vertices of Tutte's 12-cage are the points and lines (the vertices of the Heawood graph), the antiflags (vertices of the Coxeter graph) and flags of $\mathbf{PG}(2,2)$, together with the edges of the Heawood graph and the Coxeter graph. Adjacency is the natural incidence in the Heawood graph and the Coxeter graph, together with the rule: a flag (p, L) is adjacent to the unique edge of the Heawood graph containing the vertices p and L, and also with the two edges of the Coxeter graph containing vertices corresponding to antiflags the lines of which are both incident with p in $\mathbf{PG}(2,2)$, and the points of which are both incident with the line L in $\mathbf{PG}(2,2)$. Clearly Tutte's 12-cage is bipartite. In fact, it is distance regular, but not distance transitive. It is *almost* distance transitive in the sense that stabilizer of a vertex v acts transitively on each set of vertices at fixed distance from v. Tutte's 12-cage has diameter 6 and girth 12 and thus is the incidence graph of two (non-isomorphic) generalized hexagons. Hence some results of the previous section do not apply here. On the other hand, these hexagons are well studied and the following is known. Both the universal embedding rank and the generating rank of both generalized hexagons are equal to 14. Moreover, there are real embeddings of both geometries in $\mathbf{PG}(13,\mathbb{R})$. These are explicitly constructed (giving precise coordinates to each point) in [11], and one can check easily that they are barycentric. In fact, as shown in [11], the full collineation groups are induced by $\mathbf{GL}_{14}(\mathbb{R})$, and so the embeddings are automatically barycentric by Lemma 3.2. Consequently the real embedding ranks are also equal to 14. This is again a very desirable situation.

For other constructions of the generalized hexagons related to Tutte's 12-cage, see [9].

4.4. Tutte's 8-Cage and the Foster Graph

Tutte's 8-cage is the incidence graph of the unique bislim generalized quadrangle $\mathbf{W}(2)$. There are many very different constructions of this geometry and its incidence graph. We mention two of them: a less known one and a well known one, respectively.

Consider a conic $\mathcal{C}$ in the projective plane $\mathbf{PG}(2,9)$. Define a graph with vertex set being the polar triangles with respect to $\mathcal{C}$ (i.e., the triples of distinct points such that the polar line with respect to $\mathcal{C}$ of each of these points is spanned by the two other points) which consist entirely of exterior points (i.e., points incident with exactly two tangent lines to $\mathcal{C}$). Declare two vertices adjacent if the corresponding polar triangles intersect non-trivially. This graph is Tutte's 8-cage, see [9].

Now define the following bislim geometry. The points are the pairs of elements of the set $\{1,2,3,4,5,6\}$, the lines are the triples of pairs that partition $\{1,2,3,4,5,6\}$ (and incidence is the natural one). We obtain $\mathbf{W}(2)$.

An *ovoid* of a point-line geometry Γ is a set of points with the property that each line is incident with exactly one point of the ovoid. The generalized quadrangle has six ovoids. Each of them may be identified with an element of $\{1,2,3,4,5,6\}$; the ovoid consists of all pairs containing a fixed element of $\{1,2,3,4,5,6\}$. We deduce that every point is contained in exactly 2 ovoids.

A *spread* of a point-line geometry is the dual of an ovoid: it is a set of lines partitioning the point set.

It is well known and easy to verify that both the generating rank and the universal embedding rank of $\mathbf{W}(2)$ are equal to 5. Concerning the real embedding rank, we consider a symmetric incidence matrix $\mathbf{A}$ (which is not equal to the adjacency matrix of any graph because every polarity of $\mathbf{W}(2)$ has five absolute points; the points of an ovoid). Then $\mathbf{A}^2 = \mathbf{B} + 3\mathbf{I}$, where $\mathbf{B}$ is an adjacency matrix of the point graph of $\mathbf{W}(2)$ and which has -3 as an eigenvalue with multiplicity 5 (see e.g. [2]). Hence the real embedding rank is also equal to 5. We now present a direct and explicit construction of the universal barycentric embedding of $\mathbf{W}(2)$. To this end, we consider the construction of $\mathbf{W}(2)$ as given in the previous paragraph and we label the coordinates of points of $\mathbf{PG}(5,\mathbb{R})$ (but the construction will work over an arbitrary field $\mathbb{K}$) with the elements of the set $\{1,2,3,4,5,6\}$. The point $\{i,j\}$, with $i,j \in \{1,2,3,4,5,6\}$, $i \neq j$, is assigned the coordinates $(x_1,x_2,\ldots,x_6)$ with $x_k = 1$ if $k \notin \{i,j\}$ and $x_k = -2$ if $k \in \{i,j\}$. In fact, this is the result of labelling the coordinates of points of a real projective space by the ovoids of $\mathbf{W}(2)$ and assigning to any point coordinates according to the rule: the ith coordinate is 1 if the point does not belong to ovoid number i, and -2 if it does belong to that ovoid. We will use this construction later on again.

Clearly we obtain a real embedding in some subspace. Indeed, fix a line L and an entry in the coordinate tuples, i.e., an ovoid O. Since L contains exactly one point of O, the sum of the corresponding entries of the three points incident with L is equal to $1+1-2=0$, hence we already have the “barycentric property”. We leave it as an easy exercise to check that distinct points and lines of $\mathbf{W}(2)$ are represented by distinct points and lines of $\mathbf{PG}(5,\mathbb{R})$. It remains to determine the dimension of the space generated by the points of $\mathbf{W}(2)$ in $\mathbf{PG}(5,\mathbb{R})$. For each ovoid O, we consider the point p_O of $\mathbf{PG}(5,\mathbb{R})$ obtained by adding the coordinates of all points of O. We obtain a point with all coordinates equal to 2, except the entry corresponding with O, which equals -10. The coordinates of the six points p_O thus obtained form a 6×6 matrix whose rank is equal to 5 (indeed, -6 is an eigenvalue with multiplicity 5 if the numbers -10 are all put on the diagonal; hence 0 is an eigenvalue with multiplicity 1). Hence the points of $\mathbf{W}(2)$ generate a hyperplane $\mathbf{PG}(4,\mathbb{R})$ and we do have the universal barycentric embedding. The explicit equation of $\mathbf{PG}(4,\mathbb{R})$ is $X_1+X_2+X_3+X_4+X_5+X_6 = 0$ (indeed, as every point of $\mathbf{W}(2)$ belongs to exactly 2 ovoids, every point has 4 coordinates equal to 1 and two equal to -2, so the sum of coordinates is indeed 0). Note that this construction, taken over $\mathbf{GF}(2)$, produces the universal embedding over $\mathbf{GF}(2)$.

So the situation for $\mathbf{W}(2)$ is similar to the one of the two bislim generalized hexagons. This changes when considering the bislim geometry related to the Foster graph. This is the so-called *tilde geometry* $\widetilde{\mathbf{W}}(2)$; it is the unique (antipodal) triple cover of $\mathbf{W}(2)$ having no quadrangles as subgeometry (in other words, the incidence graph of which has girth >8). The following interesting property is easily deduced from the construction in [9] (note that every spread of $\mathbf{W}(2)$ gives rise to a spread of $\widetilde{\mathbf{W}}(2)$ by lifting all spread elements).

Lemma 4.3. *Let $\mathcal{S}$ be a spread of $\widetilde{\mathbf{W}}(2)$ obtained from a spread of $\mathbf{W}(2)$ by lifting. If we remove $\mathcal{S}$ from $\widetilde{\mathbf{W}}(2)$, then there remain three connected slim geometries each point of which is incident with exactly 2 lines. Each of these geometries is isomorphic to the dual of a Petersen graph, viewed as dually slim geometry.*

It is well known (as explicitly shown in [8]) that both the generating rank and universal embedding rank of $\widetilde{\mathbf{W}}(2)$ is equal to 11. Moreover, [8] contains an explicit construction of the universal embedding, and all other homogeneous embeddings over $\mathbf{GF}(2)$ are determined. Finally, it has been shown in [8] that $\widetilde{\mathbf{W}}(2)$ does not admit an embedding in $\mathbf{PG}(11,\mathbb{R})$. The question whether $\widetilde{\mathbf{W}}(2)$ admits any real embedding at all has been posed, but not answered yet to the best of my knowledge. We will answer it now. But first we claim that Corollary 3.4 implies that $\widetilde{\mathbf{W}}(2)$ does not admit any barycentric embedding. Indeed, the rank of any incidence matrix of $\widetilde{\mathbf{W}}(2)$ is equal to 40, as can be deduced from the tables in [2]. Hence the universal barycentric embedding would live in $\mathbf{PG}(4,\mathbb{R})$ and the points of this embedding are the components of the standard base vectors in the kernel of the transformation corresponding to a symmetric incidence matrix $\mathbf{A}$. Let V be the 45-dimensional vector space on which $\mathbf{A}$ acts, and let V' be the 15-dimensional vector space obtained from V be identifying the base vectors in a common fiber of the covering map $\widetilde{\mathbf{W}}(2) \to \mathbf{W}(2)$. Let $\alpha : V' \to V$ be the map defined by $\alpha(p) = p_1 + p_2 + p_3$, where p corresponds to a point of $\mathbf{W}(2)$ and p_1, p_2, p_3 are the base vectors corresponding to the points of $\widetilde{\mathbf{W}}(2)$ in the fiber of p. Since the rank of any symmetric incidence matrix of $\mathbf{W}(2)$ is equal to 10, we see that α induces an isomorphism of kernels. It follows that a collineation φ of $\widetilde{\mathbf{W}}(2)$ of order 3 inducing the identity in $\mathbf{W}(2)$ acts trivially on Ker$\mathbf{A}$, and hence the base vectors of V belonging to a common fiber have identical components in Ker$\mathbf{A}$. This is a contradiction, so there is no universal barycentric embedding and the claim follows.

We now construct a real exclusive embedding of $\widetilde{\mathbf{W}}(2)$ in $\mathbf{PG}(2,\mathbb{R})$. For the moment, it is the largest dimension we can do (without using a computer, and without wanting to make many computations). We consider a spread $\mathcal{S}$ of $\widetilde{\mathbf{W}}(2)$ (arising from a spread of $\mathbf{W}(2)$). We remove $\mathcal{S}$ from $\widetilde{\mathbf{W}}(2)$ and obtain three connected components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ each of which is isomorphic to the dual of the Petersen graph geometry. We embed each of these geometries (which are isomorphic to the geometry obtained from $\mathbf{W}(2)$ by removing the lines of a spread) in a different copy of $\mathbf{PG}(5,\mathbb{R})$. We first show that this is possible. Indeed, let e_i, $i \in \mathbb{Z}$ mod 6 be base vectors of a 6-dimensional vector space V over $\mathbb{R}$. Then the points of $\mathbf{PG}(V)$ defined by the vectors $e_i - e_{i+1}$ ($i \in \{1,2,3,4\}$), $e_5 - e_1$, $e_i + e_6$ ($i \in \{1,2,3,4,5\}$) and $e_{j-2} - e_{j-1} + e_j - e_{j+1} + e_{j+2}$ ($j \in \mathbb{Z}$ mod 5) define an embedding as desired. We now consider these three 5-spaces as mutually disjoint generating subspaces of a 17-space $\mathbf{PG}(17,\mathbb{R})$ and we obtain a representation of $\widetilde{\mathbf{W}}(2)$ (which we shall also denote by $\widetilde{\mathbf{W}}(2)$) in $\mathbf{PG}(17,\mathbb{R})$ with 30 lines and 15 "non-lines". Consider a line L of $\mathcal{S}$. The three points of $\widetilde{\mathbf{W}}(2)$ on L define a plane $\pi(L)$ of $\mathbf{PG}(17,\mathbb{R})$ not containing any line of $\widetilde{\mathbf{W}}(2)$. Also, since $\mathcal{S}$ is a spread, no two such planes intersect. Now we set $\mathcal{S} = \{L_1, L_2, \ldots, L_{15}\}$. We consider the finite set of all real subspaces of $\mathbf{PG}(17,\mathbb{R})$ generated by elements of $\widetilde{\mathbf{W}}(2)$ which do not contain L_1. Clearly, we can find a point x_1 in $\pi(L_1)$ not contained in any of these subspaces. We project $\widetilde{\mathbf{W}}(2)$ from x_1 onto a hyperplane $\mathbf{PG}(16,\mathbb{R})$ of $\mathbf{PG}(17,\mathbb{R})$ not containing x_1 and we obtain a representation of $\widetilde{\mathbf{W}}(2)$ (which we will also denote by $\widetilde{\mathbf{W}}(2)$) in $\mathbf{PG}(16,\mathbb{R})$ with 31 lines and 14 "non-lines". We now consider the finite set of real subspaces of $\mathbf{PG}(16,\mathbb{R})$ generated by elements of $\widetilde{\mathbf{W}}(2)$ and not containing the projection of the plane $\pi(L_2)$. We can find a point x_2 of that projection not contained in any of these subspaces, and we project $\widetilde{\mathbf{W}}(2)$ from x_2 onto a hyperplane $\mathbf{PG}(15,\mathbb{R})$ of $\mathbf{PG}(16,\mathbb{R})$

not containing x_2. We continue this process until we obtain a representation of $\widetilde{\mathbf{W}}(2)$ in $\mathbf{PG}(2,\mathbb{R})$. Clearly, this representation is injective on the point set of $\widetilde{\mathbf{W}}(2)$ for if two points p, q of $\widetilde{\mathbf{W}}(2)$ were represented by the same point of $\mathbf{PG}(2,\mathbb{R})$, then some x_i ($i \in \{1,2,\ldots,15\}$) would lie on some line of $\mathbf{PG}(18-i,\mathbb{R})$ containing the two different points representing p, q. We claim that it is also injective on the line set, proving that we indeed have an embedding. Suppose, by way of contradiction, that the lines M, M' of $\widetilde{\mathbf{W}}(2)$ are represented by different lines R, R', respectively, in $\mathbf{PG}(n,\mathbb{R})$, but by the same line in $\mathbf{PG}(n-1,\mathbb{R})$. Put $i = 18-n$. Hence the plane $\langle R, R'\rangle$ of $\mathbf{PG}(n,\mathbb{R})$ contains the point x_i. By the choice of x_i, the three points of the line L_i are contained in the plane $\langle R, R'\rangle$. Denote by R_1, R'_1, L^1_i the lines of $\mathbf{PG}(n+1,\mathbb{R})$ representing M, M', L_i, respectively. By the choice of x_{i-1}, the three points of the line L_{i-1} are contained in the subspace $\langle R_1, R'_1, L^1_i\rangle$, which has at most dimension 3. Continuing like this, we eventually obtain a subspace of $\mathbf{PG}(17,\mathbb{R})$ of dimension at most $i+1$ containing all points of the lines $L_1, L_2, \ldots, L_i$. This easily implies that the points of $L_1, \ldots, L_i$ in some connected component $\mathcal{C}_\ell$ ($\ell \in \{1,2,3\}$) generate a subspace of dimension at most $(i-1)/3$. For $i = 2, 3$, this contradicts the fact that 2 points of $\widetilde{\mathbf{W}}(2)$ are represented by different points in $\mathbf{PG}(17,\mathbb{R})$. For $i = 4,5,6$, we have similarly that 4 points of $\mathcal{C}_\ell$ are never contained in a line of $\mathbf{PG}(5,\mathbb{R})$. For $i = 7,8,9$, we remark that no plane of $\mathbf{PG}(5,\mathbb{R})$ contains at least 7 points of $\mathcal{C}_\ell$. Similarly, no 3-space or 4-space of $\mathbf{PG}(5,\mathbb{R})$ contains at least 10 or 13 points, respectively, of $\mathcal{C}_\ell$, as is easily seen from the explicit construction of $\mathcal{C}_\ell$ above. This proves our claim, and consequently, we indeed have an embedding of $\widetilde{\mathbf{W}}(2)$ in $\mathbf{PG}(2,\mathbb{R})$. A completely similar argument now shows that this embedding is exclusive. Hence we have shown:

Proposition 4.4. *There exists an exclusive embedding of* $\widetilde{\mathbf{W}}(2)$ *in* $\mathbf{PG}(2,\mathbb{R})$.

4.5. The Biggs-Smith graph

The self-polar bislim point-line geometry **BS** arising from the Biggs-Smith graph on 102 vertices has not yet been considered in the literature. Yet it has some beautiful properties. We begin by presenting a new construction of the graph. Remark that the original construction used 17 copies of an "H"-graph and tells one very explicitly how to join vertices of each of these copies; the construction in [2] is an application of the general way of reconstructing a geometry from its flag transitive group of automorphisms and the stabilizers of a point and a line (where the graph is seen as a point-line geometry with two points on each line and three lines through each point). We now give a third construction which is more geometric in nature, and which resembles the second construction of $\mathbf{W}(2)$ above. It will turn out that this construction is responsible for **BS** sharing many properties with $\mathbf{W}(2)$.

Consider the projective line $\mathbf{PG}(1,17)$ over the field $\mathbf{GF}(17)$. Given a pair of points $\{a,b\}$ and a third point c of $\mathbf{PG}(1,17)$, it is easy to verify that there exist a unique point d and a unique pair of points $\{e,f\}$ of $\mathbf{PG}(1,17)$ such that $|\{a,b,c,d,e,f\}| = 6$ and such that we have the following equality of cross ratios: $(a,b;c,d) = (a,b;e,f) = (c,d;e,f) = -1$. We call the triple $\{\{a,b\},\{c,d\},\{e,f\}\}$ a *harmonic triplet*. The stabilizer inside $\mathbf{PGL}_2(17)$ of a harmonic triplet is easily seen to be a group of order 24 and it is entirely contained in $\mathbf{PSL}_2(17)$. This implies that there are in total $18\,\cdot$

$17 \cdot 16/24 = 204$ harmonic triplets and that $\mathbf{PSL}_2(17)$ defines two orbits of harmonic triplets. We fix one of these orbits (and we can choose coordinates in such a way that this orbit contains the harmonic triplet $\{\infty, 0\}, \{1, -1\}, \{4, -4\}\}$). It is the vertex set of the Biggs-Smith graph. Two vertices are adjacent if the harmonic triplets share a pair of points, e.g. $\{\{\infty, 0\}, \{1, -1\}, \{4, -4\}\}$ is adjacent to $\{\{\infty, 0\}, \{2, -2\}, \{8, -8\}\}$. It follows that the lines of **BS** can be identified with the partitions of the point set of $\mathbf{PG}(1, 17)$ into the points corresponding with three harmonic triplets of our chosen orbit (in fact, a tedious computation reveals that two points are collinear if and only if the corresponding harmonic triplets do not share any element of $\mathbf{PG}(1, 17)$). Consequently, the set of harmonic triplets in the orbit containing a fixed point of $\mathbf{PG}(1, 17)$ is an ovoid of **BS**, because it contains 34 pairwise non-collinear points of **BS**. So we obtain a set of 18 ovoids on which the group $\mathbf{PSL}_2(17)$ acts doubly transitively, which implies that any two such ovoids meet in a constant number of points. Counting the triples $(\mathcal{O}, \mathcal{O}', x)$, with $\mathcal{O}$ and $\mathcal{O}'$ two distinct ovoids of **BS** and $x \in \mathcal{O} \cap \mathcal{O}'$ in two ways, we obtain $|\mathcal{O} \cap \mathcal{O}'| = 102 \cdot 6 \cdot 5/18 \cdot 17 = 10$. We can now look at real barycentric embeddings. First note that **BS** is primitive and self-polar, and that $\mathbf{PSL}_2(17)$ is a flag transitive collineation group. According to the tables in [2], the adjacency matrix of the Biggs-Smith graph has eigenvalue 0 with multiplicity 17, hence the real embedding rank is equal to 17. An explicit construction of the universal barycentric embedding goes as follows. Let the entries of the coordinates of points of $\mathbf{PG}(17, \mathbb{R})$ be indexed by the above set of ovoids of **BS** (or, alternatively, by the points of $\mathbf{PG}(1, 17)$, since these sets are in natural bijective correspondence with each other). Define for each point x of **BS** a coordinate tuple using the rule: an entry corresponding with an ovoid containing x is equal to -2; otherwise it is equal to 1. As above with $\mathbf{W}(2)$, this gives a barycentric embedding of **BS** in the hyperplane of $\mathbf{PG}(17, \mathbb{R})$ with equation $\sum X_i = 0$, where the sum is taken over the previous set of ovoids.

So, with the previous examples in mind, we expect that the generating rank and the universal embedding rank of **BS** will be equal to 17. But this is not true. An explicit calculation shows that the universal embedding rank is equal to 19. It is not known what the generating rank of **BS** is, but, again by some explicit calculation, it is one of 19 or 20 (see below for more explanation).

Hence we have here an example of a slim geometry with nonzero universal embedding rank and nonzero real embedding rank and for which these two ranks are not the same. This might be explained by the fact that the complement in **BS** of an above ovoid is not connected, while this is true for ovoids in $\mathbf{W}(2)$. To illustrate this a bit better, we now give a description of the universal embedding of **BS** (the proof is an uninterested tedious calculation exercise). Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{18}$ be the set of ovoids of **BS** obtained as above. First we note that the graph obtained from the point set of **BS** by removing all points of an $\mathcal{O}_i$, $i \in \{1, 2, \ldots, 18\}$, and by proclaiming two vertices adjacent whenever they represent collinear points of **BS**, has two connected components (each isomorphic to a graph $\{17\} + \{17/4\}$, as the reader can check for his own). We denote the point sets of these components by $\mathcal{P}_{2i-1}$ and $\mathcal{P}_{2i}$. We now let each point x of **BS** correspond to the point of $\mathbf{PG}(35, 2)$ with coordinates $(r_1, r_2, \ldots, r_{36})$, where $r_j = 1$ if x belongs to $\mathcal{P}_j$, and $r_j = 0$ otherwise (in other words, r_j is the characteristic function of $\mathcal{P}_j$), $j \in \{1, 2, \ldots, 36\}$. It can be shown that these points generate an 18-dimensional subspace of $\mathbf{PG}(35, 2)$. Since this construction does not work over the real numbers, we conclude that the fact of **BS** having two connected components when an ovoid is removed is responsible for the higher universal embedding rank. A similar phenomenon happens with $\widetilde{\mathbf{W}}(2)$. In this case, the construction using only the ovoids (and not the connected components of the complements) produces an embedding of $\mathbf{W}(2)$ in $\mathbf{PG}(4,\mathbb{K})$, for any field $\mathbb{K}$; while the construction with the connected components of the complements of the ovoids produce the universal embedding of $\widetilde{\mathbf{W}}(2)$ in $\mathbf{PG}(10,2)$. But this construction does not work over the real numbers, and so the fact that an ovoid in $\widetilde{\mathbf{W}}(2)$ has disconnected complement is responsible for the (big) difference in universal and real embedding rank.

Finally, we give some explicit information about our construction of the Biggs-Smith graph (showing that our definition indeed produces this graph!). We start with listing all vertices explicitly. The following are all harmonic triplets of one orbit under $\mathbf{PSL}_2(17)$:

$$\mathcal{T}_1 := \{\{\{\infty,x\},\{x+1,x-1\},\{x+4,x-4\}\}\,|\,x\in\mathbf{GF}(17)\},$$
$$\mathcal{T}_2 := \{\{\{\infty,x\},\{x+2,x-2\},\{x+8,x-8\}\}\,|\,x\in\mathbf{GF}(17)\},$$
$$\mathcal{T}_3 := \{\{\{x-4,x+4\},\{x-5,x+7\},\{x-7,x+5\}\}\,|\,x\in\mathbf{GF}(17)\},$$
$$\mathcal{T}_4 := \{\{\{x-2,x+2\},\{x-5,x+6\},\{x-6,x+5\}\}\,|\,x\in\mathbf{GF}(17)\},$$
$$\mathcal{T}_5 := \{\{\{x-8,x+8\},\{x-3,x+7\},\{x-7,x+3\}\}\,|\,x\in\mathbf{GF}(17)\},$$
$$\mathcal{T}_6 := \{\{\{x-1,x+1\},\{x-6,x-3\},\{x+6,x+3\}\}\,|\,x\in\mathbf{GF}(17)\}.$$

The set $\mathcal{T}_1\cup\mathcal{T}_2$ is an ovoid; one component of its complement in **BS** has vertex set $\mathcal{T}_3\cup\mathcal{T}_6$. The subgraphs induced by $\mathcal{T}_3$ and $\mathcal{T}_6$ are ordinary 17-gons (and one can now easily see that the component is isomorphic to $\{17\}+\{17/4\}$). Also, one calculates that $\mathcal{B}:=\mathcal{T}_3\cup\{\{\{-2,2\},\{-5,6\},\{-6,5\}\},\{\{0,4\},\{-3,8\},\{-4,7\}\}\}$ defines a basis for the universal embedding. Finally, $\mathcal{B}\cup\{\{\{-1,3\},\{-4,7\},\{-5,6\}\}\}$ is a generating set for **BS** and so the generating rank is either 19 or 20, as claimed above.

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