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Characterizations of quadric and Hermitian Veroneseans over finite fields

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Abstract. In Hirschfeld and Thas [5] the most important characterizations of quadric Veroneseans are surveyed. However a few difficult cases were still open, in particular the even case. In [10, 11] Thas and Van Maldeghem not only solve all open cases, but they also generalize most of these characterizations in several ways: they do not restrict themselves to the quadric Veronesean of the plane $\mathrm{PG}(2, q)$, they allow ovals instead of conics, and they also characterize projections of quadric Veroneseans. Further, Cooperstein, Thas and Van Maldeghem [1] contains some properties of Hermitian Veroneseans over finite fields and also these varieties and some of their projections are characterized. All these results on Veroneseans will be surveyed here.

1. Introduction to quadric Veroneseans

1.1. *Quadric Veroneseans*

A good reference is Chapter 25 of Hirschfeld and Thas [5]. The *Veronese variety* of all quadrics of $\mathrm{PG}(n, K)$, $n \geq 1$, and K any commutative field, is the variety

$$\mathcal{V} = \{(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, x_0x_2, \ldots, x_0x_n, x_1x_2, \ldots, x_1x_n, \ldots, x_{n-1}x_n) \parallel (x_0, x_1, \ldots, x_n) \text{ is a point of } \mathrm{PG}\,(n, K)\}$$

of $\mathrm{PG}(N_n, K)$ with $N_n = n(n+3)/2$. Clearly $\mathcal{V}$ is a variety of dimension n. It is also called the *Veronesean* of quadrics of $\mathrm{PG}\,(n, K)$, or simply the *quadric Veronesean* of $\mathrm{PG}\,(n, K)$. It can be shown that the quadric Veronesean is absolutely irreducible and non-singular. Let $\mathrm{PG}(N_n, K)$ consist of all points

$$(y_{00}, y_{11}, \ldots, y_{nn}, y_{01}, y_{02}, \ldots, y_{0n}, y_{12}, \ldots, y_{1n}, \ldots, y_{n-1,n}).$$

For y_{ij} we also write y_{ji}. Then $\mathcal{V}$ belongs to the intersection of the quadrics $F_{ij} = 0$ and $F_{abc} = 0$, with $i \neq j$ and $i, j \in \{0, 1, \ldots, n\}$, $a \neq b \neq c \neq a$ and $a, b, c \in \{0, 1, \ldots, n\}$, where

$$F_{ij} = Y_{ij}^2 - Y_{ii}Y_{jj},\ F_{abc} = Y_{aa}Y_{bc} - Y_{ab}Y_{ac}.$$

THEOREM 1.1. *The quadric Veronesean* $\mathcal{V}$ *of* $\mathrm{PG}(n, K)$ *consists of all points* $(y_{00}, y_{11}, \ldots, y_{n-1,n})$ *of* $\mathrm{PG}(N_n, K)$ *for which rank* $[y_{ij}] = 1$. *Let* $\zeta : \mathrm{PG}\,(n, K) \to \mathrm{PG}\,(N_n, K)$, *with* $N_n = n(n+3)/2$ *and* $n \geq 1$, *be defined by* $(x_0, x_1, \ldots, x_n) \mapsto (y_{00}, y_{11}, \ldots, y_{n-1,n})$, *with* $y_{ij} = x_i x_j$. *It is an easy exercise to show that* ζ *is a bijection of* $\mathrm{PG}\,(n, K)$ *onto the quadric Veronesean* $\mathcal{V}$ *of* $\mathrm{PG}(n, K)$. *It then follows that the variety* $\mathcal{V}$ *is rational.*

THEOREM 1.2. *The quadrics of* PG(n, K) *are mapped by* ζ *onto all hyperplane sections of* $\mathcal{V}$.

Theorem 1.2 explains why $\mathcal{V}$ is called the Veronesean of quadrics of PG(n, K).

COROLLARY 1.3. *No hyperplane of* PG(N_n, K) *contains the quadric Veronesean* $\mathcal{V}$.

THEOREM 1.4. *The Veronese variety* $\mathcal{V}$ *of all quadrics of* PG(n, K), $n \geq 1$, *has order* 2^n.

From now on the quadric Veronesean of PG(n, K) will be denoted by $\mathcal{V}_n^{2^n}$ or simply $\mathcal{V}_n$. For $n = 1$, the Veronesean $\mathcal{V}_1^2$ is a conic of PG$(2, K)$. For $n = 2$, the Veronesean is a surface $\mathcal{V}_2^4$ of order 4 in PG$(5, K)$. For $n = 3$, the Veronesean is a variety $\mathcal{V}_3^8$ of dimension 3 and order 8 of PG$(9, K)$.

THEOREM 1.5. *Let* Π_s *be any* s*-dimensional subspace of* PG(n, K). *Then* Π_s^ζ *is a quadric Veronesean* $\mathcal{V}_s$, *which is the complete intersection of* $\mathcal{V}_n$ *and the space* PG$(s(s+3)/2, K)$ *containing* $\mathcal{V}_s$. *Conversely, for* $K \neq$ GF(2), *any quadric Veronesean* $\mathcal{V}_s$ *contained in* $\mathcal{V}_n$ *is of the form* Π_s^ζ, *with* Π_s *some* s*-dimensional subspace of* PG(n, K).

COROLLARY 1.6. *For* $K \neq$ GF(2) *any two points of* $\mathcal{V}_n$ *are contained in a unique conic of* $\mathcal{V}_n$.

REMARK. As any three distinct points of the quadric Veronesean $\mathcal{V}_n$ of PG$(n, 2)$ form a conic, the second part of Theorem 1.5 and Corollary 1.6 do not hold for $K =$GF(2).

From now on it is assumed that $K =$ GF(q), although many of the results will also hold in the case of a general field.

THEOREM 1.7. *The quadric Veronesean* $\mathcal{V}_n$ *of* PG(n, q) *is a* $\theta_q(n)$*-cap of* PG(N_n, q), *with* $\theta_q(n) = (q^{n+1} - 1)/(q - 1)$.

1.2. *The quadric Veronesean* $\mathcal{V}_2^4$

Apart from the conic, the quadric Veronesean which is most studied and characterized is the surface $\mathcal{V}_2^4$ of PG$(5, q)$. Over $\mathbb{C}$ we have the following beautiful theorem due to Kronecker and Castelnuovo (see [8] p. 130). Any surface of PG$(m, \mathbb{C})$ which contains ∞^2 conics is the Veronesean $\mathcal{V}_2^4$ or one of its projections; any surface of PG$(3, \mathbb{C})$ having ∞^2 reducible plane sections is either the projection of a Veronesean $\mathcal{V}_2^4$ or a scroll. So let us consider the quadric Veronesean $\mathcal{V}_2^4$. By Theorem 1.5, for $K \neq$GF(2) the variety $\mathcal{V}_2^4$ contains q^2+q+1 conics, and, by Corollary 1.6, for $K \neq$GF(2) any two points of $\mathcal{V}_2^4$ are contained in a unique one of these conics. Since the conics of $\mathcal{V}_2^4$ correspond to the lines of PG$(2, q)$, $q \neq 2$, any two of these conics have a unique point in common. To the conics of PG$(2, q)$ correspond all hyperplane sections of $\mathcal{V}_2^4$. The hyperplane is uniquely determined by the conic if and only if the latter is not a single point. If the conic $\mathcal{C}$ of PG$(2, q)$ is a repeated line, then the corresponding hyperplane Π_4 of PG$(5, q)$ meets $\mathcal{V}_2^4$ in a conic; if $\mathcal{C}$ is two distinct lines, then Π_4 meets $\mathcal{V}_2^4$ in two conics with exactly one point in common; if $\mathcal{C}$ is a non-singular conic, then Π_4 meets $\mathcal{V}_2^4$ in a rational quartic curve. The planes of PG$(5, q)$ which meet $\mathcal{V}_2^4$ in a conic are called the *conic planes* of $\mathcal{V}_2^4$.

THEOREM 1.8. *Any two conic planes π and π' of $\mathcal{V}_2^4$ have exactly one point in common, and this common point belongs to $\mathcal{V}_2^4$. Any three conic planes of $\mathcal{V}_2^4$ generate* PG$(5, q)$.

The tangent lines of the conics of $\mathcal{V}_2^4$ are called the *tangents* or *tangent lines* of $\mathcal{V}_2^4$. Since no point of the surface $\mathcal{V}_2^4$ is singular, all tangent lines of $\mathcal{V}_2^4$ at the point p of $\mathcal{V}_2^4$ are contained in a plane $\pi(p)$. This plane $\pi(p)$ is called the *tangent plane* of $\mathcal{V}_2^4$ at p. Since p is contained in exactly $q + 1$ conics of $\mathcal{V}_2^4$ and since no two conic planes through p have a line in common, the tangent plane $\pi(p)$ is the union of the $q+1$ tangent lines of $\mathcal{V}_2^4$ through p. Also $\pi(p) \cap \mathcal{V}_2^4 = \{p\}$.

THEOREM 1.9. *For any two distinct points p_1 and p_2 of $\mathcal{V}_2^4$, the tangent planes $\pi(p_1)$ and $\pi(p_2)$ have exactly one point in common.*

From Theorem 1.8 easily follows the next result.

THEOREM 1.10. *Suppose that $\mathcal{C}$ is a conic on $\mathcal{V}_2^4$, that π is the plane of $\mathcal{C}$ and that $p \in \mathcal{V}_2^4 - \mathcal{C}$. Then $\pi(p) \cap \pi = \phi$.*

For q odd, conic planes and tangent planes play similar roles as shown by the following theorem.

THEOREM 1.11. *For q odd* PG$(5, q)$ *admits a polarity which maps the set of all conic planes of $\mathcal{V}_2^4$ onto the set of all tangent planes of $\mathcal{V}_2^4$. It follows that when q is odd, for any three points p_1, p_2, p_3 of $\mathcal{V}_2^4$, the intersection $\pi(p_1) \cap \pi(p_2) \cap \pi(p_3)$ of the tangent planes is empty.*

1.3. *The nucleus subspace of a quadric Veronesean*

A good reference for this section is Thas and Van Maldeghem [11]. Consider the quadric Veronesean $\mathcal{V}_n$ of PG(n, q), with q even. Let $\mathcal{G}_n$ be the set of PG(N_n, q) consisting of all nuclei of the conics on $\mathcal{V}_n$ which are images of lines of PG(n, q); for $q \neq 2$, these conics are all the conics on $\mathcal{V}_n$.

THEOREM 1.12. *The set $\mathcal{G}_n$ is the Grassmannian of the lines of* $\mathrm{PG}(n, q)$*, hence generates a subspace of dimension $(n-1)(n+2)/2$ of* $\mathrm{PG}(N_n, q)$*. This subspace Φ is called the nucleus subspace of $\mathcal{V}_n$.*

THEOREM 1.13. *The nucleus subspace Φ of $\mathcal{V}_n$ is the intersection of all hyperplanes of* $\mathrm{PG}(N_n, q)$ *which contain a unique quadric Veronesean $\mathcal{V}_{n-1} = \Pi_{n-1}^{\zeta}$, with Π_{n-1} any hyperplane of* $\mathrm{PG}(n, q)$.

1.4. *Characterizations of quadric Veroneseans*

In this paper four types of characterizations will be considered. Let X be a set of points in $\Pi = \mathrm{PG}(M, q)$, $M > 2$, spanning Π, and let $\mathcal{P}$ be a collection of planes of Π such that for any $\pi \in \mathcal{P}$, the intersection $X \cap \Pi$ is an oval in π. Then we determined under which conditions X is a quadric Veronesean $\mathcal{V}_n^{2^n}$ or one of its projections. Part of this problem was solved by Mazzocca and Melone [7], but only under the extra conditions that the intersections $X \cap \pi$ are non-singular conics, q is odd and $M = n(n+3)/2$. In Hirschfeld and Thas [5] the case q even and $M = n(n+3)/2$ is handled. In [10] we solved the problem in its most general setting. Let the quadric Veronesean $\mathcal{V}_n^{2^n}$ in $\mathrm{PG}(n(n+3)/2, q)$ be the image of the projective space $\mathrm{PG}(n, q)$. Then the image of an arbitrary hyperplane of $\mathrm{PG}(n, q)$ under the Veronesean map is a quadric Veronesean $\mathcal{V}_{n-1}^{2^{n-1}}$ and the subspace generated by it has dimension $(n-1)(n+2)/2$. Such a subspace will be called a $\mathcal{V}_{n-1}^{2^{n-1}}$-*subspace*, or, for short, a $\mathcal{V}_{n-1}$-*subspace*, of $\mathcal{V}_n^{2^n}$ or of $\mathrm{PG}(n(n+3)/2, q)$. In $\mathrm{PG}(n(n+3)/2, q)$ we now consider a set $\mathcal{S}$ of subspaces of dimension $(n-1)(n+2)/2$, and then the goal is to determine under which conditions $\mathcal{S}$ is the set of all $\mathcal{V}_{n-1}$-subspaces of a quadric Veronesean $\mathcal{V}_n$ in $\mathrm{PG}(n(n+3)/2, q)$. For $n = 2$, with q odd, the problem was solved by Tallini [9]. In [11] we solved the problem for any n, any q, and extended the problem to a larger class of objects. Here the 2-dimensional dual hyperovals defined by Huybrechts and Pasini [6] turn up. For q even, the nucleus subspace of $\mathcal{V}_n$ plays a crucial role. We emphasize that the even case is far the most difficult case. A third kind of characterization of $\mathcal{V}_2^4$ is by its number of common points with the planes and hyperplanes of $\mathrm{PG}(5, q)$. This was done by Ferri [4] for q odd and $q \neq 3$, and by Hirschfeld and Thas [5] for $q = 3$. In [11] Thas and Van Maldeghem also handle $q \in \{2, 4\}$, and then relying on the results of Ferri [4], Hirschfeld and Thas [5] and on the characterizations of the second type, Thas and Van Maldeghem obtain the characterization for all q. Finally, let X be a set of points in $\mathrm{PG}(m, q)$, $m \geq n(n+3)/2$, $n \geq 2$, $q > 2$, spanning $\mathrm{PG}(m, q)$, let $\mathcal{O}$ be a set of ovals on X, and assume that $(X, \mathcal{O})$ is the design of points and lines of a projective space. Then Thas and Van Maldeghem [11] show that X is a quadric Veronesean $\mathcal{V}_n^{2^n}$ in $\mathrm{PG}(m, q)$. In this way all open problems related to the existing characterizations of quadric Veroneseans have been solved, and moreover new characterizations, characterizations with weaker hypotheses, and characterizations of larger classes of objects were obtained.

2. Characterization of the first kind

Mazzocca and Melone [7] formulate three geometric properties ((Q1), (Q2) and (Q3) below, but they assume conics instead of ovals) that should characterize $\mathcal{V}_n$ (they call the objects satisfying these three axioms *Veronesean caps*), and they erroneously thought they indeed did; in fact they thought they proved that every Veronesean cap was a quadric Veronesean. Hirschfeld and Thas [5] pointed out some counterexamples and added a fourth axiom to make the characterization work; it should also be noted that Hirschfeld and Thas [5] modified the proof of Mazzocca and Melone [7] so as to hold also in the even case. That extra fourth axiom is just a bound on the dimension of the ambient projective space. In Thas and Van Maldeghem [10] this fourth condition is deleted again, one of the other conditions is weakened, and it is shown that the resulting geometric object is projectively equivalent either to a quadric Veronesean, or to a proper projection of some quadric Veronesean. This in particular solves the original problem of Mazzocca and Melone completely in the finite case. The proof of the characterization of the quadric Veronesean using the axioms (Q1), (Q2), (Q3) and the bound on the dimension was rather long, see Hirschfeld and Thas [5], and so Thas and Van Maldeghem [10] included a much shorter proof in their paper.

Consequently, they proved the entire classification of Veronesean caps independent of the existing literature.

THEOREM 2.1. *Let X be a set of points in $\Pi = \mathrm{PG}(M, q)$, $M > 2$, spanning Π, and let $\mathcal{P}$ be a collection of planes of Π such that for any $\pi \in \mathcal{P}$, the intersection $X \cap \pi$ is an oval in π. For $\pi \in \mathcal{P}$ and $x \in X \cap \pi$, we denote by $T_x(\pi)$ the tangent line to $X \cap \pi$ at x in π. We assume the following three properties.*

- (Q1) *Any two points $x, y \in X$ lie in a unique member of $\mathcal{P}$ which we denote by $[x, y]$;*
- (Q2) *if $\pi_1, \pi_2 \in \mathcal{P}$ and $\pi_1 \cap \pi_2 \neq \emptyset$, then $\pi_1 \cap \pi_2 \subseteq X$;*
- (Q3) *if $x \in X$ and $\pi \in \mathcal{P}$ with $x \notin \pi$, then each of the lines $T_x([x, y])$, $y \in X \cap \pi$, is contained in a common plane of Π, denoted by $T(x, \pi)$.*

Then there exists a natural number $n \geq 2$, a projective space $\Pi' = \mathrm{PG}(n(n+3)/2, q)$ containing Π, a subspace Ψ of Π' skew to Π, and a quadric Veronesean $\mathcal{V}_n$ in Π', with $\Psi \cap \mathcal{V}_n = \emptyset$, such that X is the (bijective) projection of $\mathcal{V}_n$ from Ψ onto Π. The subspace Ψ can be empty, in which case X is projectively equivalent to $\mathcal{V}_n$.

Also, note that the set of planes $\mathcal{P}$ is uniquely determined by X if $q > 2$. For $q = 2$ there are counterexamples.

3. Characterization of the second kind

Let the quadric Veronesean $\mathcal{V}_n^{2^n}$ in $\mathrm{PG}(N_n, q)$, with $N_n = n(n+3)/2$, be the image of the projective space $\mathrm{PG}(n, q)$. Then the image of an arbitrary hyperplane of $\mathrm{PG}(n, q)$

under the Veronesean map ζ is a quadric Veronesean $\mathcal{V}_{n-1}^{2^{n-1}}$ and the subspace generated by it has dimension $N_{n-1} = (n-1)(n+2)/2$. Such a subspace is called a $\mathcal{V}_{n-1}^{2^{n-1}}$-subspace, or, for short, a $\mathcal{V}_{n-1}$-subspace, of $\mathcal{V}_n^{2^n}$ or of $\mathrm{PG}(N_n, q)$. In $\mathrm{PG}(N_n, q)$ we consider a set $\mathcal{S}$ of subspaces of dimension N_{n-1}, and then the goal is to determine under which conditions $\mathcal{S}$ is the set of all $\mathcal{V}_{n-1}$-subspaces of a quadric Veronesean $\mathcal{V}_n$ in $\mathrm{PG}(N_n, q)$. For $n = 2$, with q odd, the problem was solved by Tallini [9]. In [11] Thas and Van Maldeghem solved the problem for any n and any q; in their paper they rely on the characterization of the first kind. Let $\mathcal{S}_n$ be the set of all $\mathcal{V}_{n-1}$-subspaces of the quadric Veronesean $\mathcal{V}_n$ in $\mathrm{PG}(N_n, q)$. We note the following properties of $\mathcal{S}_n$, which can easily be verified; see e.g. Hirschfeld and Thas [5].

(VS1) Every two members of $\mathcal{S}_n$ generate a hyperplane of $\mathrm{PG}(N_n, q)$.
(VS2) Every three members of $\mathcal{S}_n$ generate $\mathrm{PG}(N_n, q)$.
(VS3) No point is contained in every member of $\mathcal{S}_n$.
(VS4) The intersection of any nonempty collection of members of $\mathcal{S}_n$ is a subspace of dimension $N_i = i(i+3)/2$ for some $i \in \{-1, 0, 1, \ldots, n-1\}$.
(VS5) There exist three members S, S', S'' of $\mathcal{S}_n$ with $S \cap S' = S' \cap S'' = S'' \cap S$.

By Tallini [9], for $n = 2$, with q odd, the properties (VS1), (VS2) and (VS3) characterize the set of $\mathcal{V}_1$-subspaces, that is, the set of conic planes, of $\mathcal{V}_2^4$. In Thas and Van Maldeghem [11] the following generalization is obtained.

THEOREM 3.1. *Let $\mathcal{S}$ be a collection of $q^n + q^{n-1} + \cdots + q + 1$ subspaces of dimension $\frac{(n-1)(n+2)}{2}$ of the projective space $\mathrm{PG}(\frac{n(n+3)}{2}, q)$, with $n \geq 2$, satisfying* (VS1) *up to* (VS5). *Then either $\mathcal{S}$ is the set of $\mathcal{V}_{n-1}$-subspaces of a quadric Veronesean $\mathcal{V}_n^{2^n}$ in $\mathrm{PG}(\frac{n(n+3)}{2}, q)$, or q is even, there are two members $S_1, S_2 \in \mathcal{S}$ with the property that no other member of $\mathcal{S}$ contains $S_1 \cap S_2$, and there is a unique subspace S of dimension $\frac{(n-1)(n+2)}{2}$ such that $\mathcal{S} \cup \{S\}$ is the set of $\mathcal{V}_{n-1}$-subspaces together with the nucleus subspace of a quadric Veronesean $\mathcal{V}_n^{2^n}$ in $\mathrm{PG}(\frac{n(n+3)}{2}, q)$. In particular, if $n = 2$, then the statement holds under the weaker hypothesis of $\mathcal{S}$ satisfying* (VS1), (VS2), (VS3) *and* (VS5). *In both cases, but with $(q, n) \neq (2, 2)$ in the latter case, $\mathcal{V}_n^{2^n}$ is the set of points of $\mathrm{PG}(\frac{n(n+3)}{2}, q)$ contained in at least $q^{n-1} + q^{n-2} + \cdots + q$ members of $\mathcal{S}$; in the exceptional case there are 13 points contained in at least 2 members of $\mathcal{S}$, where 6 are coplanar while the others form $\mathcal{V}_2^4$. For q large enough we can reduce this set of axioms.*

THEOREM 3.2. *Let $\mathcal{S}$ be a collection of $q^n + q^{n-1} + \cdots + q + 1$ subspaces of dimension $\frac{(n-1)(n+2)}{2}$ of the projective space $\mathrm{PG}(\frac{n(n+3)}{2}, q)$, with $n \geq 2$, satisfying* (VS1) *up to* (VS3). *If $q \geq n$, then $\mathcal{S}$ also satisfies* (VS4). *We can also say something more in the case where $\mathcal{S}$ does not satisfy* (VS5).

THEOREM 3.3. *Let $\mathcal{S}$ be a collection of $q^n + q^{n-1} + \cdots + q + 1$ subspaces of dimension $\frac{(n-1)(n+2)}{2}$ of the projective space* $\mathrm{PG}(\frac{n(n+3)}{2}, q)$*, with $n \geq 2$, satisfying* (VS1) *up to* (VS4) *and not satisfying* (VS5)*, or satisfying* (VS1) *up to* (VS3)*, with $q \geq n$, but not satisfying* (VS5)*. Then q is even and there exists a unique subspace S of dimension $\frac{(n-1)(n+2)}{2}$ such that $\mathcal{S} \cup \{S\}$ also satisfies* (VS1) *up to* (VS4)*, and not* (VS5)*. Moreover, if $n = 2$, then $q = 2$ or $q = 4$ and $\mathcal{S}$ is uniquely determined in both cases, up to isomorphism.*

Hence, for q odd we have a most satisfying characterization, since in this case axioms (VS1) up to (VS4) really characterize the collection of $\mathcal{V}_{n-1}$-subspaces of a quadric Veronesean $\mathcal{V}_n^{2^n}$, and for $q \geq n$, axioms (VS1) up to (VS3) do this job. For q even we additionally need (VS5), although for $n = 2$ one can classify all examples that do not satisfy (VS5). This classification remains open for $n \geq 3$. In fact, if $\mathcal{S}$ does not satisfy (VS5), then Theorem 3.3 implies that it is contained in the dual of an n-dimensional dual hyperoval, as defined by Huybrechts and Pasini [6]. Also, for $n = 2$, every three distinct elements of the 2-dimensional dual hyperoval we have here, generate $\mathrm{PG}(5, q)$. These objects are classified by Del Fra [3] and only two examples turn up, respectively for $q = 4$ (related to the simple Mathieu group M_{22}) and for $q = 2$. The example for $q = 2$ can be generalized to general n as follows. Let $\mathrm{AG}(n, 2)$ be an affine space in $\mathrm{PG}(n, 2)$. Consider in the Grassmannian of the lines of $\mathrm{PG}(n, 2)$ all subspaces corresponding to the full line pencils of lines with vertex in $\mathrm{AG}(n, 2)$. Then one verifies that this gives the dual of a collection of $q^n + q^{n-1} + \cdots + q + 2$ subspaces of dimension $(n-1)(n+2)/2$ of $\mathrm{PG}(n(n+3)/2, q)$ satisfying (VS1) up to (VS4) and not satisfying (VS5). There are two corollaries of Theorems 3.2 and 3.3.

COROLLARY 3.4. *If $\mathcal{S}^*$ is a set of $q^n + q^{n-1} + \cdots + q + 2$ subspaces of dimension $\frac{(n-1)(n+2)}{2}$ of* $\mathrm{PG}(n(n+3)/2, q)$ *such that* (VS1), (VS2), (VS3) *and* (VS5) *hold for $\mathcal{S}^*$ and either also* (VS4) *holds, or $q \geq n$, then q is even and $\mathcal{S}^*$ is the set of all $\mathcal{V}_{n-1}$-subspaces together with the nucleus subspace of a quadric Veronesean $\mathcal{V}_n^{2^n}$ in* $\mathrm{PG}(N_n, q)$*. Also, $\mathcal{V}_n^{2^n}$ is the set of points of* $\mathrm{PG}(\frac{n(n+3)}{2}, q)$ *contained in $q^{n-1} + q^{n-2} + \cdots + q + 1$ members of $\mathcal{S}$.*

COROLLARY 3.5. *Let $\mathcal{S}$ be a set of $k \geq q^n + q^{n-1} + \cdots + q + 1$ subspaces of dimension $m - n - 1$ of* $\mathrm{PG}(m, q)$*, with $m \geq n(n+3)/2$ and such that $q \geq n$. Suppose every pair of elements of $\mathcal{S}$ is contained in some hyperplane of* $\mathrm{PG}(m, q)$*, no three elements of $\mathcal{S}$ are contained in a hyperplane of* $\mathrm{PG}(m, q)$*, no point is contained in all members of $\mathcal{S}$ and there exist three members S, S', S'' of $\mathcal{S}$ with $S \cap S' = S' \cap S'' = S'' \cap S$. Then $m = n(n+3)/2$ and either $k = q^n + q^{n-1} + \cdots + q + 1$ and $\mathcal{S}$ is the set of $\mathcal{V}_{n-1}$-subspaces of a quadric Veronesean $\mathcal{V}_n^{2^n}$, or q is even, $k \in \{q^n + q^{n-1} + \cdots + q + 1, q^n + q^{n-1} + \ldots + q + 2\}$ and $\mathcal{S}$ consists of k members of the set of $\mathcal{V}_{n-1}$-subspaces together with the nucleus subspace of a quadric Veronesean $\mathcal{V}_n^{2^n}$. In both cases, but with $(q, n) \neq (2, 2)$ if $\mathcal{S}$ contains the nucleus subspace of $\mathcal{V}_n^{2^n}$, $\mathcal{V}_n^{2^n}$ is the set of points of* $\mathrm{PG}(m, q)$ *contained in at least $q^{n-1} + q^{n-2}$*

$+\cdots+q$ *members of* $\mathcal{S}$*; in the exceptional case there are* 13 *points contained in at least* 2 *members of* $\mathcal{S}$*, where* 6 *are coplanar while the other* 7 *form* $\mathcal{V}_2^4$.

4. Characterization of the third kind

As an application of Section 3 one obtains a characterization of $\mathcal{V}_2^4$ by its number of common points with the planes and hyperplanes of $\mathrm{PG}(5,q)$. Recall from Hirschfeld and Thas [5] that the quadric Veronesean $\mathcal{V}_2^4$ is a cap $\mathcal{K}$ in $\mathrm{PG}(5,q)$ satisfying the following two properties.

(VC1) For every hyperplane Π of $\mathrm{PG}(5,q)$, we have $|\Pi\cap\mathcal{K}|=1, q+1$ or $2q+1$, and there exists some hyperplane Π such that $|\Pi\cap\mathcal{K}|=2q+1$.

(VC2) Any plane of $\mathrm{PG}(5,q)$ with four points in $\mathcal{K}$ has at least $q+1$ points in $\mathcal{K}$.

It is also proved in Hirschfeld and Thas [5] that these two properties characterize $\mathcal{V}_2^4$ for all odd q; Ferri [4] had proved this for odd $q\neq 3$. In [11] Thas and Van Maldeghem copy the proof in [5], except for $q\in\{2,4\}$, for which they produce a separate argument, and rely on Section 3, in order to obtain the following general characterization.

THEOREM 4.1. *Let* $\mathcal{K}$ *be a set of points of* $\mathrm{PG}(5,q)$*,* $q>2$*, satisfying* (VC1) *and* (VC2)*. Then* $\mathcal{K}$ *is projectively equivalent with the quadric Veronesean* $\mathcal{V}_2^4$ *in* $\mathrm{PG}(5,q)$*. For* $q=2$*, a set of points in* $\mathrm{PG}(5,2)$ *satisfying* (VC1) *and* (VC2) *is either a quadric Veronesean or an elliptic quadric in some subspace* $\mathrm{PG}(3,2)$.

5. Characterization of the fourth kind

As a second application of Section 3 Thas and Van Maldeghem [11] obtained a characterization of the quadric Veronesean $\mathcal{V}_n^{2^n}$ in terms of designs. This goes as follows.

THEOREM 5.1. *Let* X *be a set of points in* $\mathrm{PG}(m,q)$*,* $m\geq n(n+3)/2$*,* $n\geq 2$*,* $q>2$*, spanning* $\mathrm{PG}(m,q)$*, let* $\mathcal{O}$ *be a set of ovals on* X*, and assume that* $(X,\mathcal{O})$ *is the design of points and lines of a projective space of dimension* n*. Then* $m=n(n+3)/2$*,* X *is the quadric Veronesean* $\mathcal{V}_n^{2^n}$*, and* $\mathcal{O}$ *is the set of all conics on* $\mathcal{V}_n^{2^n}$.

6. Introduction to Hermitian Veroneseans over finite fields

This section is taken from Cooperstein, Thas and Van Maldeghem [1]; see also Cossidente and Siciliano [2] for the case $n=2$. We do not claim that we are the first to study Hermitian Veroneseans nor that (part of) Section 6.2 cannot be found elsewhere. For $n=2$, some of the properties in 6.2 are e.g. also contained in [2]. Sections 6.1 and 6.2 are just considered as an introduction to the quite deep, and difficult to prove, characterization in Section 7.

6.1. *Hermitian Veroneseans*

Set $k = \mathrm{GF}(q)$ and $K = \mathrm{GF}(q^2)$. Let γ be the generator of the Galois group of K/k so that for $a \in K$ we have $a^\gamma = a^q$. For convenience we will often denote the image of a under γ by $\bar{a}$. Recall that an $(n+1) \times (n+1)$ matrix m over K is *Hermitian* if $m^T = \bar{m}$ where T denotes the transpose map and by $\bar{m}$ we mean the result of applying γ to each of the elements of m. We shall denote the space of all $(n+1) \times (n+1)$ Hermitian matrices over K by $\mathcal{H}(n+1, q^2)$. This is a k-linear space of dimension $(n+1)^2$. The group $G = \mathrm{GL}(n+1, q^2)$ acts on $\mathcal{H}(n+1, q^2)$ with the action given by $m^g = gm\bar{g}^T$. For $1 \le i \le n+1$ let $\mathcal{H}_i(n+1, q^2)$ be the collection of matrices in $\mathcal{H}(n+1, q^2)$ with rank i and $P\mathcal{H}_i(n+1, q^2)$, or simply $P\mathcal{H}_i$, the set of 1-spaces spanned by the matrices in $\mathcal{H}_i(n+1, q^2)$. Then each $P\mathcal{H}_i$ is an orbit for G under the induced action on $\mathrm{PG}(\mathcal{H}(n+1, q^2))$ considered as a projective space of dimension $n^2 + 2n$ over k. We note that $P\mathcal{H}_1$ is canonically in one-to-one correspondence with the projective space $\mathrm{PG}(n, K)$ which can be seen as follows. Let $V = K^{n+1}$ consist of column vectors. For $\langle v \rangle \in \mathrm{PG}(V)$ set $\langle v \rangle^\pi = \langle v\bar{v}^T \rangle$. Clearly $v\bar{v}^T$ is a rank one matrix in $\mathcal{H}(n+1, K)$ and so $\langle v \rangle^\pi$ is in $P\mathcal{H}_1$. The linear group G preserves this action:

$$\langle gv \rangle^\pi = \langle (gv)(\bar{g}\bar{v})^T \rangle = \langle g(v\bar{v}^T)\bar{g}^T \rangle = (\langle v \rangle^\pi)^g.$$

Since G is transitive on $P\mathcal{H}_1$ it follows that $\mathrm{PG}(V)^\pi = P\mathcal{H}_1$. In fact, π is one-to-one from $\mathrm{PG}(V)$ onto $P\mathcal{H}_1$. Next note that $P\mathcal{H}_1$ is a *cap* in $P\mathcal{H} = \mathrm{PG}(\mathcal{H}(n+1, q^2))$, that is, no three points are collinear. We now present an alternative explicit construction of $P\mathcal{H}_1$ in $\mathrm{PG}(n^2 + 2n, k)$. In fact, this amounts to choose an explicit k-base in $\mathcal{H}(n+1, q^2)$, and then apply the previous construction, in particular, the map π. Let $r \in K \setminus k$ be arbitrary. Then the map π above can be given as (where $x_i \in K$, for all $i \in \{0, 1, \ldots, n\}$)

$$\langle (x_0, x_1, \ldots, x_n) \rangle^\pi = \langle (y_{i,j})_{0 \le i,j \le n} \rangle,$$

with $y_{i,i} = x_i\bar{x}_i$, $y_{i,j} = x_i\bar{x}_j + \bar{x}_i x_j$ for $i < j$, and $y_{i,j} = r x_i \bar{x}_j + \bar{r}\bar{x}_i x_j$ for $i > j$. From this representation, it is clear that the inverse image with respect to π of the intersection of $P\mathcal{H}_1$ with a hyperplane of $\mathrm{PG}(n^2 + 2n, k)$ is a (not necessarily non-singular) Hermitian variety, and conversely every Hermitian variety of $\mathrm{PG}(V)$ arises in this way. It follows that $P\mathcal{H}_1$ is not contained in a hyperplane of $\mathrm{PG}(n^2 + 2n, k)$. We refer to this representation as the *r-representation*, $r \in K \setminus k$. We point out that the lines of $\mathrm{PG}(V)$ have a natural interpretation in terms of the geometry of $P\mathcal{H}$: the span in $P\mathcal{H}$ of the image L^π, L a line of $\mathrm{PG}(V)$, is a 3-dimensional space and we shall denote by $\xi(L)$ the subspace $\langle L^\pi \rangle$ of $P\mathcal{H}$. Since L^π is a cap of size $q^2 + 1$ in the 3-dimensional projective space $\xi(L)$, it is an ovoid for $q > 2$, and it is easy to see that it is always an elliptic quadric, and that $\xi(L) \cap P\mathcal{H}_1 = L^\pi$. Thus the lines of $\mathrm{PG}(V)$ can be interpreted as certain 3-dimensional projective subspaces of $P\mathcal{H}$ in which the points of $P\mathcal{H}_1$ form an elliptic quadric. We will denote by $\sum$ the collection of all such subspaces. Further, for a point $p \in P\mathcal{H}_1$ and $\xi \in \sum$

with $p \in \xi$, we shall denote by $T_p(\xi)$ the tangent plane to $\xi \cap P\mathcal{H}_1$ at p in ξ (it is the union of all lines through p in ξ which intersect $P\mathcal{H}_1$ in precisely p). We now record some properties of $\mathcal{H}_{n,n^2+2n} = P\mathcal{H}_1$ which we shall refer to as a *Hermitian Veronesean of index* n (respectively, *of* PG(V)), for obvious reasons.

6.2. *Properties of the Hermitian Veronesean of index n*

Now we will list some important properties of the Hermitian Veroneseans.

PROPERTY 6.1. Let $\mathcal{H}_{n,n^2+2n}$ be a Hermitian Veronesean of index n in PG(n^2+2n, q). Then each elliptic quadric in some PG$(3, q) \subseteq$ PG(n^2+2n, q) contained in $\mathcal{H}_{n,n^2+2n}$ corresponds to a line of PG(V). Also, every n-dimensional subspace over GF(q) of PG(V) corresponds to a quadric Veronesean $\mathcal{V}_n$ over GF(q) on $\mathcal{H}_{n,n^2+2n}$ and we have $\langle\mathcal{V}_n\rangle \cap \mathcal{H}_{n,n^2+2n} = \mathcal{V}_n$. A 3-dimensional subspace generated by an elliptic quadric on $\mathcal{H}_{n,n^2+2n}$ will be called an elliptic space of $P\mathcal{H}_{n,n^2+2n}$. By the foregoing property, every elliptic space corresponds to a line of PG(V) and vice versa.

PROPERTY 6.2. Let $\mathcal{H}_{n,n^2+2n}$ be a Hermitian Veronesean of index n in PG(n^2+2n, q).

(i) Any two points p_1, p_2 of $\mathcal{H}_{n,n^2+2n}$ lie in a unique member of $\sum$ which we will denote by $\xi[p_1, p_2]$.
(ii) Two subspaces in $\sum$ are either disjoint or else meet in a (unique) point of $\mathcal{H}_{n,n^2+2n}$.
(iii) Assume $\xi \in \sum$, $p \in \mathcal{H}_{n,n^2+2n}$, $p \notin \xi$ and put $E = \xi \cap \mathcal{H}_{n,n^2+2n}$. Then $\cup_{p' \in E} T_p(\xi[p, p'])$ is a projective subspace of dimension four.

7. A characterization of Hermitian Veroneseans

Let $\Pi \cong$ PG(N, q) be a projective space. Let X be a subset of the point set of Π which spans Π and for which there exists a collection $\sum$ of 3-dimensional (projective) subspaces of Π, called the *elliptic spaces* of Π, such that for any $\xi \in \sum$, the set $X(\xi) = X \cap \xi$ is an ovoid (not necessarily an elliptic quadric) in ξ. When $\xi \in \sum$, $x \in X(\xi)$ we will denote by $T_x(\xi)$ the tangent plane to x in ξ relative to the ovoid $X(\xi)$. We say that X is a *Hermitian set* if the following properties are satisfied:

(H1) Any two points $x, y \in X$ lie in a unique member of $\sum$ which we denote by $[x, y]$;
(H2) If $\xi_1, \xi_2 \in \sum$ and $\xi_1 \cap \xi_2 \neq \emptyset$, then $\xi_1 \cap \xi_2 \subseteq X$.
(H3) If $x \in X$ and $\xi \in \sum$, with $x \notin \xi$, then each of the planes $T_x([x, y])$, with $y \in X(\xi)$ is contained in a common 4-subspace of Π which we denote by $T(x, \xi)$.

In Cooperstein, Thas and Van Maldeghem [1] it is shown that a Hermitian set is a cap and consequently in what follows a Hermitian set is referred to as a *Hermitian cap*. The reader should compare this with the definition of a Veronesean cap in Section 2. Clearly, a Hermitian Veronesean is a Hermitian cap. But one also obtains a Hermitian cap X from a Hermitian Veronesean $P\mathcal{H}_1 \subseteq P\mathcal{H}$ by setting $\Pi = P\mathcal{H}/\Phi$, where Φ is a subspace of $P\mathcal{H}$ which does not intersect any elliptic space, nor any 4-space $T(x, \xi)$ (with $x \in P\mathcal{H}_1$ and ξ an elliptic space not containing x) and letting X be the image of $P\mathcal{H}_1$. Such a Hermitian cap will be called a *quotient* of the Hermitian Veronesean $P\mathcal{H}_1$.

THEOREM 7.1. *Let X be a Hermitian cap in the projective space Π. If $\sum$ is the corresponding set of elliptic spaces, then the incidence structure (X, χ), with $\chi = \{X(\xi) \| \xi \in \sum\}$, is the point-line structure of a projective space over the field $\mathrm{GF}(q^2)$ and we refer to the dimension of this projective space as the index of the cap. If X has index r, then X is projectively equivalent to a quotient of the Hermitian Veronesean of index r.*

In order to obtain this result, one proves some particular cases and lemmas, some of which could be of independent interest. In particular we mention the following result.

THEOREM 7.2. *Let X be a Hermitian cap in the projective space $\Pi = \mathrm{PG}(N, q)$ and assume that $\sum$ is the corresponding set of elliptic spaces, where $|\sum| > 1$. Denote $\chi = \{X(\xi) \| \xi \in \sum\}$. Then the following hold*

(i) *If the index of X is r, then $N \leq (r+1)^2 - 1$.*
(ii) *If $N = (r+1)^2 - 1$, with r the index of X, then X is projectively equivalent to the Hermitian Veronesean of index r in $\mathrm{PG}(r^2 + 2r, q)$.*
(iii) *If the index r is either 2 or 3, then X is projectively equivalent to the Hermitian Veronesean of index r.*
(iv) *If X is a Hermitian cap of index r and every hyperplane Y of the r-dimensional projective space (X, χ) has the property that $X \cap \langle Y \rangle = Y$, then X is projectively equivalent to the Hermitian Veronesean of index r.*

Note that Theorem 7.1 is similar to Theorem 2.1 on Veronesean caps. Also, note that the set of elliptic spaces of a Hermitian cap X in $\mathrm{PG}(N, q)$ is uniquely determined if $q > 2$. This follows immediately from (H2) by considering two coplanar bisecants, with no common point on X, of a hypothetical ovoid contained in X and not lying in an elliptic space of X. If $q = 2$, this is not clear.

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