Moufang Polygons and Irreducible Spherical BN-pairs of Rank 2

Katrin Tent* Hendrik Van Maldeghem†

Abstract

Let G be a group with an irreducible spherical BN-pair of rank 2 satisfying the additional condition $(*)$: There exists a normal nilpotent subgroup U of B with $B = TU$, where $T = B \cap N$ and $|W| \neq 16$ for the Weyl group $W = N/B \cap N$. We show that G corresponds to a Moufang polygon and hence is essentially known.

1 Introduction

A celebrated result of Fong and Seitz [3] states that every *finite* group with an irreducible BN-pair of rank 2 satisfying $(*)$ is (an extension of) a rank 2 adjoint Chevalley group. A geometric consequence of this result is a classification of all finite Moufang generalized polygons. Recently, all Moufang generalized polygons have been classified independently from the Fong and Seitz result, and not restricted to the finite case, by Tits and Weiss (manuscript in preparation [14]), showing in particular that also in the infinite case Moufang generalized polygons give rise to irreducible BN-pairs of rank 2 satisfying $(*)$.

So the question arises naturally whether an analogue of the result by Fong and Seitz holds without the finiteness assumption, thus showing that the notions of a so-called *split* BN-pair of rank 2 and a Moufang generalized polygon are essentially equivalent. This is the aim of the present paper. The proof in [3] heavily relies on two facts:

- It was proved by Feit and Higman [2] that for a finite group with an irreducible BN-pair of rank 2, the Weyl group $W = N/B \cap N$ has order $2n$ with $n = 3, 4, 6$ or 8. As examples by Tits [12] and Tent [7] show, for infinite groups all orders $2n$ are possible for the corresponding Weyl group.
- Finite 2-transitive permutation groups satisfying a condition corresponding to $(*)$ have been classified by Hering, Kantor and Seitz [4],[5], and there is no analogue known in the infinite case.

*The first author is supported by the Bayerischer Habilitationsförderpreis

†The second author is a Research Director of the Fund for Scientific Research – Flanders (Belgium)

Thus the general situation certainly requires new tools. First steps in the direction of the theorem cited in the abstract were achieved in [10] where it was shown that for a BN-pair of rank 2 acting faithfully on the associated polygon and satisfying the somewhat weaker condition $B = UT$ with $T = B \cap N$ and $Z(U) \neq 1$, we do get restrictions on the order of the Weyl group similar to the Feit and Higman restrictions. Assuming $(*)$ we there obtained the result stated in the abstract if the order of the Weyl group is 6.

It is worthwhile to note that our arguments are geometric in nature and that they are completely elementary. The style of the arguments resembles that used by Weiss [17] to show the non-existence of Moufang 12-gons. In fact, we were very much inspired by that paper.

The result of Fong and Seitz was used in the first generation proof of the classification of finite simple groups; it is not clear to us whether it will play the same important role in the revision of that proof. However, the proof we present here together with those parts of the proof of the classification of Moufang polygons by Tits and Weiss that apply to the finite situation yield a completely independent and quite elementary proof of their result, apart from the case $n = 8$.

Let us end this introduction with the statement of our main result.

Theorem. *If G is a group with an irreducible* BN*-pair satisfying*

(*) there exists a normal nilpotent subgroup U of B such that $B = UT$, for $T = B \cap N$,

and with Weyl group W of order $2n \neq 16$, then the associated generalized n-gon Γ is a Moufang n-gon and G/R contains its little projective group, where R denotes the kernel of the action of G on Γ.

Note that using the classification of the Moufang polygons by Tits and Weiss [14], this implies that essentially G/R is (an extension of) a simple algebraic group of relative rank 2, a classical group of rank 2, or a group of mixed type B_2 or G_2.

We now give precise definitions and notation.

2 Notation and preliminary results

A *thick generalized polygon* Γ (or *thick generalized n-gon*, $n \geq 3$), or briefly a *polygon* (or *n-gon*), is a bipartite graph (the two corresponding classes are called *types*) of diameter n and girth $2n$ (the *girth* of a graph is the length of a minimal circuit) containing a proper circuit of length $2n+2$ (the latter is equivalent with saying that all vertices have valency > 2, see [15]). The vertices are called the *elements* of Γ. The distance between elements is the usual graph theoretic distance. A pair of elements $\{x, y\}$ is called a *flag* if x and y are adjacent. The set of neighbors of an element x is denoted by $\Gamma_1(x)$, and, more generally, the set of elements at distance i from x, $0 \leq i \leq n$, is denoted by $\Gamma_i(x)$. The diameter of the edge graph of Γ is also equal to n and two flags at distance n from each other are called *opposite*. Also two elements of Γ at distance n from each other are called *opposite*.

The set of elements of a circuit of length $2n$ in Γ is called an *apartment*. Two opposite flags are contained in exactly one apartment. These, and many more properties, can be found in [15]. A sequence $(x_0, x_1, \ldots, x_k)$ of elements of Γ is called a *simple path of length* k, or a (simple) k-*path*, if x_{i-1} is adjacent to x_i, for all $i \in \{1, 2, \ldots, k\}$, and if $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$.

Generalized polygons were introduced by Tits [11]. The standard examples arise from groups with an irreducible spherical BN-pair of rank 2. For the purpose of the present paper, the following geometric definition of such a BN-pair will do.

Let Γ be an n-gon, and let G be a group acting (not necessarily effectively) on Γ such that each element of G acts as a type preserving graph automorphism. If G acts transitively on the set of apartments of Γ, and if the stabilizer in G of an apartment A acts in the natural way as the dihedral group of order $2n$ on A, then we say that G is a group with an *irreducible spherical* BN-*pair of rank* 2, or briefly, with a BN-*pair*. If we fix an apartment A and a flag f contained in A, then we call the stabilizer B in G of f a Borel subgroup of G. Also, there exists a subgroup N of G stabilizing A such that $B \cap N$ is normal in N and the corresponding quotient W has order $2n$ and is isomorphic to a dihedral group. The group W is called the *Weyl group* of G. The group N is not unique; in particular one can take the full stabilizer of A in G. Groups with a BN-pair were introduced by Tits; see e.g. [13].

Let Γ be an n-gon, and let x_0 be an element of Γ. Let G be a type preserving automorphism group of Γ, and let i be some natural number, $0 \leq i \leq n$. We denote by $G_{x_0}^{[i]}$ the subgroup of G fixing all elements of $\Gamma_i(x_0)$ (and then it automatically fixes all sets $\Gamma_j(x_0)$ pointwise, for $0 \leq j \leq i$). Further, for elements $x_1, \ldots, x_k$, we set $G_{x_0,x_1,\ldots,x_k}^{[i]} = G_{x_0}^{[i]} \cap G_{x_1}^{[i]} \cap \ldots \cap G_{x_k}^{[i]}$. For $i = 0$, we sometimes omit the superscript [0]. An *elation* g of Γ is a member of $G_{x_1,\ldots,x_{n-1}}^{[1]}$ for some simple path $(x_1, x_2, \ldots, x_{n-1})$ of Γ, in which case g is also called an $(x_1, x_2, \ldots, x_{n-1})$-elation. The group $G_{x_1,\ldots,x_{n-1}}^{[1]}$ of elations acts freely on $\Gamma_1(x_0) \setminus \{x_1\}$, for every element $x_0 \in \Gamma_1(x_1) \setminus \{x_2\}$. If this action is transitive for all such x_0, then we say that the path $(x_1, x_2, \ldots, x_{n-1})$ is a *Moufang path*. If all simple $(n-2)$-paths are Moufang, then we say that Γ is a *Moufang polygon*. If n is even, and if all simple paths of length $n-2$ starting with an element of fixed type are Moufang, then we say that Γ is *half Moufang*. All Moufang polygons are classified by Tits and Weiss [14]. An elation is called *central* if it fixes $\Gamma_i(x)$ pointwise, for some element x, and for all positive $i \leq n/2$ (in which case x is called a *center* of the elation). The *little projective group* of a Moufang polygon is the group generated by all elations. It is a group with a natural BN-pair and it always contains central elations. For the notions introduced in this paragraph, see [14] and [15].

Granted the classification of finite simple groups, all finite groups with an irreducible spherical BN-pair of rank 2 can be classified, see [1]. The finiteness condition cannot be dispensed with as is shown by the 'free' and 'universal' examples of Tits [12] and Tent [7]. Hence, one must have additional hypotheses in order to classify. Hypothesis $(*)$ is one such and proved very useful in the classification of finite simple groups.

Let us remark that the motivation of our work is not only to generalize the result of Fong and Seitz to the infinite case, but also to provide an independent geometric approach.

Also, our proof for the case $n = 12$ yields an alternative proof for the nonexistence of Moufang 12-gons.

Standing Hypotheses. Throughout, let G be a group with an irreducible spherical BN-pair of rank 2 and let Γ be the associated n-gon. Let A be some apartment in Γ and let $(x_0, x_1, \ldots, x_{2n-1}, x_{2n})$, with $x_0 = x_{2n}$, be a (closed) $2n$-path containing all elements of A. Let B be the stabilizer of $\{x_0, x_1\}$, and let $N \leq G$ be such that it stabilizes A and such that $T := B \cap N \trianglelefteq N$ with $W := N/T$ isomorphic to the dihedral group of order $2n$. Finally, let R be the kernel of the action of G on Γ. Then G/R is a group with a BN-pair and with corresponding polygon Γ. The stabilizer of $\{x_0, x_1\}$ in G/R is B/R. The group N/R stabilizes A and $T/R = B/R \cap N/R \trianglelefteq N/R$, with $W \equiv (N/R)/(T/R)$. If G satisfies (*), then so does G/R. Hence, in order to show our theorem, we may assume that R is trivial and hence that G acts faithfully on Γ. We assume throughout that U is a normal nilpotent subgroup of B satisfying $B = UT$.

We will use the following results from [10] on groups satisfying our standing hypotheses.

2.1 Lemma (Cp. 2.1 of [10]) *The group U acts transitively on the set of flags opposite $\{x_0, x_1\}$.* $\square$

2.2 Lemma (Cp. Lemma 2.3 of [10]) *We have $Z(U) \leq G^{[k]}_{x_0,x_1}$, for all $k < n/2$.* $\square$

2.3 Proposition (Cp. Theorem 1 and Prop. 3.1 of [10]) *The Weyl group W has order $2n$ with $n = 3, 4, 6, 8$ or 12. Also, if $n \in \{8, 12\}$, then, up to duality (i.e., up to interchanging x_0 and x_1), $Z(U) \leq G^{[n/2]}_{x_0}$. Moreover, we have the following two facts.*

(i) If $n = 6$, then either $Z(U)$ consists of central elations (all having the same center), or, for any two elements x, y of Γ at distance 2 from each other, the group $G^{[2]}_{x,y}$ is nontrivial.

(ii) If $n = 12$, then $[G^{[6]}_{x_0}, G^{[6]}_{x_8}] \leq G^{[4]}_{x_2,x_6}$.

$\square$

2.4 Proposition (Cp. Prop. 4.1 of [10]) *Up to duality, one has that for all $x \in \Gamma_1(x_1) \setminus \{x_0\}$, and for all k, $0 < k < n/2$, the group $U^{[1]}_{x_0,x_1} \leq U$ acts transitively on the set $\Gamma_k(x) \cap \Gamma_{k+1}(x_1)$. Also, for all $y \in \Gamma_1(x_0) \setminus \{x_1\}$, and for all k, $0 < k < n/2$, the group $U^{[1]}_{x_0}$ acts transitively on $\Gamma_k(y) \cap D_{k+1}(x_0)$.* $\square$

Note that Proposition 2.4 implies our main result for the case $n = 3$ and $|W| = 6$. Thus, in light of Proposition 2.3, in order to proof our main result, we just have to deal with the cases $n = 4, 6$ and 12.

Notation. Regarding commutators and conjugation, we use the notation $[g, h] = g^{-1}h^{-1}gh$ and $g^h = h^{-1}gh$; then $g^{-h} = (g^{-1})^h = hgh^{-1}$. Also, automorphisms act on the right, so we use exponential notation.

3 Proof of the theorem for $n = 4$

3.1 Hypotheses. We use our standing hypotheses. Since G acts transitively on the set of apartments, we may apply results that we proved for the appartment A to any other apartment. Also, we may shift indices by 2 (modulo 8), or, for any $j \in \{0, \dots, 7\}$, replace the index i by $j - i$ (modulo 8) for all $i \in \{0, \dots, 7\}$. We will do so freely. By Proposition 2.4 we may assume that the indices in A are chosen in such a way that $G^{[1]}_{x_0,x_1}$ acts transitively on $\Gamma_1(x) \setminus \{x_1\}$ for any $x \in \Gamma_1(x_1) \setminus \{x_0\}$. Then $G^{[1]}_{x_0}$ acts transitively on $\Gamma_1(x') \setminus \{x_0\}$ for any $x' \in \Gamma_1(x_0) \setminus \{x_1\}$. In particular, it follows that $G^{[1]}_{x_0}$ acts transitively on $\Gamma_2(x_1) \cap D_3(x_0)$.

3.2 Proposition *G contains a nontrivial (x_7, x_0, x_1)-elation.*

Proof. First we prove that this is true if $Z(U)$ contains some nontrivial elation, in which case we clearly may assume that $Z(U)$ contains a nontrivial central elation α with center x_1. Let h be some nontrivial element of $G^{[1]}_{x_3,x_4}$ not fixing x_1 (this exists by 3.1 above). Then, again by 3.1, there is some $u \in G^{[1]}_{x_1}$ with $x_1^{hu} = x_3$. We now have

(1) both $[\alpha, h]$ and α^{hu} belong to $G^{[1]}_{x_2,x_3}$,

(2) $[\alpha, h]$ agrees with α^{hu} on $\Gamma_1(x_1)$, hence $[\alpha, h]^{-1}\alpha^{hu} \in G^{[1]}_{x_1}$,

(3) $[\alpha, h]$ acts nontrivially on $\Gamma_1(x_4)$ (indeed, α^{-1} maps x_5 onto some element of $\Gamma_2(x_7)$, while this cannot be the case for α^{-h}) and α^{hu} acts trivially on $\Gamma_1(x_4)$.

Now (1), (2) and (3) imply that $[\alpha, h]^{-1}\alpha^{hu}$ is a nontrivial (x_1, x_2, x_3)-elation. By a shift of the indices, we obtain a nontrivial (x_7, x_0, x_1)-elation.

So we may assume that $Z(U)$ does not contain elations. We remark that, in particular, this implies, together with Lemma 2.1, that $Z(U)$ does not fix any element of $\Gamma_1(x_2)\setminus\{x_1\}$.

Next we show that the proposition holds if there is a nontrivial (x_0, x_1, x_2)-elation α. Let v be any element of G fixing x_1 and mapping x_0 to x_2; we denote $U^* := U^v$. By our previous remark, there exists $u_1 \in Z(U^*)$ with $x'_7 := x_7^{u_1} \neq x_7$. By 3.1, there exists $u_2 \in G^{[1]}_{x'_7}$ with $x_1^{u_2} = x_7$. After adjusting A if necessary, we may also assume that $x_2^{u_2} = x_6$. Put $x'_6 := x_6^{u_1}$.

Now consider the product of three elations $\beta := \alpha\alpha^{-u_2}\alpha^{u_2u_1}$ and set $x'_0 = x_2^{\alpha^{-u_2}}$. An elementary verification shows that $\beta \in G^{[1]}_{x'_7,x_0,x_1}$. Moreover, if β were trivial, then u_1 would belong to $G^{[1]}_{x'_0,x_1,x_2}$, contradicting our hypothesis that $Z(U^*)$ does not contain any elation.

Thus we may now assume that $G^{[1]}_{x_0,x_1}$ does not contain any elation. Our aim is to obtain a contradiction. Our assumption in fact means that $G^{[1]}_{x_0,x_1}$ must act faithfully on both $\Gamma_1(x_7)$ and $\Gamma_1(x_2)$.

We claim that $G^{[1]}_{x_0,x_1} = Z(U)$. Let U^* be as above, and let α and β be two arbitrary nontrivial elements of $G^{[1]}_{x_7,x_0}$ and $Z(U^*) \leq G^{[1]}_{x_1,x_2}$ (see Lemma 2.2), respectively, with the only restriction that α does not fix x_2. Clearly, $[\alpha,\beta] \in G^{[1]}_{x_0,x_1}$. Our hypotheses 3.1 imply the existence of some $u \in G^{[1]}_{x_7}$ mapping x_7^β (which is different from x_7 since otherwise β would be an elation inside $Z(U^*)$) to x_1. It is easy to see that $[\alpha,\beta]$ and $\alpha^{\beta u}$ induce the same action on $\Gamma_1(x_7)$. But since they both belong to $G^{[1]}_{x_0,x_1}$, and since the latter acts faithfully on $\Gamma_1(x_7)$, we conclude that $[\alpha,\beta] = \alpha^{\beta v}$. On the other hand, our assumption that α does not fix x_2 permits to interchange the roles of α and β in the above argument, and so $[\alpha,\beta]$ is conjugate to both α and β. Hence $\alpha \in G^{[1]}_{x_7,x_0} \setminus G^{[0]}_{x_2}$ and $\beta \in Z(U^*)$ are mutually conjugate. Since $G^{[1]}_{x_7,x_0}$ acts transitively and faithfully on $\Gamma_1(x_1) \setminus \{x_0\}$, one easily sees that $G^{[1]}_{x_7,x_0}$ and $Z(U^*)$ are conjugate subgroups. We now conclude that $G^{[1]}_{x_0,x_1} = Z(U)$, showing our claim.

It now follows that $Z(U)$ acts transitively on $\Gamma_1(x_2)$ and thus every element in U fixing some element of $\Gamma_1(x_2)\setminus\{x_1\}$ must fix every element of $\Gamma_1(x_2)$. In particular, the stabilizer of x_0 in the group $G^{[1]}_{x_2} \cap U$ acts transitively on $\Gamma_2(x_0) \cap \Gamma_4(x_2)$. Since 3.1 implies that $G^{[1]}_{x_2}$ also acts transitively on $\Gamma_1(x_1) \setminus \{x_2\}$, we conclude, together with the dual arguments, that for all elements x, the group $G^{[1]}_x$ acts transitively on $\Gamma_4(x)$. It now follows from Theorem 2.1 of [16] that the stabilizer of x_6 in $G^{[1]}_{x_0,x_1}$ acts transitively on $\Gamma_1(x_2) \setminus \{x_1\}$, contradicting our assumptions on $Z(U)$.

The proposition is proved. $\square$

3.3 Proposition Γ *is a half Moufang quadrangle. More exactly, the group* $G^{[1]}_{x_7,x_0,x_1}$ *acts transitively on* $\Gamma_1(x_2) \setminus \{x_1\}$.

Proof. In the first part of the proof, we show that there are fixed elements $x_2' \in \Gamma_1(x_1) \setminus \{x_0\}$ and $x_7' \in \Gamma_1(x_0) \setminus \{x_1\}$ such that for each $h \in G^{[1]}_{x_0,x_1}$, there exists an elation α of $G^{[1]}_{x_0,x_1,x_2'}$ inducing the same action as h on $\Gamma_1(x_7')$.

We choose some nontrivial element β in $G^{[1]}_{x_5,x_6,x_7}$ (this exists by Proposition 3.2), we put $x_7' = x_1^{\beta^{-1}}$, and we let $u \in G^{[1]}_{x_1}$ be such that $x_0^{\beta u} = x_7$. Define $x_2' \in \Gamma_1(x_1)$ as $x_2'^{\beta u} = x_6$. It is easily seen that $[h,\beta]^{u^{-1}\beta^{-1}}$ and h induce the same action on $\Gamma_1(x_7')$. There are two possibilities

(1). If h fixes x_6, then $[h,\beta]$ is clearly a nontrivial (x_6,x_7,x_0)-elation, hence $[h,\beta]^{u^{-1}\beta^{-1}}$ is the desired element α.

(2). Suppose now that h does not fix x_6. By 3.1, there is some $v \in G^{[1]}_{x_6}$ mapping x_5^h onto x_1. Clearly, $[h,\beta]$ and β^{-hv} induce the same action on $\Gamma_1(x_6)$, hence $[h,\beta]\beta^{hv} \in G^{[1]}_{x_6}$. Since β^{-hv} belongs to $G^{[1]}_{x_1}$, it immediately follows that $\gamma = [h,\beta]\beta^{hv}$ is an (x_6,x_7,x_0)-elation inducing the same action on $\Gamma_1(x_1)$ as $[h,\beta]$, and hence also as $h^{\beta u}$. Consequently the conjugate $\alpha = \gamma^{u^{-1}\beta^{-1}}$ is the required elation, concluding the first part of the proof.

In order to complete the proof of the proposition, it suffices by 3.1 to show that for every element $h \in G^{[1]}_{x_0,x_1}$, there exists an (x_7',x_0,x_1)-elation θ inducing the same action on $\Gamma_1(x_2)$. So let again $h \in G^{[1]}_{x_0,x_1}$ be arbitrary. Let α be the (x_0,x_1,x_2)-elation inducing the same action on x'_7 as h and guaranteed by the first part of the proof. Then $h\alpha^{-1}$ is the desired (x'_7, x_0, x_1)-elation θ and the proof of the proposition is complete. $\square$

We can now finish the proof of our theorem in the case $n = 4$. By Proposition 3.6 in [9], we just have to show that the action of the root group $G^{[1]}_{x_7,x_0,x_1}$ on $\Gamma_1(x_2)$ does not depend on x_7 and x_0. But this follows from the fact that, given any simple 4-path (x'_7, x'_0, x_1, x_2), using 3.1 and the Proposition 3.3 there exists $v \in G^{[1]}_{x_2}$ with $v(x_7) = x'_7$. Conjugation of $G^{[1]}_{x'_7,x_0,x_1}$ with v gives us $G^{[1]}_{x'_1,x'_0,x_1}$ fixing the action on $\Gamma_1(x_2)$, and the theorem is proved.

4 Proof of the theorem for $n = 6$

The cases $n = 6$ and 12 are similar to their counterparts in [8]. The proof has two main steps: first we show that there are central elations (Proposition 4.6 below); secondly we show that the group of central elations with given center acts transitively on the appropriate set. We then use Corollary 3.8 of [6] to conclude the proof.

So our first aim is to show that there are central elations.

From now on until Proposition 4.6 we assume, besides our standard hypotheses, that $Z(U)$ contains elements that are not central elations.

Our first lemma slightly improves Proposition 2.3(i).

4.1 Lemma *The groups $G^{[2]}_{x_0,x_1,x_2}$ and $G^{[2]}_{x_1,x_2,x_3}$ are nontrivial.*

Proof. We show that $G^{[2]}_{x_1,x_2,x_3}$ is nontrivial. The non-triviality of $G^{[2]}_{x_0,x_1,x_2}$ is proved completely similarly.

Choose any $v \in G$ mapping the flag $\{x_0, x_1\}$ to the flag $\{x_3, x_4\}$, and denote $U^* := U^v$. Note that U^* only depends on the flag $\{x_3, x_4\}$, in particular, by Proposition 2.4 we can choose v in $G^{[1]}_x$, for any $x \in \Gamma_1(x_2) \setminus \{x_1, x_3\}$. Consequently we see that $Z(U)$ and $Z(U^*)$ induce the same abelian group action on $\Gamma_2(x_2) \cap \Gamma_3(x_1) \cap \Gamma_3(x_3)$. Hence, if $\alpha \in Z(U)$ and $\beta \in Z(U^*)$, and α and β are not elations, one easily verifies that $[\alpha, \beta]$ belongs to $G^{[2]}_{x_1,x_2,x_3}$. If $[\alpha, \beta]$ were trivial, then α would fix $\Gamma_2(x_0^{\beta^{-1}})$, and hence, using Lemma 2.1, α would be central with center x_1, a contradiction. The lemma is proved. $\square$

To simplify notation, we will denote by $U_{x,y}$ the conjugate U^v of U, for any $v \in G$ mapping the flag $\{x_0, x_1\}$ to the flag $\{x, y\}$.

4.2 Lemma *For every $\alpha \in Z(U)$, there exists some $g \in G^{[2]}_{x_0,x_1,x_2}$ inducing the same action on $\Gamma_1(x_{10})$.*

Proof. Choose $\alpha \in Z(U)$ and $\beta \in Z(U_{x_2,x_3})$ not central elations, and let u be in $G^{[1]}_{x_{10}}$ mapping x_1 onto x_{11}^{β} (note that $x_{11} = x_{11}^{\beta}$ would imply that β is a central elation with center x_2, a contradiction); u exists by Proposition 2.4. The commutator $g = [\alpha^{u\beta^{-1}}, \beta]$ is a nontrivial element of $G^{[2]}_{x_0,x_1,x_2}$ (by the proof of the previous lemma), and one easily verifies that $g = \alpha^{-u\beta^{-1}}\alpha^{u}$. Hence, since α^{-1} fixes $\Gamma_1(x_{10}^{\beta u^{-1}})$, we see that g and α induce the same action on $\Gamma_1(x_{10})$. $\square$

4.3 Hypothesis. By Proposition 2.4, we may assume that the group $G^{[2]}_{x_0,x_1}$ acts transitively on $\Gamma_2(x_2) \cap \Gamma_3(x_1)$.

4.4 Lemma *All nontrivial $(x_1, x_2, x_3, x_4, x_5)$-elations inside $G^{[2]}_{x_2}$ are conjugate. In particular, every such elation belongs to $G^{[2]}_{x_2,x_3,x_4}$.*

Proof. Let $\alpha \in G^{[1]}_{x_1,x_2,x_3,x_4,x_5} \cap G^{[2]}_{x_2}$ and $\beta \in G^{[1]}_{x_9,x_{10},x_{11},x_0,x_1} \cap G^{[2]}_{x_0}$ both be nontrivial. Let $u \in G^{[1]}_{x_4}$ be such that $x_4^{\beta u} = x_0$. Then $\alpha^{\beta u}$ belongs to $G^{[1]}_{x_5^{\beta u},x_0,x_1,x_2,x_3}$. But clearly $[\alpha, \beta] \in G^{[2]}_{x_0,x_2} \leq G^{[1]}_{x_5^{\beta u},x_0,x_1,x_2,x_3}$ and the action of $[\alpha, \beta]$ and $\alpha^{\beta u}$ on $\Gamma_1(x_4)$ are the same. Hence $[\alpha, \beta] = \alpha^{\beta u}$. But similarly, $[\beta, \alpha] = [\alpha, \beta]^{-1}$ is a conjugate of β. Hence α and β^{-1} are mutually conjugate and the first part of the lemma is proved.

The second statement follows from Lemma 4.1. $\square$

Up to now, we constructed elations in G. It will be important to know that these belong to appropriate conjugates of U.

4.5 Lemma *If G does not contain any central elation, then we have $G^{[2]}_{x_0,x_1,x_2} \leq U$.*

Proof. Choose an arbitrary element $v \in Z(U)$ which is not a central elation. Let $\alpha \in G^{[2]}_{x_{11},x_0,x_1}$ be the elation guaranteed by Lemma 4.2 which induces the same (nontrivial) action on $\Gamma_1(x_3)$. Since α cannot be central by assumption, we may pick $y \in \Gamma_2(x_0) \cap \Gamma_3(x_{11}) \cap \Gamma_3(x_1)$ such that α induces on $\Gamma_1(y)$ a nontrivial action. By Lemma 2.1, there exists some $u \in U$ fixing x_4 and mapping y to x_{10}. Since α in B and $U \trianglelefteq B$, the commutator $g = [\alpha, u]$ is inside U. Since the action of α on $\Gamma_1(x_3)$ agrees with an element of $Z(U)$, it is easy to see that g is an $(x_{11}, x_0, x_1, x_2, x_3)$-elation inside $G^{[2]}_{x_0,x_1}$. The choice of y guarantees that g is nontrivial. By an even shift of indices in Lemma 4.4, the result follows. $\square$

4.6 Proposition *The group G contains central elations.*

Proof. Suppose by way of contradiction that G does not contain central elations. Let $Z(U) = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq \cdots \trianglelefteq Z_k = U$ be the ascending central series for U. By Lemma 4.5, we may choose the integer i minimal with respect to the property that $Z_i \cap G^{[2]}_{x_0,x_1,x_2}$ is nontrivial. Note that this implies that $G^{[2]}_{x_0,x_1,x_2} \leq Z_i$ since by Lemma 4.4 all elations in

$G^{[2]}_{x_0,x_1,x_2}$ are conjugate under B and $Z_i \trianglelefteq B$. Note also that by assumption $i \neq 0$ since otherwise the elements of $G^{[2]}_{x_0,x_1,x_2}$ are central elations inside $Z(U)$.

Let β be any nontrivial element of $Z(U)$, and let $\alpha \in Z_i$ be the unique element of $G^{[2]}_{x_0,x_1,x_2}$, guaranteed to exist by Lemma 4.2, with the property that α and β induce the same action on $\Gamma_1(x_{10})$. Then $g = \alpha\beta^{-1} \in Z_i$ is an $(x_{10}, x_{11}, x_0, x_1, x_2)$-elation belonging to $G^{[2]}_{x_0,x_1}$, and inducing on $\Gamma_1(x_3)$ the same action as $\beta^{-1} \in Z(U)$. Since by assumption g cannot be a central elation, there is some $y \in \Gamma_2(x_0)$ such that g acts nontrivially on $\Gamma_1(y)$. Let $u \in U$ fix x_4 and map x_{10} to y. Since g agrees on $\Gamma_1(x_3)$ with an element of $Z(U)$, the commutator $[g, u] \in Z_{i-1}$ is a nontrivial $(x_{11}, x_0, x_1, x_2, x_3)$-elation belonging to $G^{[2]}_{x_0,x_1}$, and hence also to $G^{[2]}_{x_0,x_1,x_2}$. But this contradicts the minimality of i. The proposition is proved. $\square$

4.7 Lemma *All central elations in G with center of given type are conjugate. Also, if g and h are nontrivial central elations with centers x_0 and x_2, respectively, then there is some central elation α with center x_4 such that $[\alpha, g] = h$.*

Proof. The proof is almost identical to the proof of Lemma 4.4. Let there first be given two nontrivial central elations g and β with centers x_0 and x_4, respectively. Let $u \in G^{[1]}_{x_5}$ be such that $x_4^{gu} = x_2$ (u exists by Proposition 2.4). As in the proof of Lemma 4.4, we now conclude that $[\beta, g] = \beta^{gu}$. Similarly, $[g, \beta] = [\beta, g]^{-1}$ is a conjugate of g, hence g and β^{-1} are conjugated and the first part of the lemma follows easily. Now let h be an arbitrary nontrivial central elation with center x_2. Then we define $\alpha = h^{u^{-1}h^{-1}}$ and from the previous arguments follows $[\alpha, g] = h$. $\square$

In fact, we only need the second part of the previous lemma to finish the proof of our theorem in the case $n = 6$.

4.8 Proposition *Up to duality, the group $G^{[3]}_{x_1}$ acts transitively on $\Gamma_1(x_{10}) \setminus \{x_{11}\}$.*

Proof. By Proposition 4.6, we may assume that there is a nontrivial element α in $G^{[3]}_{x_9}$. Let x_9' be arbitrary in $\Gamma_1(x_{10}) \setminus \{x_9, x_{11}\}$. We show that there exists $\beta \in G^{[3]}_{x_1}$ mapping x_9 onto x_9'.

By Proposition 2.4 there are automorphisms $u \in G^{[1]}_{x_0}$ and $v \in G^{[1]}_{x_8}$ such that $x_9^u = x_9'$ and ${x_9'}^v = x_{11}$. Clearly α^{uv} is a central elation with center x_{11} and hence, by Lemma 4.7, there exists $\beta \in G^{[3]}_{x_1}$ such that $[\alpha, \beta] = \alpha^{uv}$. Also, clearly $[\alpha, u]$ belongs to $G^{[1]}_{x_9,x_{10},x_{11},x_0} \cap G^{[2]}_{x_{10}}$ and acts nontrivially on $\Gamma_1(x_8)$. Put $h := [\alpha, \beta][\alpha, u]^{-1}$. One easily verifies that h is an $(x_8, x_9, x_{10}, x_{11}, x_0)$-elation. We show that h also belongs to $G^{[1]}_{x_8^\beta}$. If h is the identity, then this is trivial. If h is not the identity, then the commutator $[h, \beta^{-1}]$ is a central elation with center x_1 fixing x_9, hence it is the identity. This implies that $\beta h \beta^{-1}$ acts as the identity on $\Gamma_1(x_8)$, implying that h acts as the identity on $\Gamma_1(x_8^\beta)$.

Since both $[\alpha, \beta]$ and $[\alpha, u]$ fix x_8^β, this now implies that the actions of $[\alpha, \beta]$ and $[\alpha, u]$ agree on $\Gamma_1(x_8^\beta)$. Since also α fixes x_8^β, the actions of α^β and α^u agree on $\Gamma_1(x_8^\beta)$. But the former acts as the identity on $\Gamma_1(x_8^\beta)$, hence α must fix every element of $\Gamma_1(x_8^{\beta u^{-1}})$. So $x_8^{\beta u^{-1}}$ must be adjacent to x_9. Consequently $x_9^\beta = x_9^u = x_9'$. The proposition is proved. $\square$

Now the theorem for $n = 6$ follows from Corollary 3.8 of [6].

5 Proof of the theorem for $n = 12$

Next we show that the case $n = 12$ cannot occur: The proof is very similar to the case $n = 6$. We show that the group of central elations would have to be transitive on the appropriate set, contradicting Proposition 5.1 in [10].

We use our standing hypotheses, and first prove an analogue of Lemma 4.7.

5.1 Lemma *All central elations in G with center of given type are conjugate. Also, if g and h are nontrivial central elations with centers x_0 and x_4, respectively, then there is some central elation α with center x_8 such that $[\alpha, g] = h$.*

Proof. The proof is identical to the proof of Lemma 4.7, if one simply doubles every index in that proof. $\square$

We now show an analogue of Proposition 4.8. The proof runs similarly, but we need a small additional argument at one place, hence we sketch the full proof. The idea is again to double most indices.

5.2 Proposition *Up to duality, the group $G_{x_2}^{[6]}$ acts transitively on $\Gamma_1(x_{20}) \setminus \{x_{21}\}$.*

Proof. By Proposition 2.3, we may assume that there is a nontrivial element α in $G_{x_{18}}^{[6]}$. Let x_{19}' be arbitrary in $\Gamma_1(x_{20}) \setminus \{x_{19}, x_{21}\}$. We show that there exists $\beta \in G_{x_2}^{[6]}$ mapping x_{19} onto x_{19}'.

By Proposition 2.4 there are automorphisms $u \in G_{x_0}^{[1]}$ and $v \in G_{x_{16}}^{[1]}$ such that $x_{19}^u = x_{19}'$ and $x_{18}^{uv} = x_{22}$. Clearly α^{uv} is a central elation with center x_{22} and hence, by Lemma 5.1, there exists $\beta \in G_{x_2}^{[6]}$ such that $[\alpha, \beta] = \alpha^{uv}$. Also, clearly $g := [\alpha, u]$ belongs to $G_{x_{20}}^{[4]} \cap G_{x_0}^{[1]}$ and acts nontrivially on $\Gamma_1(x_{16})$. We claim that $g \in G_{x_0}^{[2]}$.

Without loss of generality, we may, by way of contradiction, assume that $x_2^g \neq x_2$. Clearly the commutator $[g, \beta]$ belongs to $G_{x_1}^{[5]} \cap G_{x_{20}}^{[4]}$, hence it must be the identity, contradicting the fact that it acts nontrivially on $\Gamma_1 x_6$ (because β fixes $\Gamma_1(x_6)$ pointwise, while β^g acts nontrivially on that set). The claim is proved.

Now put $h := [\alpha, \beta][\alpha, u]^{-1}$. One easily verifies that h belongs to $G_{x_{16}, x_{17}, \ldots, x_{23}, x_0, x_1}^{[1]}$. We show that h also belongs to $G_{x_{16}^\beta}^{[1]}$. If h is the identity, then this is trivial. If h is not the identity, then the commutator $[h, \beta^{-1}]$ is a central elation with center x_2 fixing x_{19}, hence it is the identity. This implies that $\beta h \beta^{-1}$ acts as the identity on $\Gamma_1(x_{16})$, implying that h acts as the identity on $\Gamma_1(x_{16}^{\beta})$.

As in the last part of the proof of Proposition 4.8, one now easily shows that α fixes every element of $x_{16}^{\beta u^{-1}}$. So the latter must be at distance at most 4 from x_{18}. But this is only possible if $x_{19}^{\beta} = x_{19}^{u} = x'_{19}$. The proposition is proved. $\square$

Now the theorem for $n = 12$ follows from Proposition 5.1. in [10].

References

[1] F. Buekenhout and H. Van Maldeghem, Finite distance transitive generalized polygons, *Geom. Dedicata* **52** (1994), 41 – 51.

[2] W. Feit and G. Higman, The nonexistence of certain generalized polygons *J. Algebra* **1** (1964), 114 – 131.

[3] P. Fong and G. Seitz, Groups with a $(B,\, N)$-pair of rank 2. I, II. *Invent. Math.* **21** (1973), 1 – 57; ibid. **24** (1974), 191 – 239.

[4] Chr. Hering, W.M. Kantor and G.M. Seitz, Finite groups with a split BN-pair of rank 1. I, J. Algebra 20 (1972), 435–475.

[5] W.M. Kantor and G.M. Seitz, Finite groups with a split BN-pair of rank 1. II, J. Algebra 20 (1972), 476–494.

[6] M. A. Ronan, A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227 – 262.

[7] K. Tent, Very homogeneous generalized n-gons of finite Morley Rank, to appear in *J. London Math. Soc.*

[8] K. Tent, Split BN-pairs of finite Morley rank, submitted.

[9] K. Tent and H. Van Maldeghem, The Cherlin-Zil'ber Conjecture for BN-pairs with panels of Morley rank 1, submitted.

[10] K. Tent and H. Van Maldeghem, On irreducible split (B,N)-pairs of rank 2, to appear in *Forum Math.*

[11] J. Tits, Sur la trialité et certain groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13 – 60.

[12] J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.* **43** (1977), 283 – 295.

[13] J. Tits, *Buildings of spherical type and finite BN-pairs*, Springer Verlag, *Lecture Notes in Math.* **386**, Berlin, Heidelberg, New York, 1974.

[14] J. Tits and R. Weiss, *The classification of Moufang Polygons*, in preparation.

[15] H. Van Maldeghem, *Generalized polygons*, Birkhäuser Verlag, *Monographs in Mathematics* **93**, Basel, Boston, Berlin, 1998.

[16] H. Van Maldeghem, Some consequences of a result of Brouwer, *Ars. Combin.* **48** (1998), 185 – 190.

[17] R. Weiss, The nonexistence of certain Moufang polygons, *Invent. Math.* **51** (1979), 261 – 266.

Katrin Tent
Mathematisches Institut
Universität Würzburg,
Am Hubland,
D–97074 Würzburg,
Germany

E-mail: `tent@@mathematik.uni-wuerzburg.de`

Hendrik Van Maldeghem
Pure Mathematics and Computer Algebra
Universiteit Gent
Galglaan 2,
B–9000 Gent,
Belgium

E-mail: `hvm@@cage.rug.ac.be`