Slim and Bislim Geometries

Hendrik Van Maldeghem

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1. Introduction

Preliminary note. In this paper, we try to tell a coherent story without interruptions due to formal definitions. If not mentioned in the text, these will be placed in separate boxes usually near the places where they are first used. Concerning references, we have included some precise ones to refer to the original results in the literature. For the general notions on geometries a good reference is [4]. The information on eigenvalues of adjacency matrices of graphs in Section 5 is obtained from [3]. Also, the paper of Gropp [7] was used for information about some history of configurations.

Incidence Geometry is essentially a study of sets with an additional structure either made obvious by a selection of certain subsets, possibly carrying a type, and possibly with a multiplicity, or given by an incidence relation (which gives the set the structure of a labelled graph). Moreover, these *geometries* satisfy axioms that arise from certain settings. One of the most prominent classes of examples is the class of *buildings* that naturally arise in the theory of simple groups of Lie type as the structures describing the configuration of parabolic subgroups. Other classes entail various kinds of designs (for instance Steiner systems). The theory of diagrams initiated by Buekenhout provides a method to make a systematic subdivision. A first parameter is the *rank* of the geometry. This is essentially the number of different kinds of objects we are dealing with; we can also call it the *dimension*. Since a diagram of a geometry is obtained by looking at subgeometries of rank 2, the latter serve as the building bricks of all incidence geometries (of arbitrary rank). A lot of classes of rank 2 geometries are studied in the literature. They arise from different situations. For instance, partial geometries arise from studying strongly regular graph, as likewise the semipartial geometries, in contrast to for instance near polygons, which arise from studying polar spaces. Roughly speaking, one has three kinds of rank 2 geometries. There is the class of designs, where usually two different points define more than one block (but always a constant number). The class of semilinear spaces is defined as the class of rank 2 geometries where every pair of points define at most one line (but we will call these geometries *linear* below). The third class contains the geometries which do not belong to either of these two classes: sometimes two points define more than one block, but sometimes at most one. Good examples are the Hjelmslev geometries.

A triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a *geometry* if $\mathcal{P}$ and $\mathcal{L}$ are disjoint sets, and $\mathtt{I}$ is a binary symmetric relation between $\mathcal{P}$ and $\mathcal{L}$ symbolizing the incidence between the points $\mathcal{P}$ and the lines $\mathcal{L}$ of Γ. If $x \in \mathcal{P}$, $L \in \mathcal{L}$ and $x\mathtt{I}L$, then we say that x *is incident with* L, or x *lies on* L, or L *goes through* x, etc. In this case, the pair $\{x, L\}$ is called a *flag*. The *Levi graph* or *incidence graph* $\mathcal{G}(\Gamma)$ of a geometry Γ has as set of vertices $\mathcal{P} \cup \mathcal{L}$, two vertices being joint by an edge if they form a flag. The *distance* $\delta(x, y)$ between two elements x, y of Γ is by definition their graph theoretical distance in $\mathcal{G}(\Gamma)$. Then we can define $\Gamma_i(x) = \{y \in \mathcal{P} \cup \mathcal{L} \mid \delta(x, y) = i\}$. If $\Gamma_1(x)$ is constant for $x \in \mathcal{P}$, and $\Gamma_1(L)$ is constant for $L \in \mathcal{L}$, then we say that Γ has *order* (s, t), where $1 + s = |\Gamma_1(L)|$, $L \in \mathcal{L}$, and $1 + t = |\Gamma_1(x)|$, $x \in \mathcal{P}$.

In every class of rank 2 geometries, the examples with small s, t (where (s, t) is the order of the geometry) play a special and important role. They are usually small in number of elements, and hence easier to understand. They often have special properties because they can belong to different infinite classes of geometries (sporadic isomorphisms of small geometries). Also, they are often determined by the axioms and the order, which makes them very homogeneous. Special properties reflect sporadic properties of small groups. Geometries with $s = t = 1$ are easily seen to be unions of polygons and bi-infinite cycles. Geometries with $s = 1$ correspond with graphs, while for $t = 1$ one can consider the dual geometry and obtain a graph. Hence the first interesting case is the case $s = t = 2$. This is the case we will review in the present paper. These are the bislim geometries.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a geometry. Then the *dual* Γ^* of Γ is the geometry $\Gamma^* = (\mathcal{L}, \mathcal{P}, \mathtt{I})$. Two points of Γ are *collinear* if they are incident with a common line. Dually one defines *concurrent* or *confluent* lines. The *Menger graph* or *collinearity graph* $\mathcal{G}_\perp(\Gamma)$ of Γ is the graph with point set $\mathcal{P}$, where two vertices are adjacent if they represent collinear points. If Γ has an order (s, t), then Γ is called *thin* if $s = 1$, *thick* if $s, t \geq 2$ and *slim* (respectively *bislim*) if $s = 2$ (respectively $s = t = 2$).

Let us remark that the incidence graph of a geometry determines the geometry up to duality. Usually one considers geometries in dual pairs, choosing the point set in such a way to make arguments easier to understand (often for psychological, heuristic or traditional reasons). The collinearity graph, on the contrary, does not always determine the geometry uniquely. An extreme example is any projective plane; there the collinearity graph is a complete graph. Hence any two non-isomorphic projective planes of the same order have isomorphic Menger graphs. Other geometries *are* determined by their Menger graph. Good examples are the generalized n-gons with $n > 3$.

Let Γ be a geometry and n a positive integer not smaller than 2. Then we say that Γ is a *weak generalized n-gon* if the Levi graph $\mathcal{G}(\Gamma)$ has diameter n and girth $2n$ (the *girth* of a graph being the length of a minimal cycle). If a weak generalized n-gon has an order, say (s, t), then it is called an *ordinary n-gon* if $s = t = 1$, and a *generalized n-gon* if $s, t \geq 2$ (and any thick weak generalized n-gon has an order). If we do not want to emphasize n, we talk about *polygons* rather than n-gons. A generalized 3-gon is a *projective plane*. Note that, by definition, generalized polygons are connected geometries (a geometry is *connected* if its incidence graph is a connected graph).

2. Configurations and trivalent graphs

Finite geometries of order (s, t) with v points and b lines which are semilinear spaces are also known as (v_{1+t}, b_{1+s}) configurations. Especially for small values of s, t, v, b, they have been given quite a lot of attention in the literature. If $v = b$, or equivalently, if $s = t$, then one talks about symmetric configurations denoted v_{1+s}. In our case, $s = t = 2$ and hence we are concerned about v_3 configurations. For instance, the Desargues configuration (see below) is a 10_3. The projective plane of order $(2, 2)$ is the unique 7_3. Connected with this point of view is the question of *realisability* of the geometry Γ in real space, i.e., can we find a set of points and lines in real space such that the geometry induced by this set is isomorphic to Γ? We will review some results in that direction. We will connect this to the rank of an incidence matrix, hence turning to algebra in a completely other way than is usually done (by assigning coordinates and putting things on a computer).

An incidence matrix of a geometry Γ is a matrix whose rows are indexed by the point set of Γ, whose columns are indexed by the line set of Γ and whose entry (x, L) is equal to 1 if x is incident with L in Γ; otherwise the entry is 0.

Considering the incidence graph of a bislim geometry, we see that the study of bislim geometries is essentially equivalent to the study of bipartite trivalent graphs (graphs where every vertex has valency 3). What about arbitrary trivalent graphs? One can consider the *neighborhood* geometry of such a graph as defined in [8].

Let $\mathcal{G}$ be a trivalent graph. Let $\mathcal{P}$ and $\mathcal{L}$ be two copies of the vertex set (if x is a vertex, then we define the object x_p and put it in $\mathcal{P}$; the object x_ℓ is put in $\mathcal{L}$), and let a point $x \in \mathcal{P}$ be incident with a line $L \in \mathcal{L}$, $x\mathtt{I}L$, if, viewed as vertices in $\mathcal{G}$, they are adjacent. The resulting geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ is called the *neighborhood geometry* of $\mathcal{G}$.

If $\mathcal{G}$ is not bipartite, this produces a connected bislim geometry Γ and we call $\mathcal{G}$ a *polarized graph of* Γ; if $\mathcal{G}$ is bipartite, and hence isomorphic to $\mathcal{G}(\Gamma)$ for some geometry Γ, this produces two disjoint copies of Γ. The Levi graph is uniquely determined by the geometry. Is the same true for a polarized graph? When does a geometry admit a polarized graph? The answer is in the following easy proposition.

A *polarity* of a geometry is a bijection $\beta : \mathcal{P} \longrightarrow \mathcal{L}$ with the property that $x\mathtt{I}L$ if and only if $L^{\beta^{-1}}\mathtt{I}x^\beta$. An *absolute point* of a polarity β is a point $x \in \mathcal{P}$ such that $x\mathtt{I}x^\beta$. An *absolute line* is defined similarly.

Proposition 2.1. *Let Γ be a (rank 2) geometry. Then every polarized graph of Γ gives rise to a unique polarity without absolute points. Conversely, every polarity without absolute points produces a unique polarized graph.*

PROOF. If $\mathcal{G}$ is a polarized graph of Γ, then the bijection assigning the line x_ℓ to the point x_p, for every vertex x of $\mathcal{G}$ is a polarity without absolute points. Conversely, if β is a polarity of Γ, then the graph with set of vertices $\{(x, x^\beta) \mid x \in \mathcal{P}\}$ and adjacency defined by the rule "(x, x^β) is adjacent to (y, y^β) if $y\mathtt{I}x^\beta$" (which is equivalent with $x\mathtt{I}y^\beta$ by the definition of polarity) is a polarized graph of Γ (the absence of absolute points guarantees that this graph has no loops). $\square$ A polarized graph of a geometry is half as small as the Levi graph and hence provides a more condensed object to look at. We give examples below. Hence the theory of trivalent graphs is completely equivalent to the theory of bislim geometries. In the next three sections we consider constructions (Section 3), classification results (Section 4) and embeddings in real spaces and in projective spaces over the field of two elements (Section 5).

3. Constructions

It is easy to construct a v_3 out of a $(v-1)_3$. In fact, it is easy to find a construction that, in principal, yields all v_3 given all $(v-1)_3$. For instance, consider the following construction. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a bislim geometry and select two lines $L_1, L_2 \in \mathcal{L}$. Also, select two (different) points $x_i \mathtt{I} L_i$, $i = 1, 2$. Let x and L be symbols not contained in $\mathcal{P}$ and $\mathcal{L}$, respectively. Define $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ as follows. Set $\mathcal{P}' = \mathcal{P} \cup \{x\}$, $\mathcal{L}' = \mathcal{L} \cup \{L\}$. Also, if $p \mathtt{I} M$ with $p, M \in \mathcal{P} \cup \mathcal{L}$, but $\{p, M\} \not\subset \{x_1, x_2, L_1, L_2\}$, then $p \mathtt{I}' M$. By definition, $x \mathtt{I} \{L_1, L_2\}$, $\{x_1, x_2\} \mathtt{I} M$ and $x \mathtt{I} L$. It is readily checked that Γ' is a bislim geometry. This procedure obviously has an inverse, as the reader can check. Hence one can start with the trivial geometry (a generalized 2-gon or digon of order $(2,2)$) $(\{x_1, x_2, x_3\}, \{L_1, L_2, L_3\}, \mathtt{I})$, where $x_i \mathtt{I} L_j$, for every $i, j \in \{1, 2, 3\}$, and apply the above construction a few consecutive times to obtain all possible bislim geometries. For further reference, we call Γ' the *integrated geometry of* Γ *with respect to* $\{x_1, L_1\}$ *and* $\{x_2, L_2\}$. Such an idea has been used by Martinetti [9] in 1887 to construct all geometries 11_3 out of all geometries 10_3.

A geometry Γ is *linear* if no two distinct points are incident with two distinct lines. The *gonality* of a geometry is half the girth of its incidence graph. Hence a geometry is linear if and only if its gonality is at least equal to 3.

One problem with the integrated geometries is that, although starting with a linear geometry, one could end up with a nonlinear one. It could also happen that one starts with a nonlinear geometry and obtains a linear integrated one. Also, no control over the automorphism group of the integrated geometry is obtained. But in principal, it is a universal construction. The reason why we are more interested in linear geometries rather than in nonlinear ones is because a nonlinear geometry can never be represented in any projective space by points and lines of that space for the point-line geometry of a projective space is itself linear. We come back to this matter in Section 5. We now present some constructions of particular bislim geometries. They all have a large group of collineations.

A *collineation* or *automorphism* θ of a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a permutation of $\mathcal{P} \cup \mathcal{L}$ inducing permutations of $\mathcal{P}$ and $\mathcal{L}$ separately and such that $x\mathtt{I}L$ if and only if $x^\theta \mathtt{I} L^\theta$, for all $x \in \mathcal{P}$ and all $L \in \mathcal{L}$. A *duality* σ of Γ is a permutation of $\mathcal{P} \cup \mathcal{L}$ inducing bijections from $\mathcal{P}$ to $\mathcal{L}$ and from $\mathcal{L}$ to $\mathcal{P}$ such that $x\mathtt{I}L$ if and only if $x^\sigma \mathtt{I} L^\sigma$, for all $x \in \mathcal{P}$ and all $L \in \mathcal{L}$. A *correlation* is a collineation or a duality. The *full collineation group* of Γ is the group of all collineations of Γ; similarly one defines the *full correlation group*. Any subgroup of the full collineation group (full correlation group) will be called *a collineation group* (respectively *a correlation group*). A geometry which admits a duality is called *self dual*; one that admits a polarity is called *self polar*.

The *Fano geometry* is the projective plane of order $(2, 2)$. One has the classical (and general) description as nonzero vectors (points) and vector planes (lines) of a 3-dimensional vector space over the field $\mathbf{GF}(2)$ of 2 elements (with natural incidence relation), but there is also an alternative construction using difference sets: take as point set the integers modulo 7, and the lines are the translates of the 3-set $\{0, 1, 3\}$. The full collineation group is $\mathbf{PGL}_3(2)$. It contains subgroups acting sharply transitively on the set of flags. The Fano geometry is self polar.

The *Möbius-Kantor geometry* has as point set the 8 nonzero vectors of a 2-dimensional vector space over the field $\mathbf{GF}(3)$ of 3 elements; the lines are the 8 proper translates of the vector lines (so the vector lines themselves are excluded). A construction using difference sets runs as follows: take as point set the set of integers modulo 8, and the lines are the translates of the 3-set $\{0, 1, 3\}$. The full collineation group is $\mathbf{GL}_2(3)$ acting flag transitively. The Möbius-Kantor geometry is self polar. The Fano geometry and the Möbius-Kantor geometry are the unique linear bislim geometries with 7 and 8 points, respectively.

The *Pappus geometry* has as point set the 9 vectors of a 2-dimensional vector space over the field $\mathbf{GF}(3)$; the lines are the non vertical vector lines and their translates (the vertical vector line is just an arbitrarily chosen vector line as the Y-axis). The full collineation group is the stabilizer of a point at infinity of the affine group $\mathbf{AGL}_2(3)$ and acts flag transitively.

The *Desargues geometry* is the neighborhood geometry of the Petersen graph (the unique graph on 10 vertices having girth 5). We derive the following direct combinatorial construction. The points are the pairs of the 5-set $\{1, 2, 3, 4, 5\}$; the lines are the triples of that 5-set and incidence is natural. Alternatively, consider a non degenerate conic in the projective plane $\mathbf{PG}(2, 5)$. Then the points of the Desargues geometry are the 10 *internal points* of the conic (i.e., the points not incident with any tangent line to the conic) and the lines are the 10 *external lines* (i.e., the lines not meeting the conic), while incidence is natural. From these constructions it is clear that $\mathbf{PGL}_2(5) \cong \mathbf{S}_5$ is a collineation group, and that the Desargues geometry admits a polarity without absolute points. In fact this group is the full collineation group and acts flag transitively.

The *Coxeter geometry* was first discovered by Coxeter [5]. We present the two constructions taken from [17]. First, consider the 2-dimensional vector space over $\mathbf{GF}(4)$ and choose arbitrarily a *vertical* Y-axis. The points of the Coxeter geometry are the 12 non vertical vectors; the lines are the proper translates of the non vertical vector lines; incidence is natural. Secondly, the Coxeter geometry can also be defined as the neighborhood geometry of a(n at each point) truncated tetrahedron. From the first construction one can derive that the full collineation group is isomorphic to the semilinear group of permutations of the 2-dimensional vector space over $\mathbf{GF}(4)$ with a lower triangular matrix. This group acts flag transitively and contains 72 elements. From the second construction we see that the Coxeter geometry admits a polarity without absolute points. An interesting observation is that we have here an example of geometry admitting more than one polarity without absolute points, hence the automorphism group of the polarized graph is strictly smaller than the collineation group of the geometry. Note that the Coxeter geometry also admits a polarity *with* absolute points (see [17])!

The *honeycomb geometry* is the only infinite example we will define. Its incidence graph is the (bipartite) graph obtained from the tiling of the real Euclidean plane into regular hexagons. A more explicit description runs as follows. The points are the ordered pairs (i, j) of integers i, j. The lines are the triples $\{(i, j), (i, j + 1), (i + 1, j)\}$, with i, j any integers. The full collineation group is the Coxeter group of type $\tilde{A}_2$ and acts flag transitively.

Now we consider the neighborhood geometry of a series of very regular and homogeneous graphs, namely of the trivalent distance regular graphs. There are thirteen such graphs. They were classified by Biggs, Boshier and Shawe-Taylor [1] in 1986 after the classification of trivalent distance transitive graphs by Biggs and Smith [2] in 1971 (I would like to thank Nele Haelvoet for pointing this out to me).

A graph $\mathcal{G}$ is *distance regular* if for all positive integers i, j, k, and all vertices v, w of $\mathcal{G}$ at distance k from each other, the number of vertices at distance i from v and at the same time at distance j from w does not depend on the choice of v, w, but only on i, j, k.

1. First, there is the complete bipartite graph $K_{3,3}$ with six vertices and two bipartion classes of three elements. It is the incidence graph of the generalized digon of order $(2, 2)$. Clearly, this is a nonlinear geometry. It does not admit a polarity without absolute points, obviously (but it admits many polarities!). Hence there is no polarized graph.

2. The smallest trivalent graph is the complete graph on four vertices. The neighborhood geometry has four points and every 3-subset is a line. It is the integrated geometry of the slim generalized digon. We will call it the *tetrahedron geometry* for further reference. It has a unique polarity without absolute points. Hence the complete graph on four vertices is the unique polarized graph of the tetrahedron geometry.

3. The cube graph (the vertices and edges of the cube, without paying notice to the faces) is the incidence graph of the tetrahedron geometry.

4. The neighborhood geometry of the Petersen graph is the Desargues geometry, as already mentioned above. A rather strange property is that the Menger graph of the Desargues configuration is the complement of — again — the Petersen graph (i.e., the graph obtained from the Petersen

graph by taking the same set of vertices but interchanging the edges with non-edges). The complement of the Petersen graph is the Menger graph of other geometries. For instance of the bislim geometry with 10 points which can not be realized in the real affine plane (see below).

5. The incidence graph of the Desargues geometry is a distance regular graph and defines of course the Desargues geometry again.

6. The dodecahedron graph (the vertices and edges of the dodecahedron disregarding the faces) is not bipartite, hence its neighborhood geometry —which we will call the *dodecahedron geometry* — has 20 points and 20 lines. It is the unique double cover of the Desargues geometry without triangles. A peculiar feature here is the fact that the full collineation group of this geometry is twice as big as the collineation group induced by the automorphism group of the dodecahedron as a graph. This translates into the corollary that there are several polarities without absolute points.

7. The incidence graph of the Pappus geometry is a distance regular graph and defines the Pappus geometry again.

8. The Heawood graph is the incidence graph of the slim projective plane (the Fano plane).

9. The Coxeter graph is not bipartite and hence the neighborhood geometry has 28 points and 28 lines. We will call this geometry the *triangle geometry* because of the following construction. The points of the triangle geometry are the point-line pairs of the Fano plane which do not form a flag (we say that they form an *antiflag*). The lines are the triangles of the Fano plane and incidence is natural (i.e., when both the point and the line of the antiflag are contained in the triangle). One can define such a slim geometry using any projective plane, but only with the Fano plane we obtain a bislim geometry, which is also self polar (and has a polarity without absolute points). One polarity without absolute points can be described geometrically as follows: it maps a point to the unique line at (maximal) distance 7. This time the full collineation group is induced by the automorphism group of the Coxeter graph.

10. Tutte's 12-cage is the incidence graph of a pair of non-isomorphic slim generalized hexagons. It is the only trivalent distance regular graph which does not admit a transitive group. There are several constructions of these generalized hexagons. Let us present one that we will use later on to describe a real embedding. Let $\mathcal{P}_0$ and $\mathcal{L}_0$ be the point and line set, respectively, of the Fano plane, completed with the symbol 0. We define an addition on $\mathcal{P}_0 \times \mathcal{L}_0$ componentwise by (1) $A + 0 = A = 0 + A$ and $A + A = 0$, for all elements $A \in \mathcal{P}_0 \cup \mathcal{L}_0$, (2) if x, y, z are three distinct points incident with a common line, then $x + y = z$, and (3) dually, if L, M, N are three distinct lines through a common point, then $L + M = N$. The points of the generalized hexagon $\mathsf{H}(2)$ are the elements of $\mathcal{P}_0 \times \mathcal{L}_0 \setminus \{(0,0\}$. The lines are the triples $\{(x,0),(0,L),(x,L)\}$, with $\{x,L\}$ a flag of the Fano plane, together with the triples $\{(x,L),(x_1,L_1),(x_2,L_2)\}$, with $\{x,L\}$ a flag of the Fano plane, $x_1, x_2 \in \mathcal{P}$, $L_1, L_2 \in \mathcal{L}$ and with $(x_1,L_1) + (x_2,L_2) = (x,L)$. For other constructions of this generalized hexagon or its dual, see [12].

11. Tutte's 8-cage is the incidence graph of the unique bislim generalized quadrangle $\mathsf{W}(2)$. We now give one explicit construction of Tutte's 8-cage, and one of the bislim generalized quadrangle. Consider a conic $\mathcal{C}$ in the projective plane $\mathbf{PG}(2,9)$. Define a graph with vertex set the polar triangles with respect to $\mathcal{C}$ (i.e., the triples of distinct points such that the polar line with respect to $\mathcal{C}$ of each of these points is spanned by the two other points) which consist entirely of exterior points (i.e., points incident with exactly two tangent lines to $\mathcal{C}$). Declare two vertices adjacent if the corresponding polar triangles intersect non trivially. This graph is Tutte's 8-cage, see [12]. Now define the following bislim geometry. The points are the pairs of elements of the set $\{1,2,3,4,5,6\}$, the lines are the triples of pairs that partition $\{1,2,3,4,5,6\}$ (and incidence is the natural one). We obtain $\mathsf{W}(2)$. Note that $\mathsf{W}(2)$ is self polar, but all polarities are conjugate and each of them has absolute points. Hence there is no polarized graph.

12. The Foster graph is the incidence graph of the so-called *tilde geometry* $\widetilde{\mathsf{W}(2)}$. The latter is the unique triple cover of $\mathsf{W}(2)$ having no quadrangles. For many geometric properties and constructions of the tilde geometry

we refer to [10]. Here, we content ourselves with mentioning that all polarities are conjugate and each of them has exactly five absolute points. So again, there is no polarized graph. We will give an explicit construction of the tilde geometry in Section 5.

13. The Biggs-Smith graph is not bipartite and gives rise to a bislim (Biggs-Smith) geometry that has not been studied yet in the literature, as far as I know (apart from its introduction in [16]). I have given an alternative definition in [16], which I repeat here. Consider the projective line $\mathbf{PG}(1,17)$ over the field $\mathbf{GF}(17)$. Given a pair of points $\{a,b\}$ and a third point c of $\mathbf{PG}(1,17)$, it is easy to verify that there exist a unique point d and a unique pair of points $\{e,f\}$ of $\mathbf{PG}(1,17)$ such that $|\{a,b,c,d,e,f\}| = 6$ and such that we have the following equality of cross ratios: $(a,b;c,d) = (a,b;e,f) = (c,d;e,f) = -1$. Indeed, we may choose, by the triply transitivity of $\mathbf{PGL}_2(17)$ on the point set of $\mathbf{PG}(1,17)$, $(a,b,c) = (\infty,0,1)$ and then $(d,e,f) = (-1,4,-4)$ (or $(-1,-4,4)$). We call the triple $\{\{a,b\},\{c,d\},\{e,f\}\}$ a *harmonic triplet*. Three mutually disjoint harmonic triples will be called a *trisection*. The stabilizer inside $\mathbf{PGL}_2(17)$ of a harmonic triplet is easily seen to be a group of order 24 and it is entirely contained in $\mathbf{PSL}_2(17)$. This implies that there are in total $18.17.16/24 = 204$ harmonic triplets and that $\mathbf{PSL}_2(17)$ defines two orbits of harmonic triplets. We fix one of these orbits (and we can choose coordinates in such a way that this orbit contains the harmonic triplet $\{\infty,0\},\{1,-1\},\{4,-4\}\}$). It is the point set $\mathcal{P}$ of the Biggs-Smith geometry. The lines are the trisections of triplets in $\mathcal{P}$. For the Biggs-Smith graph, the vertex set is $\mathcal{P}$ and two vertices are adjacent if the harmonic triplets share a pair of points, e.g. $\{\{\infty,0\},\{1,-1\},\{4,-4\}\}$ is adjacent to $\{\{\infty,0\},\{2,-2\},\{8,-8\}\}$. In the Biggs-Smith geometry, two points are collinear if and only if the corresponding harmonic triplets do not share any element of $\mathbf{PG}(1,17)$.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a geometry and n a positive integer. Then an *n-fold cover of* Γ is a geometry $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ together with surjective maps $\varphi_1 : \mathcal{P}' \longrightarrow \mathcal{P}$ and $\varphi_2 : \mathcal{L}' \longrightarrow \mathcal{L}$ such that $x\mathtt{I}'L$, $x \in \mathcal{P}'$, $L \in \mathcal{L}'$, implies that $x^{\varphi_1}\mathtt{I}L^{\varphi_2}$, such that φ_1 (respectively φ_2) induces a bijection from $\Gamma'_1(L)$ to $\Gamma_1(L^{\varphi_2})$ (respectively from $\Gamma'_1(x)$ to $\Gamma_1(x^{\varphi_1})$) for all $L \in \mathcal{L}'$ (respectively for all $x \in \mathcal{P}'$), and such that for all points p and lines R of Γ the sizes of the *fibers* $|p^{\varphi_1^{-1}}|$ and $|R^{\varphi_2^{-1}}|$ are equal to n. For $n = 2$, we speak of a *double cover*, for $n = 3$, we speak of a *triple cover*. The geometry Γ' is called the *covering geometry*. If the fibers are equivalence classes in $\mathcal{P}'$ and in $\mathcal{L}'$ with respect to the relation *"is at maximal distance of"*, then we say that Γ' is an n-fold cover of Γ, without mentioning the maps φ_1 and φ_2.

This completes the description of twelve rather remarkable examples of bislim geometries. Of course, there are many more examples, but we have given the most symmetric ones, with emphasis on low gonality, as we will see in the next section.

4. Classification results

Viewed as v_3 configurations, linear bislim geometries with a small number of points have been classified and enumerated. We summarize the results in the next theorem.

Theorem 4.1. *There are no linear bislim geometries with* 6 *points or less.*
There is precisely 1 *linear bislim geometry with* 7 *points.*
There is precisely 1 *linear bislim geometry with* 8 *points.*
There are precisely 3 *linear bislim geometries with* 9 *points.*
There are precisely 10 *linear bislim geometries with* 10 *points.*
There are precisely 31 *linear bislim geometries with* 11 *points.*
There are precisely 229 *linear bislim geometries with* 12 *points.*
There are precisely 2036 *linear bislim geometries with* 13 *points.*
There are precisely 21399 *linear bislim geometries with* 14 *points.*

These numbers can be found in [7]. Of course, from computational point of view, this may be interesting, but from geometrical points of view, the precise numbers do not matter so much. We already know that there are a lot, that we can construct them all — in principal — and that they may have very different collineation groups. Nevertheless, an interesting conclusion will be drawn in the next section!

A geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is called *distance transitive* if it is connected and if for each positive integer i, its full collineation group acts transitively on the set of ordered pairs $(x, A) \in \mathcal{P} \times (\mathcal{P} \cup \mathcal{L})$ with $\delta(x, A) = i$, and also on the set of ordered pairs $(L, B) \in \mathcal{L} \times (\mathcal{P} \cup \mathcal{L})$ with $\delta(L, B) = i$.

An easy corollary of the classification of trivalent distance regular graphs is the following.

Theorem 4.2. *The only distance transitive bislim geometries are the tetrahedron geometry, the Desargues geometry, the Pappus geometry, the tilde geometry, and the five bislim generalized n-gons, $n = 2, 3, 4, 6$ (there are two of them for $n = 6$).*

Note that finite generalized n-gons exist only for $n = 2, 3, 4, 6, 8$, and for $n = 8$, their order (s, t) satisfies the condition that $2st$ is a square, so there is no bislim generalized octagon, see [6]. The reason why for instance the Biggs-Smith geometry is not distance transitive is that its incidence graph is not a distance transitive graph, since it is not even distance regular.

A hypothesis considerably weaker than distance transitivity is flag transitivity. But there are a lot of flag transitive bislim geometries. Nevertheless, the ones of gonality 3 can be classified. Here is the result.

Theorem 4.3. *Let Γ be a flag transitive bislim geometry with gonality 3. Then either Γ is infinite and isomorphic to the honeycomb geometry, or Γ is an explicitly defined quotient of the honeycomb (this means that the honeycomb geometry covers the geometry) considered already by Coxeter in [5], depending on two integral parameters n, m with $0 \le n \le m$ and $n + m \ge 3$, and with $n^2 + nm + m^2$ points, or Γ is the Desargues geometry or the Möbius-Kantor geometry.*

To illustrate the quotients of the honeycomb geometry, we mention that for $n = 0$ and $m = 3$, this is the Pappus geometry and for $n = 1$ and $m = 2$, it is the Fano plane. Also some other quotients with small parameters n, m have alternative constructions, see [17]. We remark that all the flag transitive bislim geometries with gonality 3 are self polar. This just comes out of the classification and I am not aware of a direct proof of this fact. Also, except for the Desargues geometry, all polarities have absolute points (hence no new polarized graphs turn up).

5. Embeddings

In this section we will review mainly two kinds of embeddings: over the reals and over the field of two elements. The first can be called *realizations*; the second *full projective embeddings*. Occasionally we will mention other projective embeddings. Note that, since we will only consider finite geometries, every realization is an affine (real) embedding.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ be two geometries. An *embedding of* Γ *in* Γ' is a pair of injective mapping $\varphi_1 : \mathcal{P} \longrightarrow \mathcal{P}'$ and $\varphi_2 : \mathcal{L} \longrightarrow \mathcal{L}'$ preserving incidence, i.e., if $x \mathtt{I} L$ in Γ, then $x^{\varphi_1} \mathtt{I}' L^{\varphi_2}$. Often the embedding is identified with the image of Γ in Γ'. If Γ' is the geometry arising from the points and lines of a projective (or affine) space, then we talk about a *projective* (or *affine*) *embedding*. In these cases we shall also assume that the image of $\mathcal{P}$ spans linearly the whole projective or affine space, and then the dimension of Γ' as a projective or affine space is the *dimension* of the embedding. A 2-dimensional embedding is called *planar*. Some particular cases: if the projective space is over the real numbers, then we have a *realization*; if for each line $L \in \mathcal{L}$ every point $x' \mathtt{I}' L^{\varphi_2}$ has an inverse image under φ_1, then we call the embedding *full*, if the projective space is over the rational numbers, then we have a *rational embedding*.

A subtle point in the definition of embedding is the fact that we (here) allow for a non incident point line pair (x, L) of Γ the images to be incident. If this is not allowed, then we call the embedding *exclusive*. A geometry is determined by any of its exclusive embeddings, but not necessarily by a non-exclusive one. We will first review some general known results on projective embeddings of bislim geometries and afterwards give some examples. We start with full projective embeddings.

5.1. Full projective embeddings of slim geometries

Concerning full projective embeddings, the situation for slim geometries is rather exceptional. Indeed, in general there is no way to construct full embeddings, far from enumerating all full embeddings. For slim geometries, however, there is a *universal embedding* and every embedding is obtained from that universal one by projection. Let us be more precise in the following theorem, due to Ronan [13].

Let $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ be a full projective embedding of a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ in the space $\mathbf{PG}(V)$, for some vector space V. Then we say that $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ is *universal* if for any other full embedding (φ_1, φ_2) in some projective space $\mathbf{PG}(W)$, there exists a semilinear mapping $\psi : V \longrightarrow W$ such that the corresponding projective mapping $\overline{\psi}$ is well defined on $\mathcal{P}^{\tilde{\varphi}_1}$ and such that $x^{\tilde{\varphi}_1 \overline{\psi}} = x^{\varphi_1}$, for all $x \in \mathcal{P}$.

Theorem 5.1. *If a slim geometry Γ admits a full embedding in some projective space, then it has a universal full embedding.*

The universal embedding of a slim geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is obtained as follows. Consider the vector space V' over $\mathbf{GF}(2)$ (the "Galois field" of 2 elements) freely generated by all elements of $\mathcal{P}$. Then there is an obvious natural embedding of Γ in the projective space $\mathbf{PG}(V)$, where V is obtained from V' by factoring out the subspace generated by the vectors $x_1 + x_2 + x_3$, where $x_i \in \mathcal{P}$, $i = 1, 2, 3$, such that there exists $L \in \mathcal{L}$ with $\Gamma_1(L) = \{x_1, x_2, x_3\}$. This may seem like a simple trick, but in general it is not easy to even determine the dimension of V. An upper bound for the dimension of V is given by the *generating rank* of Γ, i.e., the minimal number of points of $\mathcal{P}$ needed to generate Γ ("to generate" means to successively add to the set the third point on every line for which already two distinct points are in the set). Consequently, if one has a finite set of size ℓ which generates Γ, and if one constructs a full embedding of Γ in $\mathbf{PG}(V)$, with $\dim V = \ell$, then this embedding is automatically universal.

5.2. Realizations of (bi)slim geometries

Realizations of configurations have been considered by several authors. For instance, one is interested in determining which configurations n_3, $7 \leq n \leq 14$, classified by Theorem 4.1, have (planar) realizations. This has been done for $7 \leq n \leq 12$. We summarize the result in the next theorem.

Theorem 5.2. *All bislim linear geometries with n points, $7 \leq n \leq 12$, admit a planar exclusive realization except for*

(*i*) *the Fano plane ($n = 7$), which only admits projective embeddings in planes over fields of characteristic 2 (the generating rank is 3);*

(*ii*) *the Möbius-Kantor geometry ($n = 8$), which admits a planar embedding in a plane over the field $\mathbb{K}$ if and only if the polynomial $x^2 - x + 1$ is reducible over $\mathbb{K}$ (the generating rank is 3);*

(*iii*) *a unique bislim geometry with 10 points whose Menger graph is isomorphic to the complement of the Petersen graph, but which is not isomorphic to the Desargues geometry.*

An explicit construction of the example (say Γ) in (*iii*) runs as follows. We know that the collinearity graph $\mathcal{G}_\perp(\Gamma)$ is the complement of the Petersen graph, hence the point set can be identified with all (unordered) pairs of the set $\{1,2,3,4,5\}$. In $\mathcal{G}_\perp(\Gamma)$, two vertices are adjacent if the corresponding pairs are non disjoint. The Desargues geometry arises by taking as lines those triples of points that correspond with all pairs of a 3-set of $\{1,2,3,4,5\}$. For the unrealizable 10_3 we retain the lines arising from all the triples which contain 5; the other four lines arise as the complements in the set of pairs of $\{1,2,3,4\}$ of the sets of triples of pairs of 3-sets of $\{1,2,3,4\}$ (so these lines contain the points $\{\ell,i\},\{\ell,j\},\{\ell,k\}$, for $\{1,2,3,4\} = \{\ell,i,j,k\}$). Remark that the latter four lines form a unique complete quadrangle in Γ (i.e., a set of four lines which are pairwise confluent), from which we easily deduce that the symmetric group $\mathbf{S}_4$ is the full collineation group of Γ (there is also an obvious polarity mapping a point $\{i,j\}$, $i,j \in \{1,2,3,4\}$, to the line containing all triples of $\{i,j,5\}$, $i \neq j$, and mapping the point $\{i,5\}$, $i \in \{1,2,3,4\}$, to the line incident with the triples of $\{1,2,3,4\}$ containing i; this polarity has absolute lines — the lines of the complete quadrangle — and non absolute points — the points of the complete quadrangle). Further, it has been proved that Γ does not admit an embedding in a projective plane over any (commutative) field — but there are embeddings in non commutative skew field projective planes such as the real quaternion plane (as is easy to calculate). In view of the previous theorem, one would get the impression that realizability is a rather weak property, as there are only 3 exceptions amongst the 275 smallest bislim geometries. This seems to be true in general, but to really exhibit a realization for a given (rather big) bislim geometry can be hard work. A good example of this is the tilde geometry. This bislim geometry escapes from the conditions of the next theorems, and so an independent *personal* proof had to be found to show that the tilde geometry admits a(n exclusive) real planar embedding, see [16]. Moreover, this proof is not completely constructive and more work should be done to actually construct the embedding (although with the aid of a computer this should not be difficult). A general result that again points in the direction of the fact that bislim geometries easily embed in real projective plane (or spaces) was found by Steinitz [14] in 1894. He proved that every bislim geometry admits some representation in the real plane, where all points are represented by different points, and all lines by lines — except possibly for one line! For instance, the Fano plane can be drawn in the real plane using six straight lines and one "circle". But be aware of the fact that it concerns here realizations which are not necessarily exclusive! For exclusive realizations, there is a construction of Grünbaum (see Gropp [7]) that shows that for every $n \geq 16$, there exists a configuration n_3 which cannot be exclusively realized. It is made of two configurations: the Pappus geometry and an arbitrary configuration $(n-9)_3$. Let (x, L) be a flag of the Pappus geometry, and let (x', L') be a flag of the configuration $(n-9)_3$. Then the geometry obtained from the disjoint union of these geometries by declaring (x, L) and (x', L') to be antiflags and instead (x, L') and (x', L) flags (leaving the rest of the incidence relations as it is) is not exclusively realizable since the point x is forced to lie on the line L in the real plane by the properties of the Pappus configuration. Clearly one can repeat this construction with another flag of the original $(n-9)_3$ configuration and a flag of a new copy of the Pappus configuration to obtain a bislim geometry with $n+9$ points which cannot be exclusively realized and for which two lines cannot be represented as lines in the real plane. Going on like this, it is easy to see that examples can be constructed for which an arbitrary number of lines cannot be drawn as lines if one has an exclusive representation in the real (or any other Pappian) projective plane. Let me also remark that all bislim geometries with at most 12 points which are realizable also admit rational planar embeddings. This is a rather peculiar phenomenon. I do not know of any example of a bislim geometry that admits a realization, but not a rational embedding. This is in contrast with non slim geometries. Indeed, the unique generalized quadrangle of order $(4, 2)$ admits a unique embedding in $\mathbf{PG}(3, \mathbb{R})$, and it is not rational (it is described in [11]; the uniqueness is an exercise). Motivated to develop some theory rather than to check a list of small bislim geometries on realisability, we started in [16] to create some methods and criteria. The price we pay for being able to say some general things is that we make assumptions on the collineation group (flag transitivity) and correlation group (being self polar). For the moment, in joint work with Valerie Ver Gucht, we are trying to weaken these hypotheses. So let us review what can be said for now.

Let V be a vector space over any field $\mathbb{K}$, and let ρ be the natural projection map from the set $V \setminus \{\overline{O}\}$ of nonzero vectors to the point set of $\mathbf{PG}(V)$ mapping a vector $\overline{v}$ to the projective point corresponding with the vector line $\overline{v}\mathbb{K}$. If $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a slim geometry, then we call an embedding (φ_1, φ_2) of Γ in $\mathbf{PG}(V)$ *barycentric* if there is a map $\psi : \mathcal{P} \longrightarrow V \setminus \{\overline{O}\}$ with $\psi\rho = \varphi_1$ such that whenever three distinct points p_1, p_2, p_3 of Γ are on a common line of Γ then $p_1^\psi + p_2^\psi + p_3^\psi = \overline{O}$. An embedding is *semi-barycentric* provided a map ψ as above can be found with $\psi\rho = \varphi_1$ such that whenever three distinct points p_1, p_2, p_3 of Γ are on a common line of Γ then either the sum of their images under ψ is equal to the zero vector, or the sum of two of these images equals the third image. An embedding is *projectively G-homogeneous*, for $G \leq \mathrm{Aut}\,\Gamma$, if every collineation of Γ belonging to G is induced by the semi-linear projective group $\mathbf{P\Gamma L}(V)$. An $\mathrm{Aut}\,\Gamma$-homogeneous embedding is also simply called *projectively homogeneous*. If moreover G lifts to a subgroup of $\mathbf{GL}(V)$, then we say that the embedding is *vector G-homogeneous*; similarly one introduces the notion of *vector homogeneous*. For projective embeddings over $\mathbf{GF}(2)$, the notions of vector and projectively homogeneity coincide and we talk simply about *homogeneous embeddings*.

First of all, we remark that the role of the universal embedding for full embeddings can be taken over for real embeddings by a certain universal barycentric embedding. Hence this is a good reason to study barycentric embeddings. The following result is easy to prove.

Proposition 5.1. *Suppose the slim geometry Γ with n points and m lines has some barycentric embedding in $\mathbf{PG}(V)$, for some vector space V of dimension d over $\mathbb{R}$. Then the rank of any incidence matrix of Γ is at most $n-d$.*

The next result says that for bislim geometries with large collineation groups the notions of vector G-homogeneous real embedding and barycentric real embedding are more or less equivalent. A analogous statement holds for semi-barycentric real embeddings.

Proposition 5.2. *Let Γ be a flag transitive (with respect to a collineation group G) slim geometry G-homogeneously embedded in the real projective space $\mathbf{PG}(V)$, for some $G \leq \mathrm{Aut}(\Gamma) \cap \mathbf{PGL}(V)$. Then the group G lifts to a subgroup of $\mathbf{GL}(V)$ if and only if the embedding is barycentric. Also, some nontrivial central extension $2 \cdot G$ of G lifts to a subgroup of $\mathbf{GL}(V)$, with $2 \cdot G/(2 \cdot G \cap \mathbf{Sc}(V)) = G$, if and only if the embedding is semi-barycentric, but not barycentric. In the latter case there is a connected double cover Γ' of Γ.*

The next theorem provides a concrete construction of some barycentric or semi-barycentric embeddings of certain bislim geometries. Note that a geometry has a symmetric incidence matrix if and only if it is self polar.

Theorem 5.3. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a connected self polar point-line geometry with n points and let G be a flag transitive collineation group acting primitively on $\mathcal{P}$. Let $\mathbf{A}$ be a symmetric incidence matrix of Γ, and consider it as the matrix of a linear endomorphism of real n-space. If $\det \mathbf{A} = 0$, then the projection of the basis vectors from $\mathrm{Im}\,\mathbf{A}$ into $\mathrm{Ker}\,\mathbf{A}$ yields a G-homogeneous barycentric embedding of Γ. If $\det \mathbf{A} \neq 0$, and if Γ admits a self polar double cover Γ' with corresponding symmetric incidence matrix $\mathbf{A}'$ such that $\det \mathbf{A}' = 0$ and such that Γ' admits a collineation group G' which induces G in Γ, then, considering $\mathbf{A}'$ as the matrix of a linear endomorphism of real $2n$-space, the projection of the basis vectors from $\mathrm{Im}\,\mathbf{A}'$ into $\mathrm{Ker}\,\mathbf{A}'$ yields a G-homogeneous semi-barycentric (non barycentric) embedding of Γ (basis vectors corresponding to points of Γ' in the same fiber are indeed projected onto opposite vectors).*

It can be easily shown that, if a slim geometry admits a barycentric real embedding, then it admits a *universal* one, defined in the same way as universal full embedding. It turns out that this universal barycentric real embedding can be explicitly constructed using the incidence matrix for self polar bislim flag transitive geometries. In particular, the dimension of the embedding can be deduces easily from the rank of that matrix.

Corollary 5.1. *Suppose a slim self polar flag transitive point-line geometry* Γ *admits at least one real barycentric embedding. Let* $\mathbf{A}$ *be a symmetric incidence matrix of* Γ*, and consider it as the matrix of a linear endomorphism of real* n*-space, with* n *the number of points of* Γ*. Then* $\det \mathbf{A} = 0$*, and the projection of the basis vectors from* $\operatorname{Im} \mathbf{A}$ *into* $\operatorname{Ker} \mathbf{A}$ *yields the (homogeneous) universal real barycentric embedding of* Γ*.*

We now turn to some examples.

5.3. Examples of embeddings of bislim geometries

First we mention the classical construction of the **Pappus configuration** in the real projective plane. Consider three collinear points a_1, a_2, a_3, and three collinear points b_1, b_2, b_3 such that the intersection point of the corresponding lines is not amongst $a_1, \ldots, b_3$. Then the lines $a_i b_j$ and $a_j b_i$ meet in some point c_k, with $\{i, j, k\} = \{1, 2, 3\}$, and c_1, c_2, c_3 are automatically collinear. The points and lines just mentioned form the Pappus geometry. In fact, the Pappus configuration admits such an embedding for every choice of a_1, a_2, a_3 and b_1, b_2, b_3 in the projective plane over *any* commutative field with at least 3 elements (these planes are therefore called *Pappian planes*). The generating rank of the Pappus geometry is 3; its universal full embedding rank is zero, and its universal barycentric embedding rank is also zero. It does not admit semi-barycentric embeddings. Its collineation group does not act primitively; the Pappus geometry is a triple cover of the bislim generalized digon.

The *universal barycentric embedding rank* of a slim geometry is $d+1$ if the universal barycentric embedding exists and lives in d-dimensional real projective space. If the geometry does not admit any barycentric real embedding, then we say that the universal barycentric embedding rank is zero. The *universal full embedding rank* of a slim geometry is $d+1$ if the universal full embedding exists and lives in d-dimensional projective space over $\mathbf{GF}(2)$. If the geometry does not admit any full embedding, then we say that the universal full embedding rank is zero.

Also there is the classical construction of the **Desargues configuration** in the real projective plane (or real projective 3-space) with two triangles in perspective from a point, implying that the corresponding sides of the triangles meet on a common line (and hence the triangle are also in dual perspective from that line). The previous construction of the Desargues configuration can be done in every projective plane over any skew field (therefore called *Desarguesian planes*) with at least 4 elements (for exclusive embeddings; in the non exclusive case a plane over $\mathbf{GF}(3)$ qualifies and gives rise to the so-called *little Desargues configuration*), and in every projective 3-space. If we look for projectively homogeneous planar embeddings, then we have to restrict to fields with characteristic 5. The generating rank of the Desargues geometry is equal to 4; its universal embedding rank is equal to 4 and its universal barycentric embedding rank is 0.

The **dodecahedron geometry** is a double cover of the Desargues geometry and an arbitrary incidence matrix of the dodecahedron geometry (which is an adjacency matrix of the dodecahedron graph) is singular. All conditions of Theorem 5.3 are satisfied and so the Desargues geometry admits a semi-barycentric embedding in projective 3-space and 2-space. Furthermore, the dodecahedron geometry does not admit a barycentric embedding, although it admits some real planar embedding.

The **triangle geometry** behaves much like the Desargues geometry. Its generating rank is equal to 8 and its universal full embedding rank is also equal to 8. Here is an explicit construction of the universal full embedding. All 3×3 matrices over $\mathbf{GF}(2)$ form a 9-dimensional vector space V with standard basis $\{E_{ij}\,|\,i,j=1,2,3\}$. We consider the quotient space $W=V/T$, where

T is the vector line generated by the identity matrix $I = E_{11} + E_{22} + E_{33}$. Now we identify the points of $\mathbf{PG}(2,2)$ with triples $(x_1, x_2, x_3) \in \mathbf{GF}(2)^3$ and the lines with triples $[a_1, a_2, a_3] \in \mathbf{GF}(2)^3$ such that a point (x_1, x_2, x_3) is incident with a line $[a_1, a_2, a_3]$ if and only if $a_1x_1 + a_2x_2 + a_3x_3 = 0$. We identify an antiflag $\{(x_1, x_2, x_3), [a_1, a_2, a_3]\}$ with the point $\sum_{i,j} x_i a_j E_{ij} + T$ of $\mathbf{PG}(W)$. This defines the universal full embedding in $\mathbf{PG}(W)$. Any incidence matrix of the triangle geometry — which is equivalent to an adjacency matrix of the Coxeter graph — is nonsingular, hence there is no barycentric real embedding of the triangle geometry. But the latter has a double cover which satisfies the assumptions of Theorem 5.3. Hence the triangle geometry has a semi-barycentric embedding (in 7-dimensional real space, as it turns out).

An *ovoid* of a generalized quadrangle is a set of points such that every line is incident with exactly one element of that set. It follows that, if the order of the generalized quadrangle is equal to (s,t), then a potential ovoid has $1 + st$ points.

The bislim **generalized quadrangle** $\mathsf{W}(2)$ is self polar and admits both real barycentric and full embeddings. Its generating rank, its universal full embedding rank and its universal barycentric embedding rank are all equal to 5. The universal full embedding and the universal barycentric embedding have the following common description. Recall that $\mathsf{W}(2)$ has six different ovoids. Consider a vector space V of dimension 6 (over an arbitrary field $\mathbb{K}$) with the standard basis indexed by the six ovoids of $\mathsf{W}(2)$. Let the point x of $\mathsf{W}(2)$ correspond with the vector of W determined by the property that the coordinate corresponding to the ovoid $\mathcal{O}$ is equal to 1 if $x \notin \mathcal{O}$, and otherwise it is equal to -2. Let W be the subspace of V whose elements have coordinates that sum up to zero. Then we have just defined an embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(W)$. If $\mathbb{K} = \mathbb{R}$, then this is the universal barycentric embedding; if $\mathbb{K} = \mathbf{GF}(2)$, then we obtain the universal full embedding. There is also a full embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(3,2)$, the 3-dimensional projective space over $\mathbf{GF}(2)$ (and this embedding arises from a symplectic polarity). One can easily show that there are exactly 2 full embeddings of $\mathsf{W}(2)$ (the one in $\mathbf{PG}(3,2)$ and the one in $\mathbf{PG}(4,2)$ that we have just mentioned) and they are both homogeneous.

The bislim **generalized hexagon** $\mathsf{H}(2)$ has four homogeneous full embeddings. The one in the smallest dimension (in $\mathbf{PG}(5,2)$) may be described as follows. Recall the construction of $\mathsf{H}(2)$ above. The point set is $\mathcal{P}_0 \times \mathcal{L}_0 \setminus \{(0,0)\}$ (with the notation as before). This can be seen as the point set of $\mathbf{PG}(5,2)$ by declaring three distinct points collinear whenever they add up to $(0,0)$. The lines of $\mathsf{H}(2)$ are then also lines of $\mathbf{PG}(5,2)$ and we indeed have a full homogeneous embedding. The one in the biggest dimension (in $\mathbf{PG}(13,2)$) may be described in full generality over any field (in particular over $\mathbb{R}$, where it gives the universal barycentric embedding). Set $\mathcal{P} = \mathcal{P}_0 \setminus \{0\}$ and $\mathcal{L} = \mathcal{L}_0 \setminus \{0\}$ and denote incidence in the Fano plane with $\mathtt{I}$. We label the basis vectors of a 14-dimensional vector space over any field $\mathbb{K}$ with the point and line set of the Fano plane. A point $(x,0)$, $x \in \mathcal{P}$, is identified with the sum of basis vectors corresponding to the point x and all lines of the Fano plane not incident with x. Also, the point $(0,L)$, with $L \in \mathcal{L}$, is identified with the sum of basis vectors corresponding to the points of the Fano plane incident with L and the lines of the Fano plane distinct from L. Furthermore, the point $(x,L) \in \mathcal{P} \times \mathcal{L}$, with $x\mathtt{I}L$, is identified with the sum of the basis vectors corresponding with the points of the Fano plane on L, but distinct from x, and the lines of the Fano plane incident with x, but distinct from L. Finally, the point $(x,L) \in \mathcal{P} \times \mathcal{L}$, with x not on L in the Fano plane, is identified with the sum of the basis vector corresponding with x and the one corresponding with L. By projection from the origin, this defines an embedding of $\mathsf{H}(2)$ in a 13-dimensional projective space over the field $\mathbb{K}$.

A symmetric incidence matrix of the **tilde geometry** is an adjacency matrix of the Foster graph. This matrix has eigenvalue 0 with multiplicity 5, and applying the construction as given by Theorem 5.3 (ignoring the fact that we are not dealing with a primitive automorphism group), we obtain a real barycentric embedding of $\ldots \mathsf{W}(2)$. In fact, it was a hard question to find a realization of the tilde geometry, but it was proved in [16] that is has a planar one. I do not know any realization in projective dimension ≥ 3. The full embeddings of the tilde geometry behave much nicer. In fact, all homogeneous full embeddings have been classified in [10]. Let us give a description. Consider the 3-dimensional vector space $V = V(3,4)$ over the field $\mathbf{GF}(4)$. The associated projective plane $\mathbf{PG}(2,4)$ contains a *hyperoval* (here, it is just a conic with its nucleus, i.e., the intersection of all tangent lines), which corresponds to a set of six vector lines in V. Viewing $\mathbf{GF}(4)$ as a 3-dimensional vector space over $\mathbf{GF}(2)$, we obtain a 6-dimensional vector space $W = V(6,2)$ endowed with a set of 21 vector planes (corresponding to the vector lines of V) and a certain subset of six vector planes (corresponding to that hyperoval). This gives rise to a projective space $\mathbf{PG}(5,2)$ with a set $\mathcal{S}$ of 21 lines and a subset $\mathcal{O}$ of 6 lines. Any two elements $L, M \in \mathcal{S}$ span a 3-dimensional projective subspace of $\mathbf{PG}(5,2)$ containing exactly 5 elements of $\mathcal{S}$. Let $\mathcal{L}$ be the set of lines of $\mathbf{PG}(5,2)$ obtained as follows. We take two arbitrary elements L, L' of $\mathcal{O}$, consider the subspace generated by L, L', and the lines M, M', M'' of $\mathcal{S}$ in that subspace different from L, L'. There are exactly three lines of $\mathbf{PG}(5,2)$ meeting all of M, M', M'', and these belong by definition to $\mathcal{L}$. This way $\mathcal{L}$ contains 45 lines. The set of points on these lines in $\mathbf{PG}(5,2)$ is by definition $\mathcal{P}$ (and there are $63 - 6 \times 3 = 45$ such points). With $\mathtt{I}$ the incidence inherited from $\mathbf{PG}(5,2)$, the geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ is the tilde geometry, given as a full embedding in $\mathbf{PG}(5,2)$. Now consider any full embedding of $\mathsf{W}(2)$ (they are all homogeneous as mentioned above). So the points and lines of $\mathsf{W}(2)$ are points and lines of $\mathbf{PG}(d,2)$, with $d \in \{3,4\}$. Consider $\mathbf{PG}(d,2)$ and $\mathbf{PG}(5,2)$ as disjoint subspaces of $\mathbf{PG}(d+6,2)$. Let ρ be a covering map from the point set of $\widetilde{\mathsf{W}(2)}$ (inside $\mathbf{PG}(5,2)$) to the point set of $\mathsf{W}(2)$ (inside $\mathbf{PG}(d,2)$). Define the set $\mathcal{P}'$ as the set of points of either $\mathbf{PG}(d+6,2)$ obtained by considering the "third points" on the lines xx^ρ, with $x \in \mathcal{P}$. This gives two homogeneous embeddings of $\widetilde{\mathsf{W}(2)}$. The one in $\mathbf{PG}(10,2)$ is moreover the universal full embedding. Let me also add that the embedding in $\mathbf{PG}(9,2)$ can be obtained by taking the line Grassmannian of the embedding in $\mathbf{PG}(5,2)$. Remark that there are many other full embeddings of the tilde geometry by considering projections. So unlike $\mathsf{W}(2)$, not all full embeddings are homogeneous. Finally, the generating rank of the tilde geometry is equal to 11, and so is the universal full embedding rank. The universal barycentric embedding rank is zero.

The **Biggs-Smith geometry** is an example of a bislim self polar geometry with both universal embedding ranks different from zero, but with these two ranks distinct from each other. Indeed, the adjacency matrix of the Biggs-Smnith graph has rank $85 = 102 - 17$; hence by Theorem 5.3 the universal barycentric embedding rank is equal to 17. On the other hand the universal full embedding lives in $\mathbf{PG}(18, 2)$, hence the universal full embedding rank is equal to 19. Until recently, it was not known what the generating rank is, but Valerie Ver Gucht (unpublished) recently showed it is equal to 19.

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