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Forgetful polygons as generalizations of semi-affine planes

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Abstract

We generalize the notion of a semi-affine plane to structures with higher girth n. We prove that, in the finite case, for n odd, and with an additional assumption also for n even, these geometries, which we call forgetful n-gons, always arise from (finite) generalized n-gons by 'forgetting' lines.

MSC: 51E12

Keywords: Generalized polygon; Semi-affine plane; Semiplane; Finite generalized quadrangles

1. Introduction

A dual semi-affine plane is an incidence structure $(\mathscr{P}, \mathscr{L}, \mathtt{I})$, together with an equivalence relation on the point set $\mathscr{P}$ such that every two lines meet, and every two points are either collinear or equivalent. These structures were introduced by Dembowski in [5], where it is proved that every finite dual semi-affine plane Γ arises from a projective plane in such a way that the points and lines of Γ are projective points and lines, and the equivalence classes are (pieces of) lines of the projective plane. Now, the class of projective planes is part of the larger family of generalized polygons, as introduced by Tits in [11]. The question arises whether it is possible to generalize the notion of semi-affine plane (to structures that could be called *semi-affine generalized*

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polygons) such that, similar to Dembowski's result mentioned above, all such finite structures arise from finite generalized polygons. The aim of the present paper is to answer this question in a most satisfying way.

Concerning terminology, the name 'semi-affine generalized polygon' is not satisfactory. Indeed, first of all, there is no established notion of an *affine generalized polygon*, although most authors dealing with questions related to such notion define this as the subgeometry of all points "far away" from a given element in a generalized polygon. Secondly, as will follow from our results below, the new class of 'semi-affine generalized polygons' would not contain that class of affine generalized polygons as most commonly defined above. For these reasons, we have chosen a completely different terminology, inspired by the way the examples are constructed, namely, by considering a generalized polygon and deleting (forgetting) lines, or turning them, or parts of them, into equivalence classes (and hence, also forgetting them as lines). Hence the name 'forgetful polygons'. The main result of the present paper is roughly that, for n odd, every finite forgetful n-gon arises from a finite generalized n-gon (hence from a projective plane in view of the Feit and Higman result, see below), and, for n even, every finite forgetful n-gon containing a 'large' (compared to the size of the lines, see below for the exact statement) equivalence class also arises in the natural way from a generalized n-gon (and hence $n \in \{4, 6, 8\}$). This way, we characterize the class of finite generalized polygons with a priori more general notion. This is our first motivation.

However, there are other forgetful polygons (not satisfying the assumption on the size of the classes mentioned above), which we will call *short forgetful polygons*. Examples of these are related to important substructures such as subquadrangles and ovoids in generalized quadrangles, and this provides another motivation for studying them. For instance, combining the construction of a certain class of forgetful quadrangles (containing short and non-short cases) with the classification of non-short forgetful quadrangles, we obtain as an easy corollary an alternative (and natural) proof of the result of Payne [9] stating that every generalized quadrangle admitting a regular ovoid arises from the Payne construction (see below for details).

Finally we note that the finiteness conditions in our results cannot be dispensed with. Indeed, we will give a free construction producing (non-short) forgetful n-gons, for all n, which do not satisfy our classification theorems.

2. Definitions and first examples

Let $(\mathscr{P}, \mathscr{L}, \mathtt{I})$ be an incidence structure, and $\sim$ an equivalence relation on the point set $\mathscr{P}$. Denote by $\mathscr{C}$ the set of non-trivial equivalence classes of $\sim$. We say that a point x is incident with a class K of $\mathscr{C}$ if x belongs to K (and we also use the notation $x\mathtt{I}K$ for this). For two elements x and y of $\mathscr{P} \cup \mathscr{L} \cup \mathscr{C}$, a *forgetful path* of length j between x and y is a sequence $(x = z_0, z_1, \dots, z_{j-1}, z_j = y)$ of different elements of $\mathscr{P} \cup \mathscr{L} \cup \mathscr{C}$ such that $z_i \mathtt{I} z_{i+1}$, for $i = 0, \dots, j-1$. If j is the length of a shortest forgetful path connecting x and y, we say that x and y are at distance j (notation $\delta(x, y) = j$).

Now $\Gamma = (\mathscr{P}, \mathscr{L}, \mathtt{I}, \sim)$ is a *forgetful n-gon*, $n \geqslant 3$, if the following three axioms are satisfied:

(FP1) If $x, y \in \mathscr{P} \cup \mathscr{L}$ and $\delta(x, y) = k < n$, then there is a unique forgetful path of length k joining x to y.
(FP2) For every $x \in \mathscr{P} \cup \mathscr{L}$, we have $n = \max\{\delta(x, y)\colon\ y \in \mathscr{P} \cup \mathscr{L}\}$.
(FP3) Every line is incident with at least three points, every point is incident with at least three elements of $\mathscr{L} \cup \mathscr{C}$.

If $\mathscr{C} = \emptyset$, then Γ is a so-called *generalized polygon*. If $\mathscr{C} = \emptyset$, and axioms (FP1) and (FP2) are satisfied, then Γ is a *weak generalized polygon*. Generalized polygons are well-known objects in the theory of incidence geometry, the generalized 3-gons being exactly the projective planes. For an extensive survey including most proofs, we refer the reader to [13]. We say that a generalized polygon has *order* (s, t) if every line is incident with $s + 1$ points, and every point is incident with exactly $t + 1$ lines. By a theorem of Feit and Higman [7], finite generalized n-gons (this is, with a finite number of points and lines) with $n \geqslant 3$, only exist for $n \in \{3, 4, 6, 8\}$.

Note that a forgetful 3-gon is exactly a dual semi-affine plane, as defined by Dembowski [5]. Let $\Gamma = (\mathscr{P}, \mathscr{L}, \mathtt{I}, \sim)$ be a forgetful n-gon. The elements of $\mathscr{C}$ are called the *classes*. So except when mentioned differently, when talking about a class, we always mean a non-trivial class. The cardinality of the biggest class of $\mathscr{C}$ is denoted by g. A point which is only equivalent with itself is called an *isolated point*. Let x and y be two elements of $\mathscr{P} \cup \mathscr{L} \cup \mathscr{C}$ at distance $< n$. We use the notation $[x, y]$ for the shortest forgetful path between x and y. The element of this path incident with x is called the *projection of y onto x*, and denoted $\mathrm{proj}_x\, y$. We use the notation $x \bowtie y$ for the unique element of $[x, y]$ at distance $\delta(x, y)/2$ from both x and y. The set of lines incident with a point p (the set of points incident with a line L) is denoted by $\mathscr{L}_p$ ($\mathscr{P}_L$). The order of $\mathscr{L}_p$ ($\mathscr{P}_L$) is called the *degree* of p (L) and denoted by $|p|$ ($|L|$). In case Γ is a generalized n-gon, we also use the notation $\Gamma_1(p)$ and $\Gamma_1(L)$ instead of $\mathscr{L}_p$ and $\mathscr{P}_L$. If x is a point, then $x^\perp$ is the set of points collinear with x, and for a set X of points of Γ, $X^\perp = \bigcap_{x \in X} x^\perp$. If two points x and y are incident with a unique line, then we denote this line by xy. In the following, if talking about a 'path' in Γ, we will always mean a 'forgetful path'.

Let $\Delta = (\mathscr{P}', \mathscr{L}', \mathtt{I}')$ be a finite generalized n-gon of order (s, t), $n \in \{4, 6, 8\}$. Then the following construction yields a forgetful n-gon $\Gamma = (\mathscr{P}, \mathscr{L}, \mathtt{I}, \sim)$:

(I) Let D be a set of disjoint lines of Δ. Put $\mathscr{P} = \mathscr{P}'$ and $\mathscr{L} = \mathscr{L}' \setminus D$. Two different points of $\mathscr{P}$ are said to be equivalent if and only if they both lie on the same line of D.

If Δ is a finite generalized quadrangle, then we also have the following examples.

(II) Let L be a line of Δ, X_1 a subset of the points of L, $1 \leqslant |X_1| \leqslant s + 1$, and $X_2 = \Delta_1(L) \setminus X_1$. Denote by V the set of lines that intersect L in a point of X_1. Then put $\mathscr{P} = \mathscr{P}' \setminus X_1$, $\mathscr{L} = \mathscr{L}' \setminus (V \cup \{L\})$. Two different points of $\mathscr{P}$ are said to

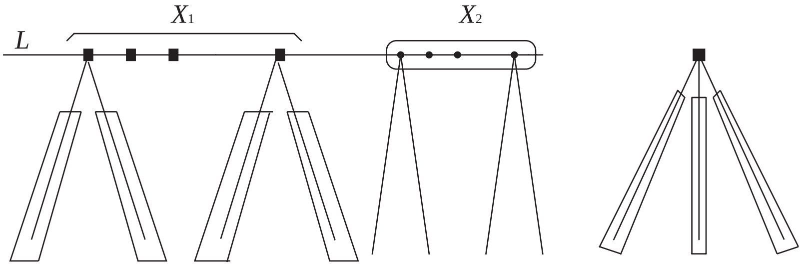

Fig. 1. A forgetful quadrangle of type (II), and the special case $|X_1| = 1$.

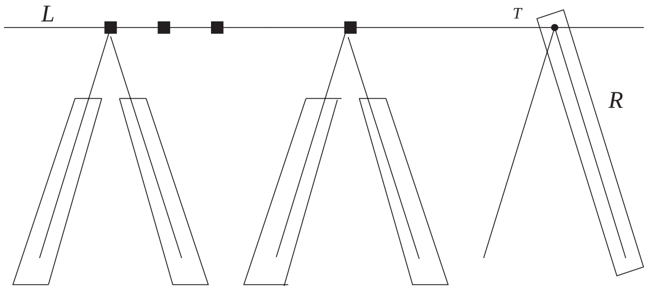

Fig. 2. A forgetful quadrangle of type (III).

be equivalent if and only if they are both incident with the same line of $V \cup \{L\}$ (see Fig. 1).

(III) Let R and L be two different lines of Δ incident with a point r. Let V be the set of lines concurrent with L, but not incident with the point r. Then put $\mathscr{P} = (\mathscr{P}' \setminus \Delta_1(L)) \cup \{r\}$, $\mathscr{L} = \mathscr{L}' \setminus (V \cup \{R, L\})$. Two different points of $\mathscr{P}$ are said to be equivalent if and only if they are both incident with the same line of $V \cup \{R\}$ (see Fig. 2).

A forgetful n-gon, n even, is called a *short forgetful n-gon* if there exist parameters g, k, d such that the following axioms are satisfied:

(S1) Every isolated point is incident with exactly $k + 1$ lines, every non-isolated point is incident with exactly k lines ($k \geqslant 2$).
(S2) Every class has the same size g, $g > 1$.
(S3) Every line is incident with $g + d$ points, $d \geqslant 1$ if $n \geqslant 6$, $d \geqslant 2$ if $n = 4$.

The parameter d is called the *deficiency* of Γ. The name *short forgetful* refers to both the short classes and the short memory of these objects. (Indeed, it will follow that their memory seems to be too short to prove that they arise from generalized polygons.)

We are now ready to state the two main theorems of this paper.

Theorem 2.1. *There does not exist a finite forgetful* n*-gon,* n *odd and* $n \geqslant 5$.

Theorem 2.2. *A finite forgetful* n*-gon,* n *even, is either a generalized* n*-gon, a forgetful* n*-gon of type* (I), (II) *or* (III), *or a short forgetful* n*-gon.*

Remark. Dembowski [6] introduces the notion of a semi-plane, which is an incidence structure together with an equivalence relation on the point set and the line set, respectively, such that every two lines (points) are either concurrent (collinear) or equivalent. Similarly, one can make the definition of a forgetful polygon self-dual. Some partial classification results about these structures for the case $n=5$ can be found in [8].

3. Classification for *n* odd

In this section, we prove Theorem 2.1. Let $\Gamma = (\mathscr{P}, \mathscr{L}, \mathtt{I}, \sim)$ be a finite forgetful n-gon, n odd and $n \geqslant 5$. The aim is to show that the geometry $(\mathscr{P}, \mathscr{L} \cup \mathscr{C}, \mathtt{I})$ is a finite generalized n-gon, hence Γ does not exist. Therefore, it suffices to prove that, if K and K' are classes, and L an arbitrary line, then $\delta(K,L) \leqslant n-1$ (this is done in Steps 1 and 2) and $\delta(K,K') \leqslant n-1$ (Step 3). In Step 0, we collect some general observations.

Step 0: Note that for two classes K and K', and an arbitrary line L, we certainly have that $\delta(K,L) \leqslant n+1$ and $\delta(K,K') \leqslant n+1$. Let K be an arbitrary class, $r \in K$, and N a line for which $\delta(N,K) = n-1$ and $\delta(N,r) = n$. Then $|N| = |r| + 1$. Indeed, the map

$$\sigma : \mathscr{L}_r \to \mathscr{P}_N \setminus \{\mathrm{proj}_N K\} : L \mapsto p, \text{ with } \delta(p,L) = n-2$$

is a bijection. Suppose M is a line at distance $n+1$ from K. Since for any point p of K, the map

$$\sigma : \mathscr{L}_p \to \mathscr{P}_M : L \mapsto p', \text{with } \delta(p',L) = n-2$$

is a bijection, $|M| = |p|$. Similarly, one shows that if z is an isolated point at distance n from a line N, then $|z| = |N|$.

Now fix a class K and suppose M is a line at distance $n+1$ from K, and $m\mathtt{I}M$. Let x be a point of K of degree k for which $N := \mathrm{proj}_m x$ is a line (note that such a point certainly exists). Then from the previous observations follows that $|M| = k$, every point of K has degree k and every line at distance $n-1$ from K is incident with $k+1$ points. Note also that $|M| = k$ implies $k \geqslant 3$, because of axiom (FP3). Let A be the element of $[x,m]$ at distance $(n+1)/2$ from x, $a = \mathrm{proj}_A x$ and $a' = \mathrm{proj}_A m$. The aim is now to show that the existence of the line M leads to a contradiction. In Step 1, we treat the case that, if $n \equiv 1 \bmod 4$, then A is not a class of size 2, and if $n \equiv 3 \bmod 4$, then a' is not a class of size 2. In Step 2, we get rid of the remaining cases.

Step 1: Suppose first that $n \neq 5$ and that in the case $n=7$, $|A| \geqslant 3$, or a is a class of size at least 3. Let z be a point at distance $(n-3)/2$ from A such that $a \neq \mathrm{proj}_A z \neq a'$, and $N' := \mathrm{proj}_z A$ is a line. Let v be a point at distance $(n-1)/2$ from a' for which $A \neq \mathrm{proj}_{a'} v \neq \mathrm{proj}_{a'} m$ and $\mathrm{proj}_v a'$ is a line (note that it is always possible to choose z and v like this because of the assumptions on the degrees of a, a' and A). We claim that both z and v are isolated points with degree $k+1$.

Let z' be a point incident with N' different from z and from $\mathrm{proj}_{N'} A$. Let x' be a point of K different from x. Then $\delta(x',z') = n-1$. We define the lines L and R as follows.

- Case $n \equiv 1 \bmod 4$. Let L be a line at distance $(n-3)/2$ from $l := x' \bowtie z'$ for which $\mathrm{proj}_l x' \neq \mathrm{proj}_l L \neq \mathrm{proj}_l z'$. Then $|L| = k+1$ since $\delta(L,K) = n-1$. Let R be a line at distance $(n-3)/2$ from a, with $\mathrm{proj}_a x \neq \mathrm{proj}_a R \neq A$. Then $|R| = k+1$ since $\delta(R,K) = n-1$.
- Case $n \equiv 3 \bmod 4$. If $n \neq 7$, let r be the point of $[x',z']$ at distance $(n+1)/2$ from x', and R a line at distance $(n-5)/2$ from r for which $\mathrm{proj}_r x' \neq \mathrm{proj}_r R \neq \mathrm{proj}_r z'$. If $n = 7$, let R be a line intersecting a, not incident with A or $\mathrm{proj}_a x$. Then in both cases, $|R| = k+1$ since $\delta(R,K) = n-1$. Let L be a line at distance $(n-1)/2$ from $a'' := \mathrm{proj}_a x$, with $\mathrm{proj}_{a''} x \neq \mathrm{proj}_{a''} L \neq a$. Then $|L| = k+1$ since $\delta(L,K) = n-1$.

Suppose z is not isolated, and let z'' be a point in the class Z containing z, $z'' \neq z$. Since $\delta(R,Z) = \delta(M,Z) = n-1$ and $\delta(R,z'') = \delta(M,z'') = n$, we obtain $|R| = |z''|+1 = |M|$ (see Step 0), a contradiction with $|R| = k+1$ and $|M| = k$. Hence z is isolated and $|z| = |L| = k+1$. The point z was arbitrarily chosen incident with N', so every point incident with N' different from $\mathrm{proj}_{N'} A$ is isolated and has degree $k+1$. From this also follows that each line T intersecting K, different from $\mathrm{proj}_x m$ is incident with $k+1$ points (indeed, one can always find a point of $N' \setminus \{\mathrm{proj}_{N'} A\}$ lying at distance n from T).

Suppose v is not isolated, and let v' be a point in the class V containing v, $v' \neq v$. Then $|M| = |v'| + 1 = |N'|$, a contradiction with $|N'| = k+1$ and $|M| = k$, hence v is isolated. If $n \equiv 1 \bmod 4$, then $|v| = |R| = k+1$. If $n \equiv 3 \bmod 4$, then consider a line X incident with x, $\mathrm{proj}_x m \neq X \neq \mathrm{proj}_x v$ (note that this is possible since $|x| = k \geqslant 3$). Since $\delta(v,X) = n$, we obtain $|v| = |X| = k+1$. This shows that both z and v are isolated points of degree $k+1$, as claimed.

Let R_z be a line incident with z, different from N'. Because $\delta(v,R_z) = n$, this line is incident with $k+1$ points, hence $\delta(R_z,K) = n-1$. The map

$$\sigma : \mathscr{L}_z \setminus \{N'\} \to K \setminus \{x\} : R_z \mapsto y, \text{ with } \delta(y,R_z) = n-2$$

is a bijection, from which follows that $|K| = |z| = k+1$.

Finally, we consider the point m. If m is isolated, then, with X a line incident with x different from $\mathrm{proj}_x m$, $\delta(m,X) = n$ implies that $|m| = |X| = k+1$. Every point of K lies at distance $n-1$ from m, hence we need $k+1$ lines incident with m lying at distance $n-1$ from K, a contradiction since there are at most k such lines (indeed, $\delta(M,K) = n+1$). If m is not isolated, let m' be a point of the class K' containing m, $m' \neq m$. Then $\delta(m',x) = n-1$. Put $B = \mathrm{proj}_x m'$. If $B \neq K$, then since $|B| = k+1$, $\delta(B,K') = n-1$ and $\delta(m,B) = n$, the point m has degree k. If $B = K$, put $y = \mathrm{proj}_K m'$. Let Y be any line incident with y different from $\mathrm{proj}_y m'$. Then, since $|Y| = k+1$, $\delta(Y,K') = n-1$ and $\delta(m,Y) = n$, the point m again has degree k. Since $|K| = k+1$, we need at least k lines incident with m lying at distance $n-1$ from K, a contradiction since there are at most $k-1$ such lines.

Case $n = 5$: Let z be a point incident with the line A, $a \neq z \neq m$, and L a line intersecting ax not incident with a or x. If a would be equivalent with a point a'',

$a'' \neq a$, then $|M| - 1 = |a''| = |L| - 1$ (by Step 0), a contradiction with $|L| = k + 1$ and $|M| = k$. Hence the point a is isolated. Let N'' be a line incident with a, $ax \neq N'' \neq am$ (such a line exists, since the degree of the isolated point a is at least 3). Note that $|N''| = k + 1$. Similarly as above for the point a, we obtain that z is isolated (now using N'' instead of L). From $\delta(z, L) = 5$ and $|L| = k + 1$ follows $|z| = k + 1$. By considering a line at distance 5 from both a and z, we conclude that $|a| = k + 1$. Suppose first that m has degree at least 3. Then let v be a point collinear with m, v not incident with the lines M or A. Again by using the same argument as above for the point a, we obtain that v is isolated and has degree $k + 1$. From this follows that every line R_z incident with z, $R_z \neq A$, is incident with $k + 1$ points. Suppose now m has degree 2, and let r be a point equivalent with m, $r \neq m$. Then $|r| = k$ since $|N''| = k + 1$, implying that also in this case every line R_z incident with z, $R_z \neq A$, is incident with $k + 1$ points. We now proceed similarly as in the case $n > 5$.

Case $n = 7$: $|A| = 2$ *and* a, a' *are lines*. We show that this case cannot occur. Let Z be the class containing A, and $z \in Z$, $z \neq A$. Put $a'' = \mathrm{proj}_a x$. Let Y be a line at distance 4 from $a''x$, and at distance 5 from both x and a''. Then $|Y| = k + 1$. Let y be a point incident with a, $a'' \neq y \neq A$ and Y' a line at distance 3 from a'', and at distance 4 from $a''x$ and a. Then $|Y'| = k + 1$. If $y \sim y'$, $y' \neq y$, then $|M| - 1 = |y'| = |Y'| - 1$, a contradiction, hence y is isolated. Since $\delta(y, Y) = 7$, $|y| = k + 1$. Let finally x' be a point of K, $x' \neq x$. Then $\delta(x', z) = 6$. The line $\mathrm{proj}_{x'} z$ is incident with $k + 1$ points (because it lies at distance 7 from y), implying that $|A| = k \geqslant 3$. This is a contradiction with the assumption.

Case $n = 7$: *a a class of size* 2: Define z and v as in the general case. We have to give another argument to conclude that the points z and v are isolated (since the line R defined in the general case cannot be found). Let $x' \in K$, $x' \neq x$. Then $\delta(x', m) = 6$. Put $R = \mathrm{proj}_m x'$.

- Suppose first R is a line. Then $|R| = k + 1$. If $z \sim z'$, $z' \neq z$, then $|M| - 1 = |z'| = |R| - 1$, a contradiction, hence the point z is isolated. Also, $|z| = |R'| = k + 1$, with R' a line at distance 3 from $a'' = \mathrm{proj}_a x$ for which $a \neq \mathrm{proj}_{a''} R' \neq a''x$.
- Suppose now R is a class and let w be the point of $[x', R]$ at distance 2 from x'. Using the same argument as for the point z above (with a' in the role of R), we obtain that w is isolated. Also, $|w| = k + 1$ (since $|w| = |Az|$). Hence an arbitrary line incident with $a'' = \mathrm{proj}_a x$, different from $a''x$, is incident with $k + 1$ points. Again using the same argument as above, it now follows easily that also in this case the point z is isolated. By considering a line intersecting $a''x$ not incident with x or a'', we obtain $|z| = k + 1$.

If $v \sim v'$, $v \neq v'$, then $|Az| - 1 = |v'| = |M| - 1$, a contradiction, hence v is isolated. For a line X intersecting M, not incident with m, holds that $|z| = |X| = |v|$, hence $|v| = k + 1$. The rest of the proof is similar as in the general case.

This finishes the case $n \equiv 1 \bmod 4$ and A not a class of size 2, or $n \equiv 3 \bmod 4$, and a' not a class of size 2. Note that for $n = 5$ (7), the element A (a') can be chosen to be a line, so Step 2 does not concern these cases.

Step 2: (a) $n \equiv 1 \bmod 4$, $n > 5$ *and* A *a class of size* 2.

In the following, we will construct a point p' such that $\delta(p',A)=n$, $\delta(p',a)=n-1$ for which both $\mathrm{proj}_{p'}a$ and the element of $[a,p']$ at distance $(n+1)/2$ from a are lines, and such that $|p'|=k$ if p' is not isolated, and $|p'|=k+1$ if p' is isolated. Step 1 then implies that every line incident with p' lies at distance $n-2$ from a point of A. Hence $|A|\geqslant k\geqslant 3$, a contradiction.

Case $n=9$: Let p' be a point at distance 5 from the line $X=\mathrm{proj}_x m$, with $x\neq \mathrm{proj}_X p'\neq \mathrm{proj}_X m$ and such that the path $[X,p']$ only consists of points and lines. If p' is isolated, then let y be a point at distance 3 from K and 4 from x, and y' the point of $[y,p']$ at distance 2 from y. By considering a line R at distance 3 from y', $\mathrm{proj}_{y'}y\neq \mathrm{proj}_{y'}R\neq \mathrm{proj}_{y'}p'$, we see that $|p'|=|R|=k+1$. If p' is contained in a non-trivial class Z, then let $p''\in Z$, $p''\neq p'$, and $x'\in K$, $x'\neq x$. Let L be a line at distance 3 from the point $z:=x'\bowtie p''$, $\mathrm{proj}_z p''\neq \mathrm{proj}_z L\neq \mathrm{proj}_z x'$. Since $|L|=k+1$, p' has degree k. So we constructed a point p' as claimed.

Case $n\equiv 1 \bmod 8$, $n>9$: Let p be the point of $[x,a]$ at distance $\delta(x,a)/2-2=(n-9)/4$ from x, and p' a point at distance $(3n-11)/4$ from p such that $\mathrm{proj}_p a\neq \mathrm{proj}_p p'\neq \mathrm{proj}_p x$ and such that the path $[p,p']$ only consists of points and lines, except possibly for the element $\mathrm{proj}_p p'$. Suppose first that p' is isolated. We show that $|p'|=k+1$. Consider a point y at distance 6 from x such that $\mathrm{proj}_x p\neq \mathrm{proj}_x y\neq K$. Then $\delta(p',y)=n-1$. Let γ be the union of the paths $[p',y]$ and $[y,x]$. Let z be the element of γ at distance $(n+3)/2$ from x, and z' a point at distance $(n-9)/2$ from z such that $\mathrm{proj}_z p'\neq \mathrm{proj}_z z'\neq \mathrm{proj}_z y$. Then any line incident with z' different from the projection of K onto z' is incident with $k+1$ points (since it lies at distance $n-1$ from K), and lies at distance n from p', hence $|p'|=k+1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p'. We show that $|p'|=k$. Consider a point y at distance 4 from x such that $\mathrm{proj}_x p\neq \mathrm{proj}_x y\neq K$. Then $\delta(p'',y)=n-1$. Let c be the point of $[y,p'']$ at distance $(n-5)/2$ from y and let c' be a point at distance $(n-9)/2$ from c such that $\mathrm{proj}_c p''\neq \mathrm{proj}_c c'\neq \mathrm{proj}_c y$. Then any line $R_{c'}$ incident with c' different from the projection of K onto c' is incident with $k+1$ points (because it lies at distance $n-1$ from K). Since $\delta(R_{c'},K')=n-1$ and $\delta(R_{c'},p')=n$, Step 0 implies that $|p'|=|R_{c'}|-1=k$. Now the point p' satisfies all the conditions above.

Case $n\equiv 5 \bmod 8$, $n>5$: Let p be the point of $[x,a]$ at distance $(n-5)/4$ from x, and p' a point at distance $(3n-7)/4$ from p such that $\mathrm{proj}_p a\neq \mathrm{proj}_p p'\neq \mathrm{proj}_p x$ and such that the path $[p,p']$ only consists of points and lines, except possibly for the element $\mathrm{proj}_p p'$. Suppose first that p' is isolated. Let z be the point of $[x,p']$ at distance $(n-5)/2$ from x, and z' a line at distance $(n+1)/2$ from z such that $\mathrm{proj}_z x\neq \mathrm{proj}_z z'\neq \mathrm{proj}_z p'$. Then $|p'|=|z'|=k+1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p'. Let y be a point of K, $y\neq x$. Then $\delta(p'',y)=n-1$. Consider a line z' at distance $(n-3)/2$ from the point $z=y\bowtie p''$ for which $\mathrm{proj}_z p''\neq \mathrm{proj}_z z'\neq \mathrm{proj}_z y$. Then $|p'|=|z'|-1=k$.

(b) $n\equiv 3 \bmod 4, n>7$ *and* a' *a class of size* 2.

Similarly as in the case $n\equiv 1 \bmod 4$, we construct a point p' at distance n from a' and at distance $n-1$ from A, for which both $\mathrm{proj}_{p'}A$ and the element of $[A,p']$ at distance $(n+3)/2$ from A are lines, and such that either $|p'|=k+1$ if p' is isolated, or $|p'|=k$ if p' is not isolated. The result will then follow.

Case $n \equiv 3 \bmod 8$: Let p be the point of $[x, A]$ at distance $(n-3)/4$ from x, and p' a point at distance $(3n-9)/4$ from p such that $\mathrm{proj}_p A \neq \mathrm{proj}_p\, p' \neq \mathrm{proj}_p\, x$ and such that the path $[p, p']$ only consists of points and lines except possibly for the element $\mathrm{proj}_p\, p'$. Suppose first that p' is isolated. Let y be a point at distance 4 from x such that $\mathrm{proj}_x A \neq \mathrm{proj}_x\, y \neq K$. Then $\delta(p', y) = n-1$. Let γ be the union of the paths $[x, y]$ and $[y, p']$, and z the point of γ at distance $(n+1)/2$ from x. Let finally z' be a line at distance $(n-5)/2$ from z such that $\mathrm{proj}_z\, y \neq \mathrm{proj}_z\, z' \neq \mathrm{proj}_z\, p'$. Then $|p'| = |z'| = k+1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p. Consider a point y at distance 2 from x such that $\mathrm{proj}_x\, p \neq \mathrm{proj}_x\, y \neq K$. Let z be the point of $[y, p'']$ at distance $(n-3)/2$ from y, and z' a line at distance $(n-5)/2$ from z such that $\mathrm{proj}_z\, y \neq \mathrm{proj}_z\, z' \neq \mathrm{proj}_z\, p''$. Then $|p'| = |z'| - 1 = k$.

Case $n \equiv 7 \bmod 8,\ n > 7$: Let p be the point of $[x, A]$ at distance $(n-7)/4$ from x, and p' a point at distance $(3n-13)/4$ from p such that $\mathrm{proj}_p A \neq \mathrm{proj}_p\, p' \neq \mathrm{proj}_p\, x$ and such that the path $[p, p']$ only consists of points and lines, except possibly for the element $\mathrm{proj}_p\, p'$. Suppose first that p' is isolated. Let z be the point of $[x, p']$ at distance $(n-7)/2$ from x, and z' a line at distance $(n+3)/2$ from z such that $\mathrm{proj}_z\, x \neq \mathrm{proj}_z\, z' \neq \mathrm{proj}_z\, p'$. Then $|p'| = |z'| = k+1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p'. Let y be a point at distance 3 from K and at distance 4 from x. Then $\delta(p'', y) = n-1$. Let z be the point of $[y, p'']$ at distance $(n-3)/2$ from y, and z' a line at distance $(n-5)/2$ from z such that $\mathrm{proj}_z\, y \neq \mathrm{proj}_z\, z' \neq \mathrm{proj}_z\, p''$. Then $|p'| = |z'| - 1 = k$.

Hence we have shown that for any class K and an arbitrary line M, $\delta(M, K) \leqslant n-1$.

Step 3: Suppose there exist two classes K_1 and K_2 at distance $n+1$ from each other. We look for a contradiction. Let $x \in K_1$ be arbitrary. Since by the results of Steps 1 and 2, any line incident with x lies at distance $n-1$ from K_2, the map

$$\sigma : \mathscr{L}_x \to K_2 : L \mapsto y, \text{ with } \delta(L, y) = n-2$$

is a bijection, hence $|K_2| = |x| =: k$ and all points in K_1 have the same degree k. Fix points $x \in K_1$ and $y \in K_2$. Note that $\delta(x, y) = n-1$. Let z be the element of $[x, y]$ at distance $(n-3)/2$ from x. If $n \equiv 1 \bmod 4$, then we can assume without loss of generality that z is a line (indeed, if z is a class, then interchange the roles of K_1 and K_2). If $n = 5$, let w be a point incident with z, different from x or $x \bowtie y$. If $n > 5$, let w be a point at distance $(n-3)/2$ from z such that $\mathrm{proj}_z\, x \neq \mathrm{proj}_z\, w \neq \mathrm{proj}_z\, y$ (this is possible because z was assumed not to be a class) and $\mathrm{proj}_w\, z$ is a line (this is not possible if $n = 7$, z has degree 2 and $x \bowtie y$ is a line, see case (1) below). Let finally W be a line incident with w, $W \neq z$ if $n = 5$, $W \neq \mathrm{proj}_w\, z$ if $n > 5$. Then $|W| = k+1$, since $\delta(K_1, W) = n-1$. But also $\delta(K_2, W) = n-1$ by Steps 1 and 2, hence there is a point in K_2 which has degree k. By repeating the argument at the beginning of this step, we obtain that also $|K_1| = k$, and every point of K_2 has degree k. Note that this implies that every line intersecting K_1 or K_2 is incident with $k+1$ points.

Let the point w be as above. If w is isolated then, by considering a line intersecting K_1, but not incident with x, we obtain that $|w| = k+1$. Since every line incident with w lies at distance $n-1$ from K_2, we need at least $k+1$ points in K_2, a contradiction. Suppose now w is contained in a non-trivial class K, and let w' be a point of K, $w' \neq w$. Then $|w'| = k$. Indeed, $|w'| = |R| - 1$, with R an arbitrary line incident with x

different from $\mathrm{proj}_x\, y$. But then the map

$$\sigma : \mathscr{L}_{w'} \to K_1 \setminus \{x\} : L \mapsto v, \text{ with } \delta(L, v) = n - 2$$

is a bijection, hence $|K_1| = k + 1$, the final contradiction since $|K_1| = k$.

(1) $n = 7$, $x \bowtie y$ a line and z non-isolated.

Let L be a line intersecting $x \bowtie y$, $\delta(L, K_1) = \delta(L, K_2) = 6$. Then $|L| = k + 1$, hence every point in $K_2 \setminus \{y\}$ has degree k. By repeating the argument at the beginning of this step, we obtain that also $|K_1| = k$, and every point of K_2 has degree k. Let w be a point in the class containing z, $w \neq z$. Let Y be a line intersecting K_1, Y not incident with x. Then $|w| = |Y| - 1 = k$. Since every line incident with w lies at distance 6 from a point of $K_2 \setminus \{y\}$, $|K_2| = k + 1$, a contradiction.

This shows that for two classes K_1 and K_2, $\delta(K_1, K_2) \leqslant n - 1$. Now Theorem 2.1 is proved.

4. Classification results for *n* even

Let for the rest of this section, Γ be a forgetful n-gon, n even, admitting non-isolated points.

Lemma 4.1. *Every line is incident with the same number l of points, and $l \geqslant g$.*

Proof. Let L and L' be two arbitrary lines. Note that $\delta(L, L') \leqslant n$ by axiom (FP2). If $\delta(L, L') = n$, then the projection map defines a bijection between $\mathscr{P}_L$ and $\mathscr{P}_{L'}$ (by axiom (FP1)), hence $|L| = |L'|$. If L and L' meet in a point p, then consider a line M at distance $n - 1$ from p for which $L \neq \mathrm{proj}_p\, M \neq L'$ (such a line exists by axiom (FP3)). Then $\delta(L, M) = \delta(L', M) = n$, hence $|L| = |M| = |L'|$. If finally $2 < \delta(L, L') < n$, then let N be a line at distance $n - \delta(L, L')$ from L such that $\mathrm{proj}_L\, N \neq \mathrm{proj}_L\, L'$, for which the path between L and N only contains points and lines. Then $|L| = |N|$; and because $\delta(N, L') = n$ we also have $|N| = |L'|$. So all lines are incident with the same number of points. Now let G be a class of size g, and L a line at distance n from G. Since for every point x of G, there is a unique point x' incident with L for which $\delta(x, x') = n - 2$, and since all these points are different, we obtain $l \geqslant g$. □

Lemma 4.2. *Let K be a class of size at least* 3. *Then all the points in K have the same degree or K is a forgetful quadrangle of type* (III).

Proof. Let K be a class of size at least 3, and suppose that there are two points $p_1, p_2 \in K$ for which $|p_1| \neq |p_2|$. Suppose K' is a class different from K and x a point of K'. Since every point of K lies at distance $< n$ from any line incident with x, $\delta(K, K') \leqslant n + 2$.

Suppose $\delta(K, K') = n + 2$. Since the map

$$\sigma : \mathscr{L}_x \to \mathscr{L}_{p_i} : L \mapsto L', \text{ with } \delta(L, L') = n - 2$$

is a bijection, we have $|x| = |p_i|$, $i = 1, 2$, a contradiction.

Suppose $\delta(K,K')=n$, and $\delta(p_1,K')=\delta(p_2,K')=n+1$. Let y be a point of K' at distance $n-1$ from K. Then the map

$$\sigma' : \mathscr{L}_y \setminus \{\mathrm{proj}_y K\} \rightarrow \mathscr{L}_{p_i} : L \mapsto L', \text{ with } \delta(L,L') = n-2$$

is a bijection, hence $|y|=|p_i|+1$, $i=1,2$, a contradiction.

Suppose $\delta(K,K')=n$, and $\delta(p_1,K')=\delta(p_2,K')=n-1$. Let $r_i=\mathrm{proj}_{K'}\,p_i$, $i=1,2$, and let $p_3\in K\setminus\{p_1,p_2\}$. If $\delta(p_3,K')=n+1$, then $|r_1|-1=|p_3|=|r_2|-1$. If $\delta(p_3,K')=n-1$, then $|r_1|=|p_3|=|r_2|$. Since $|p_1|=|r_2|$ and $|p_2|=|r_1|$, we obtain a contradiction in both cases.

We conclude that $\delta(K,K')<n$, or $\delta(K,K')=n$ but then exactly one of p_1 and p_2 lies at distance $n-1$ from K'. Completely similar, one shows that any isolated point w lies at distance at most $n-1$ from K, and that if $\delta(w,K)=n-1$, then either $\delta(w,p_1)=n-2$ or $\delta(w,p_2)=n-2$.

Suppose first that no point at distance $n-1$ from K is isolated. Let $p_3\in K\setminus\{p_1,p_2\}$. Without loss of generality, we can assume $|p_2|\neq|p_3|$. Let K' be a class at distance n from K for which $\delta(p_1,K')=n-1$ and put $r_1=\mathrm{proj}_{K'}\,p_1$. Then $\delta(p_2,K')=n+1$. If $\delta(p_3,K')=n+1$, then $|p_3|=|r_1|-1=|p_2|$, a contradiction, hence $\delta(p_3,K')=n-1$. Put $r_3=\mathrm{proj}_{K'}\,p_3$. Now $|p_1|=|r_3|=|p_2|+1$. Let K'' be a non-trivial class at distance n from K for which $\delta(p_2,K'')=n-1$, and put $r_2=\mathrm{proj}_{K''}\,p_2$. Let finally r be a point of K'' different from r_2. If $\delta(r,K)=n-1$, then $|p_2|=|r|=|p_1|+1$. If $\delta(r,K)=n+1$, then $|p_1|=|r|=|p_2|-1$. So in both cases, we obtain a contradiction with $|p_2|+1=|p_1|$.

So we may assume that there exists an isolated point w at distance $n-1$ from K for which $\delta(p_1,w)=n-2$. Then $|w|=|z|+1$, for all points z of K different from p_1. Let v be a point at distance $n-2$ from p_2 and $n-1$ from K for which $\mathrm{proj}_v\,p_2$ is a line. Let p_3 be an arbitrary point of K different from p_1 and p_2. If v would be isolated, then we obtain $|p_1|+1=|v|=|p_3|+1$, hence (since $|p_2|=|p_3|$) also $|p_1|=|p_2|$, a contradiction. So v is contained in a class K'. By the first paragraph of the proof, $\delta(p_1,K')=n+1$ and hence $|v|=|p_1|+1$. Since the degree of an arbitrary point z of K different from p_1 is $|p_2|$, such a point z lies at distance $n-1$ from K' (if not, the degree of z would be $|v|-1=|p_1|$). Also, it is now clear that $|v|=|p_3|$, hence $|p_3|=|p_1|+1$. Note that any isolated point at distance $n-1$ from K necessarily lies at distance $n-2$ from p_1. Now let w' be an arbitrary point at distance $n-1$ from K and at distance $n-2$ from p_1 such that $\mathrm{proj}_{w'}\,p_1$ is a line. We show that w' is isolated. Suppose by way of contradiction that w' is equivalent with a point w'', $w''\neq w'$. Since w'' does not lie at distance $n-2$ from any point of K, we see that $|p_2|=|w''|=|p_1|-1$, a contradiction.

Finally, we claim that every point u of K' (with K' as above) lies at distance $n-2$ from a point of $K\setminus\{p_1\}$, and $|u|=|p_2|$. Suppose by way of contradiction that the class K' contains a point v' at distance $n+1$ from K. Then $|v'|=|p_2|-1=|p_1|$. Since K' contains at least two points of degree $|p_2|$ (namely the projections of p_2 and p_3 onto K'), every isolated point at distance $n-1$ from K' has to lie at distance $n-2$ from v'. Let γ be a fixed n-path between p_1 and v', and x the element of γ at distance $n/2+1$ from v'. If x is not a point of degree two or a class containing only two points, then consider a point y at distance $n/2-1$ from x for which $\mathrm{proj}_y\,p_1$ is a line and $\mathrm{proj}_x\,p_1\neq\mathrm{proj}_x\,y\neq\mathrm{proj}_x\,v'$. The point y is isolated (since $\delta(y,K)=n-1$

and $\delta(y, p_1) = n-2$) and lies at distance $n-1$ from K' (if $\delta(y, K') = n+1$, all the points of K' would have equal degree) but at distance n from v', a contradiction. If x is a class or a point of degree two, then let R be a line at distance $n/2-1$ from $x' = \mathrm{proj}_x\, p_1$ for which $\mathrm{proj}_{x'}\, p_1 \neq \mathrm{proj}_{x'}\, R \neq \mathrm{proj}_{x'}\, v'$. Then the line R is incident with at least two isolated points, hence at least one isolated point at distance n from v', a contradiction. This shows the claim.

So we obtained the following situation ($\diamondsuit$): if K' is an arbitrary non-trivial class at distance n from K, then $|K'| = |K| - 1$ and each point of K' lies at distance $n-2$ from a unique point of $K \setminus \{p_1\}$. The degree of a point in K' is equal to $|z| = |p_1| + 1$, with z an arbitrary point of $K \setminus \{p_1\}$. A point w' at distance $n-1$ from K for which $\mathrm{proj}_{w'}\, K$ is a line is isolated if and only if $\delta(w', p_1) = n-2$. Moreover, $|K| = g = l$. Indeed, consider a line L at distance n from K. Suppose $l > |K|$. Then L is incident with a point y which lies at distance n from all the points of K. This contradicts the observations at the beginning of the proof. Consequently, $l = |K| = g$.

Suppose $n = 4$. We show that Γ is of type (III). Note that every non-trivial class different from K lies at distance 4 from K and hence has size $g-1$. It is also easy to see that, if two classes K' and K'' lie at distance 4, $K' \neq K \neq K''$, then every point of K' lies at distance 3 from a point of K'' and conversely. Indeed, let $z \in K'$, $\delta(z, K'') = 3$, and suppose y is a point of K'' at distance 5 from K'. Then $|y| = |z| - 1$, a contradiction with the fact that all points in K' and K'' have degree $|p_1| + 1$.

We define the following equivalence relation $\sim_C$ on the classes of size $g-1$:

$$K_1 \sim_C K_2 \Leftrightarrow \delta(K_1, K_2) = 6.$$

The transitivity of $\sim_C$ is shown as follows: suppose $K_1 \sim_C K_2$, $K_1 \sim_C K_3$, but $\delta(K_2, K_3) = 4$. Let L be a line intersecting both K_2 and K_3. Every point of K_1 has to lie at distance 2 from a unique point of L, not belonging to K_2 or K_3, hence $|L| \geqslant |K_1| + 2 = g+1$, a contradiction. We associate a symbol ∞_i, $i = 1, \ldots, s$ to each equivalence class C_i of $\sim_C$. Now define the following geometry $\Delta = (\mathscr{P}', \mathscr{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i = 1, \ldots, s$. A line of Δ is either a line of Γ; the set K (with K the unique class of size g); the set of points of a class of size $g-1$ together with the symbol of its equivalence class, or the set of points $\{\infty_1, \ldots, \infty_s, p_1\}$. Incidence is the incidence of Γ if defined, or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (s, k). Now clearly, Γ is a forgetful quadrangle of type (III).

Suppose $n \geqslant 6$. We look for a contradiction. Let K' be a class at distance n from K (such a class exists by ($\diamondsuit$)), and $z \in K'$. We construct a line M for which $\delta(M, K) = n-2$, $\delta(M, K') = n$, $\delta(M, p_1) = n-3$ and such that there exists a point $r \in K'$ for which $\delta(\mathrm{proj}_M\, p_1, r) = n-2$. Fix a line L incident with p_1 and put $z' = \mathrm{proj}_L\, z$. If $n \equiv 2 \bmod 4$, let m be the point of $[z, z']$ at distance $n/2-3$ from z' and let M be a line at distance $n/2-2$ from m for which $\mathrm{proj}_m\, p_1 \neq \mathrm{proj}_m\, M \neq \mathrm{proj}_m\, z$. (Note that, for $n = 6$, $z' = m$, but since $|z'| \geqslant |z''| = |p_1| + 1 \geqslant 3$, with $z'' \in K' \setminus \{z\}$, the line M exists.) If $n \equiv 0 \bmod 4$, consider the element N of $[z, z']$ at distance $n/2-3$ from z'. If N is a line or a class incident with or containing at least three points, then let M be a line at distance $n/2-2$ from N such that $\mathrm{proj}_N\, p_1 \neq \mathrm{proj}_N\, M \neq \mathrm{proj}_N\, z$. If N is a class of size 2, then let N' be a line at distance $n/2-3$ from $x = \mathrm{proj}_N\, z'$ such that

$\text{proj}_x\, p_1 \neq \text{proj}_x N' \neq \text{proj}_x z$. Fix a point $z'' \in K' \setminus \{z\}$ and put $y = \text{proj}_{N'} z''$. Note that if $\text{proj}_y z''$ is a line, then $|y| \geqslant |z| = |p_1| + 1 \geqslant 3$. Hence it is possible to choose a line M incident with y different from N' or $\text{proj}_y z''$. So in each case, we constructed a line M as claimed. Now all the points incident with M different from $\text{proj}_M\, p_1$ are isolated (see ($\Diamond$)). Since $|K'| = |M| - 1$, there exists a point a incident with M which lies at distance n from all the points of K'. Hence $|a| = |r| = |p_2|$, the final contradiction since a is isolated (which implies $|a| = |p_2| + 1$). □

From now on, we assume that two points belonging to a class of size at least 3, have the same degree.

Lemma 4.3. *The points of a class K of size 2 have the same degree.*

Proof. Let $K = \{p_1, p_2\}$ and suppose by way of contradiction that $|p_1| \neq |p_2|$. For an arbitrary class K', one shows similarly as in the proof of Lemma 4.2 that $\delta(K, K') \leqslant n$, that $\delta(K, K') = n$ implies that $|K'| = 2$ and that the two points of K' have different degrees. Also, any isolated point lies at distance at most $n-1$ from K. We first claim that the degrees of p_1 and p_2 differ by one. Let L be a line at distance n from K. Since $l \geqslant 3$, there is at least one point x on L at distance n from p_1 and p_2. Since x is not isolated, this point is contained in a class K' of size 2 at distance n from K. Without loss of generality, we can assume that the point y of K' different from x lies at distance $n-2$ from p_2. From this follows that $|p_2| - 1 = |x| = |p_1| = |y| - 1$, hence the claim. From now on, we assume that $|p_2| = |p_1| + 1$.

Now the following observation ($\Diamond$) can easily been shown. If $K_1 = \{q_1, q_2\}$ and $|q_1| \neq |q_2|$, then (up to interchanging q_1 and q_2) $|q_1| = |q_2| - 1$, and for any class $K_2 = \{r_1, r_2\}$ at distance n from K_1, we have (up to interchanging r_1 and r_2) either $\delta(q_1, r_1) = \delta(q_2, r_2) = n-2$ with $|q_1| = |r_2|$ and $|q_2| = |r_1|$ or $\delta(q_2, r_2) = n-2$, $\delta(q_1, K_2) = \delta(r_1, K_1) = n+1$ and $|q_i| = |r_i|$, $i = 1, 2$.

First consider the case $n = 4$. We start by showing that there are no isolated points. Let L be an arbitrary line not intersecting K, and put $r_i = \text{proj}_L\, p_i$, $i = 1, 2$. Let z be a point incident with L, $r_1 \neq z \neq r_2$. Then z is not isolated (because of the observations at the beginning of the proof), hence z is contained in a class $K_z = \{z, z'\}$ at distance 4 from K and by ($\Diamond$), $|z| = |p_1|$ and $|z'| = |p_2|$. If r_2 would be isolated, then $|p_1| + 1 = |r_2| = |z'| + 1$, a contradiction since $|z'| = |p_1| + 1$, hence r_2 is contained in a class $K_{r_2} = \{r_2, r_2'\}$, and $|r_2| = |p_2|$ (indeed, if $|r_2| = |p_1|$, then the two points r_2 and z of degree $|p_1|$ would be collinear, contradicting ($\Diamond$)). If r_1 would be isolated, then $|r_2'| + 1 = |r_1| = |z'| + 1$, again a contradiction. Now it is clear that no point of Γ is isolated. Hence every point is contained in a class of size 2, and has degree $|p_1|$ or $|p_2|$. Put $h = |\mathscr{P}|/2$. We count the number of pairs (p, L), L a line of Γ incident with the point p, with p a point of degree $|p_i|$. Since by ($\Diamond$) every line is incident with at most 1 point of degree $|p_1|$, we obtain

$$h|p_1| \leqslant |\mathscr{L}| \ (i = 1), \quad \text{and} \quad h(|p_1| + 1) \geqslant 2|\mathscr{L}| \quad (i = 2).$$

Hence $|p_1| \leqslant 1$, the final contradiction.

Now consider the case $n \geqslant 6$ (in fact, the argument below also works for $n = 4$ except when $l = 3$). Choose an element x at distance $n/2 + 1$ from p_1 and at distance $n/2 + 2$ from K such that x is a line if $n \equiv 0 \bmod 4$. Suppose first that, if $n = 6$, the point x can be chosen such that either x is isolated (implying $|x| \geqslant 3$) or $\mathrm{proj}_x\, p_1$ is a class. Let M_1 and M_2 be two lines at distance n from K and at distance $n/2 - 2$ from x such that $\mathrm{proj}_x M_1 \neq \mathrm{proj}_x M_2$ (note that such lines exist because of the assumptions just made). On each of the lines M_i, $i = 1, 2$, there is at least one point m_i that lies at distance $n + 1$ from the class K. Because of the first paragraph of the proof, the point m_i is contained in a class K_i of size 2, and by ($\diamondsuit$), $|m_1| = |p_1| = |m_2|$ (recall that $|p_1| = |p_2| - 1$). But since $\delta(m_1, m_2) = n - 2$, this is a contradiction with ($\diamondsuit$). Suppose now that $n = 6$ and that we cannot choose a point x as above. This implies in particular that every point collinear with p_1 and at distance 3 from K is isolated. Then let again x be a point at distance 5 from K for which $\delta(x, p_1) = 4$, M_1 a line incident with x, $M_1 \neq \mathrm{proj}_x K$ and m_1 a point incident with M_1 at distance 6 from both p_1 and p_2. Again, the point m_1 is not isolated, and the class K_1 containing m_1 lies at distance 6 from K. Hence $K_1 = \{m_1, m_1'\}$ and by ($\diamondsuit$), also $|m_1| = |m_1'| - 1$. Now the point $x \bowtie p_1$ lies at distance 5 from K_1 and 4 from m_1, and has degree at least 3. Hence we can apply the argument of the general case above (with K_1 in the role of K and $x \bowtie p_1$ in the role of x) to obtain a contradiction with $|m_1| \neq |m_1'|$. □

Lemma 4.4. (i) *If two classes K and K' lie at distance n, then every point of K lies at distance $n - 1$ from K' and vice versa, hence $|K| = |K'|$.*

(ii) *All non-isolated points have the same degree k.*

Proof. First we note that, if K and K' are two classes for which the points have degree k and k', respectively, and for which $\delta(K, K') = n + 2$, then, for every $x \in K$ and every $x' \in K'$ one has $k = |x| = |x'| = k'$.

We now show (i). So let K and K' be two classes with $\delta(K, K') = n$. There are points $x \in K$ and $x' \in K'$ such that $\delta(x, x') = n - 2$. If there exist points $y \in K$, $y \neq x$ and $y' \in K'$ such that $\delta(y, y') = n - 2$, then $|x'| = |y|$, hence $k = k'$. But if no such points exist, then, for an arbitrary point $y \in K$, $y \neq x$, and a point $y' \in K'$, $y' \neq x'$, we have $|y'| = |y| = |x'| - 1$, contradicting $|x'| = |y'|$. Hence $k = k'$. Note that if K would contain a point z at distance $n + 1$ from K', then $k = |z| = |x'| - 1 = k - 1$, a contradiction. This shows (i).

We now prove (ii). Choose a class K' at minimal distance from K (if such a class does not exist, (ii) is proved). We show that the points in K and the points in K' have the same degree. We can assume $\delta(K, K') \leqslant n - 2$ (by the two previous paragraphs). Let X be the element at distance $(\delta(K, K')/2)$ from both K and K' (note that X cannot be a class because of the minimality of $\delta(K, K')$). Consider a point x at distance $n - 1 - (\delta(K, K')/2)$ from X such that $\mathrm{proj}_X K' \neq \mathrm{proj}_X x \neq \mathrm{proj}_X K$ (such a point exists since again by the minimality of $\delta(K, K')$, X is not a class of size 2 or a point of degree 2) and such that $\mathrm{proj}_x X$ is not a class. If x is isolated, then it is easy to see that $k = k'$ (indeed, then $|x| = |z| + 1$, for z an arbitrary point of K or K' different from the projection of X onto K or K'). If x is contained in a non-trivial class K'', then K'' lies at distance n from both K and K', hence the result. Now (ii) easily follows. □

Lemma 4.5. *One of the following situations occurs*:

(i) *There is a unique isolated point of degree* k. *In this case*, Γ *is a generalized quadrangle of type* (II), *with* $|X_1| = s$.
(ii) *Any isolated point has degree* $k+1$, *and lies at distance at most* $n-1$ *from any class*.

Proof. Let w be an isolated point. We first prove that $|w| \in \{k, k+1\}$. For an arbitrary class K, we have $\delta(w,K) \leqslant n+1$. If there is a class K at distance $n+1$ from w, then $|w| = k$, and if there is a class K at distance $n-1$ from w, then $|w| = k+1$. Now choose a class K at minimal distance from w. We can assume $\delta(w,K) \leqslant n-3$. Put $v = \mathrm{proj}_K\, w$. Let X be the element on the shortest path between v and w at distance $\delta(v,w)/2$ from w. Then X is not a class, hence it is possible to choose a point x at distance $n - \delta(v,w)/2$ from X such that $\mathrm{proj}_X\, w \neq \mathrm{proj}_X\, x \neq \mathrm{proj}_X\, v$. If x is isolated, then (since $\delta(x,K) = n \pm 1$), $|x| = k$ or $|x| = k+1$. Since obviously isolated points at maximal distance from one another have the same degree, also $|w| \in \{k, k+1\}$. If x is not isolated, then, with K' the class containing x, $\delta(w,K') = n \pm 1$, hence also $|w| \in \{k, k+1\}$. So we conclude that every isolated point has degree k or $k+1$; if it has degree $k+1$, then it cannot lie at distance $n+1$ from any class; if it has degree k, then it cannot lie at distance $n-1$ from any class.

$n = 4$: Suppose there exists an isolated point w of degree k. We show that all other isolated points have degree $k+1$. Note that all points collinear with w are isolated. Let K be an arbitrary class, and $x, y \in K$. Let L_1 and L_2 be two different lines incident with w and put $x_i = \mathrm{proj}_{L_i}\, x$, $y_i = \mathrm{proj}_{L_i}\, y$, $i = 1,2$. Then x_i and y_i are isolated points of degree $k+1$ (since they lie at distance 3 from K). If w' is a second isolated point of degree k, then w' is collinear with the points x_i and y_i, $i = 1,2$ (since isolated points at distance $n = 4$ have the same degree), hence $w = w'$. So w is the unique isolated point of degree k. From this it immediately follows that a point is isolated if and only if it is collinear with w. Let G be a class of size g. Since every point of a line L incident with w, different from w, is collinear with a unique point of G, it follows that $l = g+1$. Note that this implies that all classes have size g. Indeed, for an arbitrary non-trivial class K, every point of L different from w has to be collinear with a unique point of K and vice versa, hence $|K| = g$. As in the proof of Lemma 4.2, the following relation $\sim_C$ is an equivalence relation on the classes of size g:

$$K_1 \sim_C K_2 \Leftrightarrow \delta(K_1, K_2) = 6.$$

We associate a symbol ∞_i, $i = 1, \ldots, s$ to each equivalence class C_i of $\sim_C$. Now define the following geometry $\Delta = (\mathscr{P}', \mathscr{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i = 1, \ldots, s$. A line of Δ is either a line of Γ, the set of points of a class of $\mathscr{C}$ together with the symbol ∞_j of its equivalence class or the set of points $\{\infty_1, \ldots, \infty_s,\ w\}$. Incidence is the incidence of Γ if defined or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (s,k). Hence Γ is a forgetful quadrangle of type (II), with $|X_1| = s$.

$n \geqslant 6$: We show that an isolated point of degree k cannot exist. So let by way of contradiction, w be an isolated point of degree k. Let S be the set of points x at distance $n-2$ from w for which $\mathrm{proj}_x\, w$ is a line. Clearly, S is non-empty and consists of isolated points. We first show that all the points of S have degree $k+1$. Suppose by way of contradiction that S contains a point x of degree k. Since isolated points at maximal distance from one another have the same degree, it is easy to see that all the points of S then have the same degree k. We can now always find a point y of S at distance $n-1$ from a certain class, which is a contradiction. Indeed, let K be an arbitrary non-trivial class at minimal distance from w. If $\delta(w,K)=n+1$, let v be a point of K and γ a fixed n-path between v and w. If $\delta(w,K)<n+1$, let $v=\mathrm{proj}_K\, w$ and $\gamma=[v,w]$. Let X be the element of γ at distance $\delta(v,w)/2$ from both v and w. Since X cannot be a class or a non-isolated point, it is possible to choose a point y at distance $n-2-\delta(v,w)/2$ from X such that $\mathrm{proj}_X\, w \neq \mathrm{proj}_X\, y \neq \mathrm{proj}_X\, v$ and $\mathrm{proj}_y\, X$ is not a class. Now the point y belongs to S and lies at distance $n-1$ from K, the contradiction. We conclude that all the points of S have degree $k+1$.

Now let x be a point at distance n from w. Then x cannot be isolated. Indeed, if x is isolated, then $|x|=|w|=k$, but it is easy to see that there exists a point y of S at maximal distance from x, hence $|x|=|y|=k+1$, a contradiction. Also, the class K_x containing x cannot lie at distance $n-1$ from w. Let L be a line incident with w. Since $\delta(w,K_x)=n+1$, projecting the points of K_x onto L shows that $l \geqslant |K_x|+1$. Let γ be a fixed n-path between x and w, and X the element of γ at distance $n/2+1$ from x. If $n>6$ and X is not a class of size two, then consider a line M at distance $n/2-2$ from X for which $\mathrm{proj}_X\, x \neq \mathrm{proj}_X\, M \neq \mathrm{proj}_X\, w$. Since the points of M, different from $\mathrm{proj}_M\, w$, are contained in S, they all have to lie at distance $n-1$ from K_x, hence $|K_x|=|M|=l$, a contradiction. If $n=6$, then the same argument can be applied except if X is a point of degree 2 for which both $\mathrm{proj}_X\, x$ and $\mathrm{proj}_X\, w$ are lines. In this case, consider a line M at distance 3 from w and at distance 6 from the class K' containing X. Then similarly as above, we obtain $|K'|=|M|=l$, but this is a contradiction since by Lemma 4.4(i), $|K'|=|K_x| \leqslant l-1$. Finally, if X is a class containing exactly two points, we proceed as follows. Let Y be the element of γ at distance $(\delta(w,X)-3)/2$ from w and, if Y is not a class of size 2, M a line at distance $n-(\delta(w,X)+3)/2$ from Y such that $\mathrm{proj}_Y\, w \neq \mathrm{proj}_Y\, M \neq \mathrm{proj}_Y\, X$. Note that $\delta(w,M)=n-3$ and $\delta(M,X)=n$. The points of M different from $\mathrm{proj}_M\, w$ (and there are at least 2 of them) are contained in S, hence lie at distance $n-2$ from a point of X (different from $\mathrm{proj}_X\, w$). This is a contradiction, since $|X|=2$. If Y is a class of size 2, then we repeat the argument above with Y in the role of X. In this way, we obtain that there are no isolated points of degree k, and the lemma is proved. □

From now on, we assume that any isolated point has degree $k+1$.

Lemma 4.6. *If there exists a class X, with $1<|X|<g$, then Γ is a forgetful quadrangle of type* (II), *with $X=X_2$ and $1<|X_1|<s$.*

Proof. $n=4$: Let X be a class of size $<g$, and G a class of size g. Note that $\delta(X,G)=6$, because of Lemma 4.4(i). By projecting the points of G onto a line L intersecting X, we see that $|L| \geqslant g+1$. Suppose there is a second class X' with $|X'|<g$. If $\delta(X,X')=4$, then let (X,x,M,x',X') be a 4-path between X and X'. Every

point of G is collinear with a point of M different from x or x', and all these points are isolated (indeed, if $\text{proj}_M\, p$, with $p \in G$, is not isolated, then the class K' containing $\text{proj}_M\, p$ would satisfy $|G| = |K'| = |X|$ because of Lemma 4.4(i)). Hence there are at least g isolated points incident with M. Now let L be a line incident with x different from M, and assume there is a class K intersecting L, but not containing x. Because of Lemma 4.5(ii), every isolated point of M is collinear with a point of K, different from $\text{proj}_K\, x$. Hence $|K| \geqslant g+1$, a contradiction. So all the points incident with L different from x are isolated (this makes at least g isolated points incident with L). But since, again by Lemma 4.5(ii), every isolated point of L is collinear with a point of X', $|X'| \geqslant g+1$, a contradiction. So all points at distance 3 from X are isolated, and $\delta(X, X') = 6$. Let N be a line intersecting X. Since N is incident with at least g isolated points, $|X'| \geqslant g$, a contradiction. Hence X is the unique class of size $< g$, and a point is isolated if and only if it lies at distance 3 from X. By projecting the points of a class G of size g onto a line L intersecting X, we see that $l = g+1$. As in the proof of Lemma 4.5, it is now possible to define the following equivalence relation on the classes of size g:

$$K_1 \sim_C K_2 \;\Leftrightarrow\; \delta(K_1, K_2) = 6.$$

Note that it is possible to find two classes of size g at distance 4 (indeed, consider the points incident with a line at distance 4 from X), hence $\sim_C$ defines at least two equivalence classes. We associate a symbol ∞_i, $i = 1, \ldots, r$ to each equivalence class C_i of $\sim_C$. Now define the following geometry $\Delta = (\mathscr{P}', \mathscr{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i = 1, \ldots, r$. A line of Δ is either a line of Γ, the set of points of a class of size g together with the symbol ∞_j of its equivalence class or the set of points $\{\infty_1, \ldots, \infty_r\} \cup X$. Incidence is the incidence of Γ if defined or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (s, k) with $s = r + |X| - 1$, hence Γ is a forgetful quadrangle of type (II), with $1 < |X_1| < s$.

$n = 6$: We treat this case separately, because here the reasoning is slightly different from the general case.

Let by way of contradiction X be a class for which $|X| < g$. Let G be a class of size g. Then $\delta(X, G) \in \{4, 8\}$. If $\delta(X, G) = 8$, then $l \geqslant g+1$. Indeed, choose $x \in X$ and L a line incident with x. By projecting the points of G onto the line L, we see that $|L| \geqslant g+1$.

We next show that, if $\delta(X, G) = 4$, then $l = g$. Let (X, x, M, y, G) be the 4-path between X and G. Consider a line N concurrent with M not incident with x or y. Then the points incident with N different from $\text{proj}_M\, N$ are isolated (indeed, a class K intersecting M but not containing $\text{proj}_M\, N$ would satisfy $|X| = |K| = |G|$ by Lemma 4.4(i)). So N is incident with at least $g-1$ isolated points. Now let L be a line incident with y different from M, and suppose there is a class K intersecting L, different from G. Since $\delta(X, K) = 6$, $|K| = |X| < g$. By Lemma 4.5(ii), there is a bijection between the points of K different from $\text{proj}_L\, y$ and the points incident with N different from $\text{proj}_M\, N$. This implies that $|K| = l$, hence $|K| = g = l$, a contradiction. So all the points incident with L are isolated. Now let y' be a point of G different from y, and L' a line incident with y'. Not all points incident with L' different from y' can be isolated (since otherwise Lemma 4.5(ii) would imply $|X| = |L'| \geqslant g$), so there is a class K' intersecting L', $K' \neq G$. Then $|K'| = |L| \geqslant g$, and hence $|K'| = |L| = g$.

First, suppose $\delta(X,G) = 8$ (so $l \geqslant g+1$ in this case), and let $\gamma = (x,\dots,y)$ be a fixed 6-path between points $x \in X$ and $y \in G$. Let M be the element of γ at distance 3 from both x and y. Suppose first M is a class. If $|M| = g$, then $\delta(X,M) = 4$ implies $l = g$. If $|M| < g$, then $\delta(M,G) = 4$ implies $l = g$. Hence we obtain a contradiction in both cases, so M is necessarily a line. Clearly, the points incident with M different from the projection of x or y onto M are isolated. Also the point $z = \mathrm{proj}_M x$ is isolated. Indeed, this point cannot be contained in a class of size $< g$ (since such a class would lie at distance 6 from G). But z cannot lie in a class of size g either, since such a class would lie at distance 4 from X, implying $l = g$. Hence we have at least g isolated points incident with M. Now let N be a line incident with y different from $\mathrm{proj}_y M$, and suppose there is a class K intersecting N in a point different from y. Since there is a bijection between the points of M and the points of K, we obtain $|K| = g+1$, a contradiction. Hence N is incident with at least g isolated points, but this implies (using Lemma 4.5(ii)) that $|X| \geqslant g$, a contradiction. This shows that $\delta(X,G) \neq 8$.

Next, suppose $\delta(X,G) = 4$ (so $l = g$ in this case). Let again (X,x,M,y,G) be the 4-path between X and G. Let L'' be a line at distance 3 from y such that $\mathrm{proj}_y L''$ is a line different from M. Suppose there is a class K intersecting L'', but not containing the point $\mathrm{proj}_{L''} y$. Since $\delta(G,K) = 6$, $|K| = g$, but this contradicts $\delta(K,X) = 8$ and the previous paragraph. Hence all $g-1$ points of L'' different from $\mathrm{proj}_{L''} y$ are isolated. Consequently, $|X| = g$ (again by Lemma 4.5(ii)), a contradiction. We conclude that all classes have size g. Hence X cannot exist.

$n > 6$: Let by way of contradiction X be a class for which $|X| < g$. Let G be a class of size g. Then $\delta(X,G) \leqslant n+2$ and $\delta(X,G) \neq n$.

We first claim that if $\delta(X,G) = n+2$, $\gamma = (x,\dots,y)$ is an arbitrary n-path between points $x \in X$ and $y \in G$, and M is the element of γ at distance $n/2$ from both x and y, then either $n \equiv 2 \bmod 4$ and M is a class of size two or $n = 8$ and there does not exist a line incident with M different from $\mathrm{proj}_M x$ and $\mathrm{proj}_M y$. $(*)$ So assume $\delta(X,G) = n+2$, and suppose by way of contradiction that there is a path $\gamma = (x,\dots,y)$ between points $x \in X$ and $y \in G$ such that the element M (with M as above) is not a class of size two if $n \equiv 2 \bmod 4$ or such that there does exist a line incident with M different from $\mathrm{proj}_M x$ and $\mathrm{proj}_M y$ if $n = 8$. By projecting the points of G onto the line $\mathrm{proj}_x M$, we obtain $l \geqslant g+1$. Let L be a line at distance $n/2-3$ from M such that $\mathrm{proj}_M x \neq \mathrm{proj}_M L \neq \mathrm{proj}_M y$ (L exists because of the assumptions on M). Then the points incident with L different from $\mathrm{proj}_L M$ are isolated (indeed, a class K intersecting L but not containing $\mathrm{proj}_L M$ would satisfy $\delta(X,K) = \delta(G,K) = n$, so $|X| = |K| = |G|$, a contradiction). Now let N be a line incident with y different from $\mathrm{proj}_y M$. Since a class K' intersecting N, $K' \neq G$, would satisfy $|K'| = |L| \geqslant g+1$ (using Lemma 4.5(ii)), every point incident with N is isolated. But (again using Lemma 4.5(ii)) this implies $|X| \geqslant |N| - 1 \geqslant g$, a contradiction. This shows the claim.

We next show that $\delta(X,G) \neq j$, with $j \leqslant n/2+1$.

Suppose that $\delta(X,G) = 4$, and let (X,x,xy,y,G) be a 4-path between X and G. Let L be a line at distance $n-3$ from y such that $\mathrm{proj}_y L \neq xy$, and such that the element M of the path $[x,L]$ at distance $n/2$ from x is not a class of size 2 if $n \equiv 2 \bmod 4$ or such that there can be chosen a line incident with M not belonging to $[x,L]$ if $n=8$. Let r be a point incident with L, $r \neq \mathrm{proj}_L\, y$. Then r is isolated. Indeed, suppose by way of contradiction that r is contained in a class K. Since $\delta(G,K)=n$, $|K|=|G|=g$. This implies that $\delta(X,K)=n+2$. But now, by considering the n-path between $x\in X$ and $r\in K$ containing L, we see that K cannot exist because of $(*)$. So the line L is incident with at least $g-1$ isolated points. By Lemma 4.5(ii), there is a bijection between the points of $X\setminus\{x\}$ and the points incident with L different from $\mathrm{proj}_L\, y$, a contradiction with $|X|<g$.

We proceed by induction on $\delta(X,G)$. Let $4\leqslant k<n/2+1$ and suppose that $\delta(X,G') > k$, for any class G' of size g. Then we first claim that there does not exist a class G'' of size g at distance $n+2-k$ from X such that the element of the path $[X,G'']$ at distance $n/2-k+1$ from G'' is not a class of size 2 if $n\equiv 2 \bmod 4$ $(**)$. Suppose by way of contradiction a class G'' as above does exist. Let z be a point of G'' not belonging to $[X,G'']$, and L a line at distance $k-3$ from z such that $\mathrm{proj}_z\, L \neq G''$. All the points incident with L different from $\mathrm{proj}_L\, z$ are isolated. Indeed, if K' would be a class intersecting L, $\mathrm{proj}_L\, z\notin K'$, then $|K'|<g$ contradicts $\delta(K',G'')\leqslant k$ and the previous paragraph, but $|K'|=g$ contradicts $\delta(K',X)=n+2$ and $(*)$. So there are at least $g-1$ isolated points incident with L. By Lemma 4.5(ii), there is a bijection between the points of $L\setminus\{\mathrm{proj}_L\, z\}$ and the points of $X\setminus \mathrm{proj}_X\, G''$. Hence $|X|=g$, a contradiction. This shows $(**)$. Now we show that $\delta(X,G'')\neq k+2$, $k<n/2$, for any class G'' of size g. So suppose by way of contradiction that $\delta(X,G'')=k+2$, and let y be the element of G'' at distance $k+1$ from X. Let L be a line at distance $n-1-k$ from y, and $n-k$ from G'' such that $\mathrm{proj}_y\, L\neq \mathrm{proj}_y\, X$ and such that the element of the path $[G'',L]$ at distance $n/2-k+1$ from G'' is not a class of size 2 if $n\equiv 2 \bmod 4$. Then every point r incident with L different from $\mathrm{proj}_L\, y$ is isolated. Indeed, suppose by way of contradiction that r is contained in a class K. Since $\delta(K,G'')=n+2-k$, $(**)$ implies that $|K|=g$. But if $|K|=g$, then $\delta(X,K)=n+2$, which contradicts $(*)$. Hence r is isolated. Now again by Lemma 4.5(ii), there is a bijection between the points of L and X, implying $|X|\geqslant g$, a contradiction.

So we obtained the following: if $\delta(X,G)=j$, then $j>n/2+1$ and the element of a j-path between X and G at distance $j-(n/2+1)$ from G is a class K of size two. But now, applying this result on the classes G and K (note that $|K|<|G|$ and $\delta(K,G)<n/2-1$ if $j\neq n+2$) leads to the final contradiction. This shows that X cannot exist. □

From now on, we assume that every non-trivial class has the same size g. Put $d := l-g$.

Lemma 4.7. *If* $d=0$, *then* Γ *is a forgetful* n*-gon of type* (I).

Proof. Suppose $d=0$. Define the geometry $\Delta=(\mathscr{P},\mathscr{L}\cup\mathscr{C},\mathtt{I})$. Then Δ is a generalized n-gon. Indeed, we only have to check that a point p and a class K lie at distance at most $n-1$ from each other. Suppose by way of contradiction that $\delta(p,K)=n+1$. But then projecting the points of K onto an arbitrary line L incident with p shows that $|L|\geqslant g+1$, a contradiction. □

Lemma 4.8. *If $n=4$ and $d=1$, then Γ is a forgetful quadrangle of type* (II), *with* $|X_1| \in \{1, s+1\}$.

Proof. Suppose $d=1$, so $l=g+1$. As in Lemma 4.6, we define an equivalence relation $\sim_C$ on the classes (which are all of size g), and associate a symbol ∞_i, $i=1,\ldots,r$ to each equivalence class C_i of $\sim_C$.

Suppose first there are no isolated points. Then define the following geometry $\Delta = (\mathscr{P}', \mathscr{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i=1,\ldots,r$. A line of Δ is a line of Γ, the set of points of a class K together with the symbol of its equivalence class or the set of points $\{\infty_1,\ldots,\infty_r\}$. Incidence is the incidence of Γ if defined or symmetrized containment otherwise. Similarly as in Lemma 4.6, one shows that Δ is a finite generalized quadrangle of order (g,k), hence Γ is a forgetful quadrangle of type (II), with $|X_1| = s+1$.

Suppose now there is an isolated point w. We show that any two classes of size g lie at distance 6 (hence $\sim_C$ has only one equivalence class). Note first that no line incident with w is incident with only isolated points. Indeed, if L_0 would be a line incident with w full of isolated points, then for a class G of size g (which necessarily lies at distance 3 from w), Lemma 4.5(ii) implies that $|G|=|L_0|=g+1$, a contradiction. Now let $L_0,\ldots,L_k$ be the lines incident with w and X_0 a class of size g intersecting L_0. Since $l=g+1$, there is a unique point x_i incident with L_i, $i=1,\ldots,k$ that is not collinear with any point of X_0. Hence by Lemmas 4.5(ii) and 4.4(i), x_i is contained in a class X_i of size g for which $\delta(X_0,X_i)=6$. Since $\sim_C$ is an equivalence relation, also $\delta(X_j,X_{j'})=6$, for $j,j' \in \{1,\ldots,k\}$, $j \neq j'$. Suppose now there exists a class K of size g, $K \neq X_i$, $i=0,\ldots,k$. Then without loss of generality, we can assume that K intersects L_0 in the point y. Because of the construction of the classes X_i, the point y lies at distance 3 from every class X_i, $i=0,\ldots,k$. Hence there exists a line N_i incident with y, $i=0,\ldots,k$ such that $\delta(N_i,X_i)=2$ (note that all these lines N_i are different because the classes X_i mutually lie at distance 6). But since y is not isolated, it has degree k, a contradiction. So $\sim_C$ has a unique equivalence class with associated symbol ∞. Now define the following geometry $\Delta = (\mathscr{P}', \mathscr{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or the symbol ∞. A line of Δ is either a line of Γ or the set of points of a class K together with the symbol ∞. Incidence is the incidence of Γ if defined or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (g,k), hence Γ is a forgetful quadrangle of type (II), with $|X_1|=1$. □

Now Theorem 2.2 is proved.

5. Short forgetful quadrangles

5.1. General properties

Let Γ be a finite short forgetful quadrangle, with parameters (g,k,d). We denote by l the common size of lines (and by shortness, $l \geqslant g+2$). Recall that an isolated point of Γ lies at distance 3 from any class, and that, if two classes K_1 and K_2 lie at distance 4, then any point of K_1 (K_2) lies at distance 3 from K_2 (K_1).

Lemma 5.1. *Every line of Γ is incident with a constant number ρ of isolated points.*

Proof. From $l \geqslant g+2$, it is easily seen that there does not exist a line of Γ only containing isolated points. Let K be a fixed class of size g, and L_1 (L_2) a line intersecting K and incident with ρ_1 (ρ_2) isolated points. Each point of Γ at distance 5 from K lies at distance 3 from L_1 and L_2, and each line at distance 4 from K is incident with exactly d points at distance 5 from K. So counting the number of points of Γ at distance 5 from K, we obtain

$$\rho_1 kd + (g+d-\rho_1-1)(k-1)d = \rho_2 kd + (g+d-\rho_2-1)(k-1)d,$$

hence $\rho_1 = \rho_2$. Now let K_1 (K_2) be a class of size g such that every line intersecting K_1 (K_2) is incident with exactly ρ_1 (ρ_2) isolated points. We count the number of points of Γ:

$$\begin{aligned} |\mathscr{P}| &= g + gk(g+d-1) + \rho_1 kd + (g+d-\rho_1-1)(k-1)d \\ &= g + gk(g+d-1) + \rho_2 kd + (g+d-\rho_2-1)(k-1)d, \end{aligned}$$

implying $\rho_1 = \rho_2$. This shows the lemma. □

Lemma 5.2. (i) *Either $\rho = 0$ or $\rho = g-(d-1)(k-1)$.*
(ii) *If $\rho \neq 0$, then $|\mathscr{L}| = gk(k+1)$. If $\rho = 0$, then $|\mathscr{L}| = k((d-1)(k-1)+gk)$.*

Proof. Let $\mathscr{I}$ be the set of isolated points. If K is a class of size g, then every isolated point lies at distance 3 from K, hence $|\mathscr{I}| = gk\rho$. Also, every line of Γ intersects K, or lies at distance 3 from a fixed point of K, hence

$$|\mathscr{L}| = gk + k(\rho k + (g+d-\rho-1)(k-1)).$$

Counting the number of pairs (i,L), $i \in \mathscr{I}$, $L \in \mathscr{L}$, $i\,\mathtt{I}\,L$, we obtain:

$$gk\rho(k+1) = (gk + k(\rho k + (g+d-\rho-1)(k-1))\rho.$$

If $\rho \neq 0$, this simplifies to $\rho = g-(d-1)(k-1)$, showing (i). Now by using this in the expression for $|\mathscr{L}|$ above, we obtain (ii). □

Define the following graph G_Γ. The vertices of G_Γ are the classes of Γ. Two vertices are adjacent if and only if the corresponding classes lie at distance 6.

Lemma 5.3. (i) *If $\rho \neq 0$, then G_Γ is a*

$$\mathrm{srg}((k+1)(kd+1-k), kd, k-1, d)).$$

(ii) *If $\rho = 0$, then, with $f = d(d-1)(k-1)/g$, G_Γ is a*

$$\mathrm{srg}(1+k(g+d-1)+(k-1)d+f, (k-1)d+f, k-d-1+f, f).$$

Proof. We determine the parameters of the dual or complementary graph G_Γ^C. The number of classes follows from $|\mathscr{P}| = g + gk(g+d-1) + (g+d-1-\rho)(k-1)d + \rho kd$ and $|\mathscr{I}| = gk\rho$, with $\mathscr{I}$ the set of isolated points. Now let K be a fixed class, and $r \in K$. A class K' lies at distance 4 from K if and only if K' contains a point collinear with r, hence there are $k(g+d-1-\rho)$ classes lying at distance 4 from K. Let K' be a fixed class, with $\delta(K,K') = 4$, and (K,r,R,r',K') a 4-path between K and K'. A class K'' lies at distance 4 from both K and K' if and only if K'' intersects R (not in r or r' of course) or K'' intersects a line L incident with r, $L \neq R$, in a point v ($v \neq r$) that lies at distance 3 from K'. Note that every isolated point incident with such a line L necessarily lies at distance 3 from K'. Hence there are $g+d-2-\rho+(k-1)(g-1-\rho)$ classes at distance 4 from K and K'. Let finally $\bar{K}$ be a class at distance 6 from K. A class K'' lies at distance 4 from both K and $\bar{K}$ if and only if there exists a line N incident with r such that K'' intersects N in a point at distance 3 from $\bar{K}$. Since any line N incident with r is incident with exactly $g-\rho$ non-isolated points at distance 3 from $\bar{K}$, we have in total $k(g-\rho)$ classes at distance 4 from K and $\bar{K}$. □

Let Γ be a short forgetful quadrangle without isolated points. For a point x of Γ, we denote by K_x the class containing x. We define the following relations $\mathscr{R} = (R_0, R_1, R_2, R_3, R_4)$ on $\mathscr{P}$:

$$R_0 = \{(x,x) \mid x \in \mathscr{P}\},$$

$$R_1 = \{(x,y) \in \mathscr{P}^2 \mid x \perp y\},$$

$$R_2 = \{(x,y) \in \mathscr{P}^2 \mid \delta(x,y) = 4 \text{ and } \delta(K_x,K_y) = 4\},$$

$$R_3 = \{(x,y) \in \mathscr{P}^2 \mid \delta(x,y) = 4 \text{ and } \delta(K_x,K_y) = 6\}.$$

Then it is easy to check that the pair $(\mathscr{P}, \mathscr{R})$ is an association scheme. Expressing the Krein conditions, one obtains the inequality $k \leqslant (l-1)^2$.

5.2. *Examples of short forgetful quadrangles*

5.2.1. *Subquadrangle type*

Let Δ be a finite generalized quadrangle of order (s,t), having a (possibly weak) subquadrangle Δ' of order (s',t), $s' \geqslant 1$ (for examples of such quadrangles, we refer to [10]). Then we define the following geometry $\Gamma = (\mathscr{P}, \mathscr{L}, \mathtt{I}, \sim)$. The points of Γ are the points of Δ not contained in the subquadrangle Δ'. The lines of Γ are the lines of Δ that do not intersect Δ'. Incidence is the incidence of Γ. Two points of Γ are equivalent if and only if they are incident with a line of Δ'. So the equivalence classes correspond to the sets $\Delta_1(L) \setminus \Delta'_1(L)$, with L a line of Δ that is also a line of the subquadrangle Δ'. It is now easy to see that Γ is a short forgetful quadrangle, with parameters $g = s - s'$, $k = t$, $d = s' + 1$ and $\rho = s - s't$. Note that in this example, G_Γ corresponds with the line graph of Δ'. If a short forgetful quadrangle Γ arises from this construction, we say that Γ is of *subquadrangle type*.

5.2.2. Ovoid type

An *ovoid* of a generalized quadrangle Γ of order (s,t) is a set of points such that each line is incident with a unique point of Γ. The ovoid $\mathcal{O}$ is called *regular* if for any two points o_1 and o_2 of $\mathcal{O}$, $|\{o_1,o_2\}^{\perp\perp}|=t+1$ and $\{o_1,o_2\}^{\perp\perp}\subset\mathcal{O}$.

Let Δ be a finite generalized quadrangle of order (s,t), admitting a regular ovoid $\mathcal{O}$. Then we define the following geometry $\Gamma=(\mathcal{P},\mathcal{L},\mathtt{I},\sim)$. The points of Γ are the points of Δ not belonging to $\mathcal{O}$. The lines of Γ are the lines of Δ. Two points of Γ are equivalent if and only if they are contained in a set $\{o_1,o_2\}^{\perp}$, for $o_1,o_2\in\mathcal{O}$. Incidence is the incidence of Δ. Then Γ is a forgetful quadrangle with parameters $g=k=t+1$, $d=s-t-1$ and $\rho=0$. If a short forgetful quadrangle Γ arises from this construction, we say that Γ is of *ovoid type*.

Note that a forgetful quadrangle of ovoid type has the following property:

(O) If there is a line intersecting three different (possibly trivial) classes K, K' and K'', then any line intersecting two of these classes, also intersects the third one.

The known regular ovoids giving rise to a short forgetful quadrangle (thus with $d\geqslant 2$) are the following.

- Let Δ be the $(q+1)\times(q+1)$-grid (so a thin generalized quadrangle of order $(q,1)$), $q\geqslant 4$, and $\mathcal{O}$ the points on one of the diagonals. Then the associated short forgetful quadrangle has parameters $g=k=2$ and $l=q$.
- Let Δ be the generalized quadrangle $\mathsf{H}(3,q^2)$, and $\mathcal{O}$ the points of a Hermitian curve $\mathcal{H}$ lying on $\mathsf{H}(3,q^2)$ (for the definition of this quadrangle, we refer to [10, Section 3.1]). Then the associated short forgetful quadrangle has parameters $g=k=q+1$ and $l=q^2$.

Application: The other known examples of regular ovoids all occur in generalized quadrangles of order $(q+1,q-1)$ (see for instance [3] Section 2.6.2). By applying the above construction on a generalized quadrangle Δ of order $(q+1,q-1)$ admitting a regular ovoid $\mathcal{O}$, one obtains a forgetful quadrangle Γ with $g=k=q$ and $d=1$. By Lemma 4.8, Γ then arises from a generalized quadrangle Δ' by applying construction (II), with $|X_1|=s+1$. Clearly, Δ' has order (q,q). Since property (O) holds, the line L of Δ' corresponding to the set X_1 has to be regular. One can now easily see that the quadrangle Δ arises by applying the construction of Payne (as explained in [10], Section 3.1.4) on the generalized quadrangle Δ' with L as regular line. So any generalized quadrangle of order $(q+1,q-1)$ having a regular ovoid $\mathcal{O}$, arises from the construction of Payne. This result can be found in [9], Section 3.

Note: The classes of a short forgetful quadrangle of ovoid type are not lines of the ambient generalized quadrangle. However, it remains possible that also these examples arise from generalized quadrangles in the 'usual' way, see Section 3.6.4 in [8].

5.3. Characterization results

Lemma 5.4. *Let Γ be a short forgetful quadrangle with $k\leqslant g$ and satisfying property* (O). *Then Γ is of ovoid type.*

Proof. Let Γ be a short forgetful quadrangle satisfying the conditions of the lemma. We first claim that Γ does not contain isolated points. Indeed, suppose first $0 < \rho < l-1$ and let w be an isolated point. Let L be a line incident with w. Then there are at least two non-trivial classes K and K' intersecting L. Since K and K' lie at distance 4, there is a line N different from L intersecting K and K', hence, by property (O), N is incident with w. Now a 'triangle' arises, the contradiction. Suppose $\rho \geqslant l-1$. Let L be a line at distance 4 from a non-trivial class K. Note that $l-1 \geqslant g+1$. Now by Lemma 4.5, every isolated point of L lies at distance 3 from K, implying $|K| \geqslant g+1$, again a contradiction. Hence $\rho = 0$.

We now prove that $g=k$. Let L_1 be a line of Γ and $K_1,\dots,K_l$ the classes intersecting L_1. Then by Lemma 4.4(i) there exists a set of lines $\mathscr{S} = \{L_1,\dots,L_g\}$ such that every line of $\mathscr{S}$ intersects $K_1,\dots,K_l$. Since every line of Γ intersects at most one line of $\mathscr{S}$ (otherwise a 'triangle' would arise), there are $(g+d)g(k-1)$ lines intersecting a line of $\mathscr{S}$. Hence $g+(g+d)g(k-1) \leqslant |\mathscr{L}| = k((d-1)(k-1)+gk)$. Using $k \neq 1$, this simplifies to

$$(g-k)(g+d-1) \leqslant 0,$$

hence $g \leqslant k$. Since also $k \leqslant g$, we obtain $g=k$. Note that in particular, this implies that every line of Γ intersects exactly one line of $\mathscr{S}$.

Define the following equivalence relation on the set of lines of Γ. Two lines L_1 and L_2 of Γ are equivalent if and only if there exist at least two classes intersecting both L_1 and L_2 (the fact that this is an equivalence relation immediately follows from property (O)). To each equivalence class C_i of lines of Γ, we associate a symbol ∞_i, $i=1,\dots,r$. Then define the following geometry $\Delta = (\mathscr{P},\mathscr{L},\mathtt{I})$. A point of Δ is either a point of Γ or a symbol ∞_i. The lines of Δ are the lines of Γ. Incidence is the incidence of Γ if defined, and symmetrized containment otherwise. Then it is easily checked that Δ is a generalized quadrangle of order $(l,k-1)$.

The points ∞_i form an ovoid $\mathscr{O}$ of Δ. Moreover, $\mathscr{O}$ is a regular ovoid. Indeed, let ∞_i and ∞_j be two different equivalence classes of lines. Let L and L' be two lines of the class ∞_i. Then by the second paragraph of the proof, L (L') meets a unique line M (M') of the class ∞_j in a point r (r'). Note that $M \neq M'$, since otherwise a 'triangle' would arise. Suppose that r and r' are not equivalent. Let K be the class containing r. Since $L \sim L'$, K intersects the line L' in a point a, $a \neq r$, and since $M \sim M'$, K intersects M' in a point b, $r \neq b \neq r'$. But now a 'triangle' arises through the points a, b and r', a contradiction. Hence the lines of ∞_i and ∞_j meet in the set of points of a class K. From $\{\infty_i,\infty_j\}^{\perp} = K$ follows easily that the ovoid $\mathscr{O}$ is regular. Now clearly, Γ is of ovoid type. □

Theorem 5.5. *Let Γ be a short forgetful quadrangle without isolated points, such that $g=k$ and $l \geqslant (g-1)^2$. Then Γ is of ovoid type.*

Proof. Let Γ be a short forgetful quadrangle without isolated points, such that $g=k=: q+1$, and $l = q^2+r$, $r \geqslant 0$. We show that Γ has property (O). Suppose first $g=2$. Let L be a line of Γ, and $K_i = \{r_i, r_i'\}$, $i=1,2,3$, three different classes intersecting the line L in the point r_i. Since $g=2$, $r_1'r_2'$, $r_1'r_3'$ and $r_2'r_3'$ are lines. This gives rise to a triangle, unless $r_1'r_2'r_3'$ is a line. So in this case, property (O) is satisfied. From now on, we assume $g \geqslant 3$. Let K be a fixed class, and put $K=\{a_0,\ldots,a_q\}$. Let L be a fixed line incident with a_0 and $C=\{K_1,\ldots,K_{q^2+r-1}\}$ the set of classes different from K intersecting L. Note that, since $\delta(K,K_i)=4$, $i=1,\ldots,q^2+r-1$, every point of K is collinear with exactly one point of each class K_i, $i=1,\ldots,q^2+r-1$. Put b_i, $i=0,\ldots,q$ the point of K_1 collinear with the point a_i of K. We claim that if at least one class of C intersects a line concurrent with K, then at least $q+r-1$ classes of C intersect this line. Indeed, consider the class K_1 and the line a_1b_1. Let V be the set of the q^2+r-2 points of $K_2,\ldots,K_{q^2+r-1}$ that are collinear with b_1 (these points exist, since $\delta(K_1,K_i)=4$, $i=2,\ldots,q^2+r-1$). No point of V is incident with a line a_ib_i, $i=0,2,\ldots,q$, since these lines already are incident with a point that is equivalent with b_1 (otherwise a 'triangle' would arise). Hence every point of V is incident either with a_1b_1 or with a line incident with a_i, different from a_ib_i, with $2 \leqslant i \leqslant q$. Since a line incident with a_i, $2 \leqslant i \leqslant q$, different from a_ib_i can be incident with at most one point of V, the line a_1b_1 is incident with at least $q^2+r-2-q(q-1)=q+r-2$ points of V. So at least $q+r-1$ classes of C intersect the line a_1b_1. This shows the claim. Note also that if the line a_1b_1 is incident with exactly $q+r-2$ points of V, then each line incident with a_i, $2 \leqslant i \leqslant q$, different from a_ib_i, is incident with exactly one point of V $(*)$.

Suppose first that there is a line M concurrent with K that is intersected by exactly $q+r-1$ classes of C. Without loss of generality, we can assume that $M=a_1b_1$. By $(*)$, this means that every line incident with a_i, $i=2,\ldots,q$, intersects at least one, and hence at least $q+r-1$ classes of C. So in total, a point a_i, $2 \leqslant i \leqslant q$, lies at distance 3 from at least $(q+1)(q+r-1)$ classes of C. Hence $q^2+r-1 \geqslant (q+r-1)(q+1)$, implying that $r=0$ and that every line incident with a_i, $2 \leqslant i \leqslant q$, intersects exactly $q-1$ classes of C. By symmetry, also every line incident with a_1 intersects exactly $q-1$ classes of C. Let, without loss of generality, $C'=\{K_1,\ldots,K_{q-1}\}$ be the set of $q-1$ classes of C intersecting the line a_1b_1. The point b_1 has to be collinear with a point $p_i \in K_i$, for $q \leqslant i \leqslant q^2-1$ (note that these K_i are the classes of $C \setminus C'$). By $(*)$ each line incident with a_i, $2 \leqslant i \leqslant q$, different from a_ib_i is incident with a point collinear with b_1 and belonging to a class of $C \setminus C'$. Let C_1 be the set of $q-1$ classes of C intersecting the line a_2b_2, and C_2 the set of q^2-q classes intersecting a_2b_2, different from K and not belonging to C_1. Since exactly $q-1$ classes intersecting a_2b_2 different from K also intersect the line a_0b_0 (namely the classes of C_1), every line concurrent with K but not incident with a_2 is intersected by exactly $q-1$ classes of $C_1 \cup C_2$ (this follows from the argument above, applied on $C_1 \cup C_2$ instead of C). Now since K_1 lies at distance 4 from every class of C_2, there is a set $V'=\{v_1,\ldots,v_{q^2-q}\}$ of q^2-q points collinear with b_1 and belonging to a class of C_2. These points cannot be collinear with a_i, $i \geqslant 2$ (since the points collinear with b_1 incident with lines incident with a_i, $i \geqslant 2$, belong to classes of C, and $C \cap C_2=\emptyset$). So all the points of V' are collinear with a_0 (and there are at most q such points) or are incident with the line a_1b_1. So at least q^2-2q+1 classes of $C_1 \cup C_2$ intersect the line a_1b_1. This implies $q^2-2q+1 \leqslant q-1$, hence $q=1$ (then $g=2$, a contradiction) or $q=2$. But if $q=2$, then $l=4$, hence $d=l-g=1$, a contradiction.

We may now assume that if at least one class of C intersects a line concurrent with K, then at least $q+r$ classes of C intersect this line. For each point a_i, $2 \leqslant i \leqslant q$, let A_i be the number of lines incident with a_i that do not intersect any class of C. If $A_i = 0$ for some i, then every line incident with a_i is incident with at least $q+r$ points belonging to one of the q^2+r-1 classes of C. This would imply that $(q+1)(q+r) \leqslant q^2+r-1$, a contradiction. Hence $M := \min_{2 \leqslant i \leqslant q} A_i \neq 0$. We next claim that every line incident with a_i, $2 \leqslant i \leqslant q$, that intersects at least one class of C, intersects at least $(q-1)(M+1)+r$ classes of C. Let V again be the set of the q^2+r-2 points of $K_2, \ldots, K_{q^2+r-1}$ that are collinear with b_1. Now the number of points of V that are not incident with the line a_1b_1 is at most $(q-1)(q-M)$, hence there are at least $q^2+r-1-(q-1)(q-M) = (q-1)(M+1)+r$ classes of C that intersect the line a_1b_1. This shows the claim. Now consider a point a_j of K, $2 \leqslant j \leqslant q$, for which $A_j = M$. Then there are $q+1-M$ lines incident with a_j such that each of these lines intersects at least $(q-1)(M+1)+r$ classes of C, hence

$$(q+1-M)(qM+q+r-M-1) \leqslant q^2+r-1.$$

Combined with $0 < M \leqslant q$, we obtain $M = q$, meaning that if a line concurrent with K intersects a class of C, it intersects every class of C. This is exactly property (O), so the conditions of Lemma 5.4 are satisfied, and Γ is of ovoid type. □

Corollary 5.6. *If Γ is a short forgetful quadrangle without isolated points, satisfying $k = g \neq 2$, then $l \leqslant (g-1)^2$.*

Proof. Suppose by way of contradiction that Γ is a short forgetful quadrangle without isolated points, with $k = g =: q+1$ and $l = (g-1)^2 + r = q^2 + r$, $r \geqslant 1$. Then because of Theorem 5.5, Γ is of ovoid type, and there exists a generalized quadrangle of order (l, q), with $l > q^2$, a contradiction since $q \neq 1$. □

Lemma 5.7. *Let Γ be a short forgetful quadrangle for which G_Γ is the line graph of a generalized quadrangle. Then Γ is of subquadrangle type.*

Proof. Let Γ be a short forgetful quadrangle for which G_Γ is the line graph of a generalized quadrangle Δ' of order (s,t). Then one calculates (using Lemma 5.3) that $s = d-1$, $t = k$ and that, if $\rho = 0$, $g = (k-1)(d-1)$. Each point of Δ' corresponds to a (maximal) clique of size $k+1$ in the graph G_Γ, so to $k+1$ classes lying at distance 6 from each other. Also, every two classes at distance 6 from each other are contained in a unique clique of size $k+1$, and every class belongs to exactly d $(k+1)$-cliques. We denote the points of Δ' by ∞_i, $i = 1, \ldots, v$. Now define the following geometry $\Delta = (\mathscr{P}, \mathscr{L}, \mathtt{I})$. The points of Δ are the points of Γ and the symbols ∞_i. There are two types of lines of Δ. The lines of type (A) are the lines of Γ. A line of type (B) consists of the points of a class K of Γ, together with the symbols $\infty_1, \ldots, \infty_d$ of the $(k+1)$-cliques containing K. Incidence is the incidence of Γ if defined, and symmetrized containment otherwise. Then one easily checks that Δ is a generalized quadrangle of order $(l-1, k)$. Since Δ' is a subquadrangle of Δ with $t = k$, Γ is of subquadrangle type. □

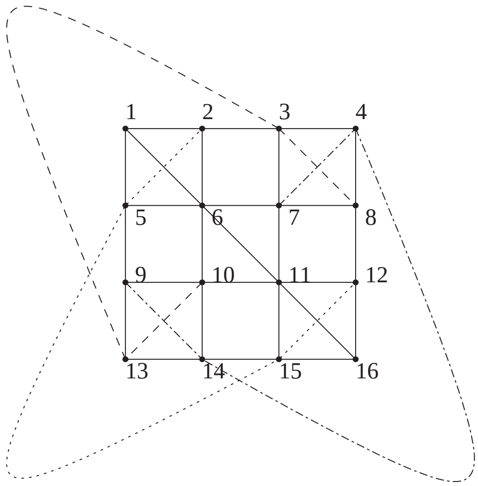

Fig. 3. The complement of the Shrikhande graph.

Application 5.8. *Let Γ be a short forgetful quadrangle with parameters $(g,k,2)$.*

(i) *If Γ contains isolated points, then Γ is of subquadrangle type.*
(ii) *If $g = k - 1$, then Γ is of subquadrangle type.*

Proof. Let Γ be a short forgetful quadrangle with $d = 2$, and suppose that either Γ contains isolated points, or the parameters of Γ satisfy $g = k - 1$ (which implies that there are no isolated points because of Lemma 5.2(i)). Put $l = s + 1$, $g = s - 1$ and $k = t$. Then in both cases, G_Γ is a srg $((t+1)^2, 2t, t-1, 2)$.

Suppose first $t \neq 3$. Then G_Γ is the $(t+1) \times (t+1)$-grid (or equivalently, the line graph of a thin generalized quadrangle of order $(1,t)$) (see [1] or [4, Theorem 4]). By Lemma 5.7, Γ is of subquadrangle type, which proves the theorem in this case. Suppose now $t = 3$. Then G_Γ is a srg (16,6,2,2). Any strongly regular graph with these parameters is either the 4×4-grid or the Shrikhande graph (see for example [2, Theorem 3.12.4]). In the former case, the theorem again follows as before, so assume that G_Γ is the Shrikhande graph. We show that this leads to a contradiction. For convenience, we work with the complementary graph G_Γ^{C}. Label the classes of G_Γ^{C} with $K_1, \ldots, K_{16}$ as in Fig. 3. Remember that Γ is a short forgetful quadrangle with $g = s - 1$, $l = s + 1$, $k = 3$ and $\rho = s - 3$, so every line is incident with exactly 4 non-isolated points. Also, two vertices are adjacent in G_Γ^{C} if and only if the corresponding classes lie at distance 4 from each other. We now make the following observations.

(a) Every vertex v of G_Γ^{C} is contained in exactly 3 maximal cliques of size 4 (which only intersect in the vertex v). This implies the following property for Γ: if there exists a line of Γ intersecting the four classes K_i, K_j, K_m and K_n, then there exist exactly $g = s - 1$ lines intersecting the classes K_i, K_j, K_m and K_n.

(b) If p is an isolated point of Γ, then p lies at distance 3 from every non-trivial class, so the point p will determine four cliques in G_Γ^C, each of size 4 (corresponding to the four lines incident with p). Hence p will determine a partition of the vertices of G_Γ^C into four disjoint maximal cliques.

We will call a line intersecting the four classes K_i, K_j, K_m and K_n, an (i,j,m,n)-line.

Suppose $g \neq 2,4$. Note that incident with every point of K_1, there is a line of type (1,2,3,4), (1,5,9,13) and $(1,6,11,16)$. Since $g \neq 2$, the geometry Γ contains isolated points. Let $a_1 \in K_1$ and let r be an isolated point incident with the (1,2,3,4)-line L incident with a_1. Each line incident with r is incident with $s-3$ isolated points, which necessarily lie at distance 3 from K_1, and hence each line incident with r different from L is incident with exactly 2 non-isolated points at distance 3 from K_1. So in total there are 6 non-isolated points collinear with r, at distance 3 from K_1 and not incident with L. These 6 points belong to the classes K_5, K_6, K_9, K_{11}, K_{13} and K_{16} (since according to observation (b), the point r determines the partition $(1,2,3,4),(5,6,7,8)$, $(9,10,11,12)$, $(13,14,15,16)$). Hence we need at least 6 lines concurrent with K_1, not incident with a_1 and not of type (1,2,3,4), implying that $g \geqslant 4$. Now we claim that every line incident with r different from L intersects $g-3$ lines of type $(1,2,3,4)$ different from L. Indeed, consider for example the (5,6,7,8)-line R incident with r. The points b_5 and b_6 of K_5 and K_6 incident with R lie at distance 3 from K_1, hence R has to intersect a (1,5,9,13)-line and a (1,6,11,16)-line concurrent with K_1. But R cannot intersect two (1,5,9,13)-lines concurrent with K_1, since this would give rise to a 'triangle'. Similarly R cannot intersect two (1,6,11,16)-lines concurrent with R. Hence every line concurrent with both R and K_1 and not incident with b_5 or b_6 is necessarily a (1,2,3,4)-line. This shows the claim. So each of the three lines incident with r different from L intersects $g-3$ lines of type $(1,2,3,4)$ different from L. Since all these lines of type (1,2,3,4) at distance 3 from r are different, $3(g-3) \leqslant g-1$, implying that $g=4$. This contradicts our assumption. The case $g=2$ is left to the reader. The case $g=4$ can be ruled out by a similar but more detailed analysis as above of the classes at distance 4 from a given class, see [8], Theorem 3.6.10. □

Application 5.9. *Let Γ be a short forgetful quadrangle with $k=2$. If Γ contains isolated points, then Γ is of subquadrangle type. If Γ does not contain isolated points, then Γ is of ovoid type. In both cases, the corresponding generalized quadrangles are uniquely determined.*

Proof. Let Γ be a short forgetful quadrangle with $k=2$. Suppose first that Γ contains isolated points. Since every two adjacent vertices of G_Γ are contained in a clique of size $k+1=3$ (see Lemma 5.3(i)), Γ is of subquadrangle type by Lemma 5.7. Using the notation of Section 5.2, the associated generalized quadrangle Δ has order $t=2$. Hence Δ is isomorphic with $\mathsf{H}(3,4)$. Suppose now Γ does not contain isolated points. If $g=k=2$, then Γ is of ovoid type, by Lemma 5.4. Again using the notation of Section 5.2, we see that the associated generalized quadrangle Δ is an $(l+1)\times(l+1)$-grid. We now show that the case $g>2$ leads to a contradiction.

The cases $g=3$ and 4 can easily be ruled out by a detailed analysis the classes at distance 4 from a given class, see [8, Application 3.6.11]). From now on, we assume $g \geqslant 5$. Let $G=\{r_1,r_2,\dots,r_g\}$ be a class and L a line at distance 4 from G. For $i=1,\dots,g$, put $a_i=\mathrm{proj}_L\, r_i$, K_i the class containing a_i, $R_i=a_ir_i$ and R_i' the line incident with r_i different from R_i. Clearly, K_2 intersects the lines R_i', $i\neq 2$, K_3 intersects the lines R_i', $i\neq 3$, and K_4 intersects the lines R_i', $i\neq 4$. Since K_3 and K_4 both have a point incident with R_1', $\delta(K_3,K_4)=4$. Let a_3' be the point of K_3 incident with the line R_4', and a_4' the point of K_4 incident with the line R_3'. The line L and the lines R_i', $i\neq 3,4$, intersect both K_3 and K_4. So we already have $g-1$ lines intersecting K_3 and K_4. Since none of these lines is incident with the points a_3' or a_4', a_3' and a_4' have to be collinear. Put $L'=a_3'a_4'$. Clearly, the line L' intersects R_1, say in a point of a class B. Note that $B\neq K_i$, $i=1,\dots,g$, and that B does not intersect the line L. Since L' intersects the lines R_2 and R_i, $i=5,\dots,g$, the class B intersects the lines R_2', R_3, R_4 and R_i', $i=5,\dots,g$. Now $\delta(B,K_2)=4$, since these classes have collinear points incident with the line R_5'. This is a contradiction, since the point a_2 is not collinear with any point of B (indeed, the line L incident with a_2 does not intersect R_i', $i=1,\dots,g$). $\square$

6. Addendum: the infinite case

We describe a free construction yielding examples of infinite forgetful n-gons, for all n, that are not of any of the types given earlier. The construction is similar to the one of free polygons, see [12] or [13], Section 1.3.13. Let $\Gamma=(\mathscr{P},\mathscr{L},\mathtt{I},\sim)$ be an incidence structure, with $\sim$ equivalence relation on the point set $\mathscr{P}$. As before, we define distances and forgetful paths in Γ. Then Γ is a *partial forgetful n-gon*, $3\leqslant n$, if the following axioms are satisfied:

(PFP1) If $x,y\in\mathscr{P}\cup\mathscr{L}$ and $\delta(x,y)=k<n$, then there is a unique forgetful path of length k joining x to y.

(PFP2) The geometry contains a closed path of length $2n$ consisting of points and lines, such that every point of this path is incident with at least three lines, and every line of this path is incident with at least three points.

Now fix n and let $\Gamma^{(0)}$ be a partial forgetful n-gon, which is not a forgetful n-gon. We define a geometry $\Gamma^{(i)}$, $i\in\mathbb{N}$, by induction on n as follows. The geometry $\Gamma^{(i)}$ arises from $\Gamma^{(i-1)}$ by adding a completely new forgetful path of length $n-1$ between every two elements of $\mathscr{P}\cup\mathscr{L}$ of $\Gamma^{(i)}$ which lie at distance $n+k$ from each other, k odd if k is finite (and the paths respect the axioms of the geometry, i.e. a point is incident with at most one non-trivial class), and by adding a finite number of points to the non-trivial classes of $\Gamma^{(i)}$. It is now readily seen that, for $i\geqslant 1$, $\Gamma^{(i)}$ is a partial forgetful n-gon, and the union Γ of the family $\{\Gamma^{(i)}\colon i\in\mathbb{N}\}$ is a forgetful n-gon. Suppose that the partial forgetful n-gon $\Gamma^{(0)}$ contains two finite non-trivial equivalence classes C_1, C_2 and that in the following steps, we do not add points to C_1 or C_2. Suppose moreover that in each step, we add a point to a fixed class C_0 of $\Gamma^{(0)}$, $C_0\neq C_1,C_2$. Then the obtained forgetful n-gon Γ is not of any of the types (I), (II) or (III) as constructed in

Section 2 (considering the appropriate infinite analogs of these constructions), although, for n even, it satisfies the additional assumption that at least one equivalence class of points (namely C_0) has the same size of the lines. This shows that a classification of all forgetful polygons without the finiteness restriction is hopeless.

References

[1] R.C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, Pacific J. Math. 13 (1963) 389–419.

[2] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer, Berlin, 1989.

[3] B. De Bruyn, Constructions and Characterizations of Near Polygons, Ph.D. Thesis, Ghent University, 2000.

[4] F. De Clerck, H. Van Maldeghem, Some classes of rank 2 geometries, in: F. Buekenhout (Ed.), Handbook of Incidence Geometry, Buildings and Foundations, North-Holland, Amsterdam, 1995, pp. 443–475 (Chapter 10).

[5] P. Dembowski, Semiaffine Ebenen, Arch. Math. 10 (1962) 120–131.

[6] P. Dembowski, Finite Geometries, Springer, Basel, 1968.

[7] A. Feit, G. Higman, The nonexistence of certain generalized polygons, J. Algebra 1 (1964) 114–131.

[8] E. Govaert, Some contributions to the (geometric) theory of generalized polygons, Ph.D. Thesis, Ghent University, 2001.

[9] S.E. Payne, M.A. Miller, Collineations and characterizations of generalized quadrangles of order $(q+1, q-1)$ (Combinatorics '98 (Mondello)), Rend. Circ. Mat. Palermo (2) Suppl. No. 53 (1998) 137–166.

[10] S.E. Payne, J.A. Thas, Finite Generalized Quadrangles, in: Research Notes in Mathematics, Vol. 104, Pitman, London, 1984.

[11] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, Inst. Hautes Etudes Sci. Publ. Math. 2 (1959) 14–60.

[12] J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, Invent. Math. 43 (1977) 283–295.

[13] H. Van Maldeghem, Generalized Polygons, Monographs in Mathematics, Vol. 93, Birkhäuser Verlag, Basel, Boston, Berlin, 1998.