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Sharp homogeneity in some generalized polygons

By

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Abstract. We show that, if a collineation group G of a generalized $(2n+1)$-gon Γ has the property that every symmetry of any apartment extends uniquely to a collineation, then Γ is the unique projective plane with 3 points per line (the Fano plane) and G is its full collineation group. A similar result holds if one substitutes "apartment" with "path of length $2k \leqq 2n+2$".

1. Introduction. The classification of geometries satisfying certain homogeneity conditions has a rich history. Especially in the finite case, many classes of geometries are characterized by a transitive action on certain substructures. For example, Ostrom & Wagner [3] proved in 1959 that a finite projective plane admitting a collineation group that acts doubly transitively on the point set is necessarily Pappian. In the infinite case, however, such transitivity assumptions are usually not enough to classify, if one does not assume some extra "finiteness structure" such as, for example, a compact connected topology. In fact, many classes of infinite geometries tend to admit free or universal constructions with large groups of collineations, and the existence of these examples prevent a classification of the highly transitive cases. One way to overcome this is to hypothesize a sharply transitive action on a set of substructures. This has proved to be successful in the case of projective planes, where the substructures are the non degenerate quadrangles; see [1] and [8]. Recently a similar approach to generalized $(2n+1)$-gons [7] lead to the nonexistence of generalized $(2n+1)$-gons, $n > 1$, admitting a collineation group acting sharply transitively on the set of ordered ordinary $(2n+2)$-gons, or on the set of ordered ordinary $(2n+1)$-gons, or on paths of fixed even length not exceeding $2n$. It was conjectured in [7] that the only projective plane admitting a collineation group acting sharply transitively on the family of all ordered ordinary triangles is the unique plane of order 2, but no proof seemed within reach. In the present paper, we prove this conjecture. Moreover, the arguments show that a slightly weaker hypothesis suffices, and we also apply this to the case of generalized $(2n+1)$-gons, $n > 1$. Hence we obtain a global classification of all generalized $(2n+1)$-gons, $n \geqq 1$, satisfying some condition on the stabilizer of any ordinary $(2n+1)$-gon (the latter is just an apartment in the language of buildings); the condition says that every symmetry of every

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ordinary (2n+1)-gon has a unique extension to a collineation. Also, we mention some related results, where conditions on the stabilizers of short paths are hypothesized.

Let us finally mention that the proof of our main result is very different from the proofs of the related result for generalized $(2n+1)$-gons, $n > 1$.

We now get down to precise definitions.

Let $m \geqq 2$ be a positive integer. A *(thick) generalized m-gon* is a point-line incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with point set $\mathcal{P}$, line set $\mathcal{L}$ and symmetric incidence relation $\mathcal{I}$, whose incidence graph (i.e., the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and adjacency relation $\mathcal{I}$) has diameter m (the maximal distance between two vertices) and girth $2m$ (the length of the smallest cycle), and which contains an ordinary $(m+1)$-gon as a subgeometry (see the monograph [5]). In this paper, we will only be concerned with the case m odd, and so we put $m = 2n + 1$. Below we motivate this restriction. If $n = 1$, then Γ is a projective plane in the usual sense.

An ordinary m-gon (viewed as a subgeometry) of Γ is called an *apartment* of Γ. An *ordered ordinary k-gon* of Γ (with $k \geqq m$) is a cycle $(x_0, x_1, \ldots, x_{2k-1}, x_0)$ in the incidence graph of Γ, with $x_0 \in \mathcal{P}$, with $x_i \mathcal{I} x_{i+1}$ (indices to be considered modulo $2k$), and such that $x_i \neq x_j$ for $i \not\equiv j \bmod 2k$. We emphasize that we view a cycle (and hence an ordered ordinary k-gon) as a closed path with a distinguished origin (x_0) and a distinguished direction (from x_0 to x_1 and *not* to x_{2k-1}). A *simple path* in Γ *of length k* is a sequence $(x_0, x_1, \ldots, x_k)$ of points and lines, with x_0 a point, such that $x_{i-1} \mathcal{I} x_i$ for all $i \in \{1, 2, \ldots, k\}$, and such that $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$.

A *collineation* of Γ is a pair of permutations, one of $\mathcal{P}$ and one of $\mathcal{L}$, such that two elements are incident if and only if their images are incident. The set of all collineations forms a group, the *full collineation group of* Γ. Every subgroup of that full collineation group will be called a *collineation group*.

For each field $\mathbb{K}$ there is a unique projective plane $\mathbf{PG}(2, \mathbb{K})$ constructed from a 3-dimensional vector space V over $\mathbb{K}$ by taking as point set the set of vector lines of V, as line set the set of vector planes of V, with natural incidence relation. This plane is often referred to as the *Pappian projective plane over* $\mathbb{K}$. For finite $\mathbb{K}$ of order q, we usually write $\mathbf{PG}(2, q)$ for $\mathbf{PG}(2, \mathbb{K})$. The plane $\mathbf{PG}(2, 2)$ is sometimes called the *Fano plane*.

2. Statement of the results. Originally, our aim was to prove the following result (which we state as a corollary because it will follow from Theorem 2.2 below).

Corollary 2.1. *Let Γ be a generalized m-gon, where $m \geqq 3$ is odd, and let G be a collineation group of Γ which acts sharply transitively on the ordered ordinary m-gons of Γ. Then $m = 3$, Γ is the Pappian projective plane* $\mathbf{PG}(2, 2)$*, and G is the full collineation group* $\mathbf{PGL}_3(2)$ *of* Γ.

Note that, to prove this result, only the case $m = 3$ has to be considered, since the part $m > 3$ is treated in [7]. However, we will show a slightly more general theorem by weakening the hypotheses (but it leads to the same conclusion).

Theorem 2.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a generalized m-gon, where $m \geqq 3$ is odd, and let G be a collineation group of Γ with the following property: for every apartment Σ of Γ, the stabilizer G_Σ acts faithfully on Σ as the dihedral group D_{2m} of order $2m$ in its natural action on Σ. Then $m = 3$, Γ is isomorphic to the Pappian projective plane* $\mathbf{PG}(2, 2)$ *and G is the full collineation group* $\mathbf{PGL}_3(2)$ *of Γ.*

If Γ is a generalized m-gon, with m odd, and if Γ admits a collineation group which acts sharply transitively on the ordered ordinary $(m + 1)$-gons, then $m = 3$ and Γ is a Pappian projective plane, see [1], [7]. Also, Corollary 2.1, together with Proposition 5.2 of [7] will imply the following.

Corollary 2.3. *Let Γ be a generalized m-gon, with $m = 2n + 1$ odd, and let k be an integer with $1 \leqq k \leqq n + 1$. If G is a collineation group of Γ acting either sharply transitively on the set of simple paths of length $2k$ of Γ, or in such a way that, for each simple closed path $(x_0, x_1, \ldots, x_{2k})$ of length $2k$ of Γ, there exists a unique collineation $\sigma \in G$ with $x_i^\sigma = x_{2k-i}$ for all $i \in \{0, 1, \ldots, k\}$, then $m = 3$, $k = 2$, Γ is the Pappian plane* $\mathbf{PG}(2, 2)$ *and G is the full collineation group of Γ.*

All the foregoing results are restricted to odd values of m because it is difficult to control the involutive collineations if m is even (cp. the results in [7] for $m = 4$ and $m = 6$), in particular, there are too many possibilities for the fixed point structure of an involution.

3. Proofs. First we prove Theorem 2.2.

It is well known, as already mentioned, that, if a generalized $(2n + 1)$-gon Γ admits a group acting regularly on the set of ordered apartments, then $n = 1$. The proof of that result, which is Proposition 5.2 in [6], however, does not use the full strength of the assumptions. Indeed, let $n > 1$ and suppose that for each apartment Σ of the generalized $(2n + 1)$-gon Γ the stabilizer G_Σ of any apartment acts faithfully on Σ as the dihedral group D_{4n+2} in its natural action. Clearly, G contains involutions. Since no involution of Γ can fix an apartment pointwise, Theorem 3.2 of [6] implies that every involution is a *central collineation*, i.e. every involution fixes all elements at distance at most n from some point p (called the *center* of the involution) of Γ, and fixes all elements at distance at most n from some line L (called the *axis* of the involution) of Γ, with $p\mathcal{I}L$. So let σ be a nontrivial central collineation of Γ and let Σ be any apartment in Γ containing the center p and the axis L of σ. Put $p = x_0$ and $L = x_1$. We label the elements of Σ as $x_0, x_1, \ldots, x_{2m-1}, x_0$, with $x_i\mathcal{I}x_{i+1}$, for all integers i mod $2m$. We conjugate σ with an element of G_Σ which maps (x_0, x_1) onto (x_{n+1}, x_{n+2}) and obtain a second nontrivial central collineation σ' of Γ, with center and axis x_{n+1}, x_{n+2} (or vice versa). Since σ (respectively σ') fixes x_{n+1} (respectively x_1), the commutator $[\sigma, \sigma']$ fixes $x_{3n+3}, x_{3n+4}, \ldots, x_{4n+1}, x_0, x_1, \ldots, x_{2n+1}$, and so, since $(3n + 3) - (2n + 1) = n + 2 < 2n + 1$, this commutator fixes Σ pointwise. This is a contradiction because σ' fixes x_{2n+2}, but σ' cannot fix x_{2n+2}^σ (cp. Section 3, Case 1 of [4]).

So, for the rest of the proof, we may assume that $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a projective plane. Since we will concentrate on the action of the collineation group G on $\mathcal{P}$, we will consider *triangles* in Γ rather than apartments (a triangle being a set of three points of Γ not incident with a common line). Hence we assume that G is a collineation group acting on Γ in such a way that, for every triangle T of Γ, the stabilizer G_T has order 6 and acts faithfully on the set of vertices of T (as the full symmetric group on three letters). For later purposes, we assume that $\mathcal{P}$ does not contain the symbol 0.

First we note that G acts 2-transitively on $\mathcal{P}$. Indeed, let $x, y, y' \in \mathcal{P}$ be arbitrary points with $y \neq x \neq y'$. If y' is not incident with the line xy, then there is an involution in G fixing x and interchanging y and y'. If y' is incident with xy, then choose an arbitrary point y'' not on xy and compose the involutions fixing x and interchanging y and y', and y' and y'', respectively.

Let $\{a, b, c\}$ be a triangle. The hypothesis implies that there is a unique involution $\sigma(a, b, c) \in G$ fixing c and interchanging a with b. Clearly $\sigma(a, b, c)$ is not a Baer involution, since this would imply that it fixes a triangle pointwise, and hence should be the identity. So $\sigma(a, b, c)$ is a central collineation with center z on the line ab. If the axis A of $\sigma(a, b, c)$ were not incident with the center z, then $\sigma(a, b, c)$ again would fix a triangle consisting of c, z and the intersection of A with ab. Hence A is incident with both c and z. Now let c' be any point not on the line ab, then the product $\sigma := \sigma(a, b, c)\sigma(a, b, c')$ fixes the points a and b. If the axis A' of $\sigma(a, b, c')$ were not incident with z, then σ would fix the triangle $\{a, b, A \cap A'\}$. But clearly σ acts nontrivially on both A and A', a contradiction. Hence A' is incident with z and σ also fixes all lines through z; hence it is a central collineation and its axis must be ab. This shows that the action of $\sigma(a, b, c)$ on the point set of the line ab is independent of c. So may define a binary function "+" on the set $\mathcal{P} \cup \{0\}$ as follows. Given two distinct points a, b, we define $a + b = z$, where z is the unique fixed point on the line ab of the involution $\sigma(a, b, c)$, for any choice of c not on ab. We also put $a + a = 0$ and $a + 0 = a = 0 + a$, for all $a \in \mathcal{P} \cup \{0\}$. Our goal is to show that this "addition" turns $\mathcal{P} \cup \{0\}$ into a commutative group of exponent 2. Clearly we already have $a + b = b + a$, since $\sigma(a, b, c) = \sigma(b, a, c)$. Since we also have an identity by definition, and an inverse for every element a (being a itself by definition), we only have to show that the addition is associative. First, we show this for three points a_1, a_2, a_3 not incident with a common line. Let $\{i, j, k\} = \{1, 2, 3\}$. Put $T = \{a_1, a_2, a_3\}$.

We claim that a_1, a_2, a_3 are contained in a subplane of order 2. Let z_k be the center of the central collineation $\sigma(a_i, a_j, a_k)$ (and note that z_k is incident with the line $a_i a_j$). By the previous paragraph we know that the axis A_k of $\sigma(a_i, a_j, a_k)$ is the line $a_k z_k$. Let p be the intersection of A_1 and A_2. The element $\sigma(a_2, a_3, a_1)\sigma(a_3, a_1, a_2)\sigma(a_2, a_3, a_1)$ fixes a_3 and interchanges a_1 and a_2, hence, by the assumptions on G_T, it coincides with $\sigma(a_1, a_2, a_3)$. But now $\sigma(a_2, a_3, a_1)\sigma(a_3, a_1, a_2)\sigma(a_2, a_3, a_1)$ fixes the point p (which is the intersection of the axes of $\sigma(a_2, a_3, a_1)$ and $\sigma(a_3, a_1, a_2)$) and the line $z_1 z_2$ (which joins the centers of $\sigma(a_2, a_3, a_1)$ and $\sigma(a_3, a_1, a_2)$), implying that p is incident with A_3, and hence A_1, A_2, A_3 are concurrent, and that z_1, z_2, z_3 are incident with a common line L. Both p and L are fixed under $\sigma(a_i, a_j, a_k)$. It is now clear that the points $a_1, a_2, a_3, z_1, z_2, z_3, p$ together with the lines $A_1, A_2, A_3, a_1 a_2, a_2, a_3, a_3, a_1, L$ form a projective subplane of order 2 of Γ and our claim is proved.

The central involution $\sigma(a_i, a_j, z_i)$ with center z_k fixes z_j (because z_j is incident with the axis L of $\sigma(a_i, a_j, z_i)$) and hence maps the intersection a_k of the lines $z_i a_j$ and $z_j a_i$ onto the intersection p of the lines $z_i a_i$ and $z_j a_j$. We have shown that $\sigma(a_i, a_j, z_i) = \sigma(a_k, p, z_i)$ and so $a_k + p = z_k$.

Now clearly $\sigma(z_j, a_j, a_k) \neq \sigma(a_i, a_j, a_k)$ and so $\sigma' := \sigma(z_j, a_j, a_k)\sigma(a_i, a_j, a_k)$ is not the identity. But σ' clearly fixes the lines $a_i a_k$, $a_j a_k$ and $z_k c$, where c is the center of $\sigma(z_j, a_j, a_k)$. So, if $c \neq p$, then σ' fixes an ordered triangle, a contradiction. Consequently $c = p$ and we deduce $z_j + a_j = p$. Since in the previous paragraph we showed $a_j + p = z_j$, we deduce from the 2-transitive action of G (and the fact that the addition of distinct points is clearly invariant under G) that, if $a + b = c$, then $a + c = b$ and $b + c = a$. In other words, we have

$$(1) \qquad a + (a + b) = b,$$

for all $a, b \in \mathcal{P}$.

From the result in the previous paragraph we also deduce

$$(2) \qquad (a_1 + a_2) + a_3 = z_3 + a_3 = p = a_1 + z_1 = a_1 + (a_2 + a_3),$$

which shows associativity for a_1, a_2, a_3.

Now we show that $(a + b) + c = a + (b + c)$ whenever $0 \in \{a, b, c\}$ or $|\{a, b, c\}| \leqq 2$. This is trivial when $0 \in \{a, b, c\}$, and it follows from the identity (1) if $a = b$ or $b = c$. If $a = c$, it follows from the commutativity of the addition.

Finally we show that $(a+b)+c = a+(b+c)$ for every triple of distinct collinear points a, b, c. We choose an arbitrary point d not on the line abc and we remark that $\{a+b, c, d\}$, $\{a, b, c+d\}$, $\{b, c, d\}$ and $\{a, b+c, d\}$ are all triangles of Γ. Hence an easy application of the identities (1) and (2) yields

$$\begin{aligned}(a + b) + c &= (((a + b) + c) + d) + d = ((a + b) + (c + d)) + d \\ &= (a + (b + (c + d))) + d = (a + ((b + c) + d)) + d \\ &= ((a + (b + c)) + d) + d = a + (b + c),\end{aligned}$$

which completes the proof of the fact that $\mathcal{P} \cup \{0\}$ is a commutative group of exponent 2 for the addition "+".

Now we show that every element of Γ is incident with exactly three other elements. Suppose by way of contradiction that Γ has a line M of size $\geqq 4$. Select two distinct points a, a' on M and a point z not incident with M. Let the center of the involution $\sigma(a, a', z)$ be the point c, and let d be a fourth point on the line M (d is different from a, a' and c). The point $a + d$ is different from c since otherwise $d = a'$. Let us denote by d' the image of d under $\sigma(a, a', z)$. Since the addition is invariant under G, the image $(a+d)'$ of $a+d$ under $\sigma(a, a', z)$ is nothing else but $a' + d'$. Hence we have (using associativity of the addition!) $c = (a+d)+(a+d)' = a+d+a'+d' = a+a'+d+d' = c+c = 0$, a contradiction.

Theorem 2.2 is proved. □

Now Corollary 2.1 is a direct consequence of Theorem 2.2. As for Corollary 2.3, it suffices to remark that G contains involutions (the collineation σ in the statement is an involution since the assumptions imply $\sigma = \sigma^3$) and every involution fixes at least one simple path of length m pointwise.

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