Classification of Finite Veronesean Caps

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Abstract

We show that all Veronesean caps in finite projective spaces, as defined by Mazzocca & Melone [3], are projections of quadric Veroneseans. In fact we prove a slightly stronger result by weakening one of the conditions of Mazzocca and Melone.

1 Introduction

Let $\mathbb{K}$ be a field and n a natural number greater than or equal to 1. The *quadric Veronesean* $\mathcal{V}_n$ *of index* n is the set of points of the projective space $\mathbf{PG}(n(n+3)/2, \mathbb{K})$ with generic element

$$(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, x_0x_2, \ldots, x_0x_n, x_1x_2, \ldots, x_1x_n, \ldots, x_{n-1}x_n),$$

where $(x_0, x_1, \ldots, x_n)$ is a point of $\mathbf{PG}(n, \mathbb{K})$. Equivalently, if we consider a point of $\mathbf{PG}(n(n+3)/2, \mathbb{K})$ with projective coordinates

$$(y_{00}, y_{11}, \ldots, y_{nn}, y_{01}, y_{02}, \ldots, y_{0n}, y_{12}, \ldots, y_{1n}, \ldots, y_{n-1,n}),$$

then it belongs to $\mathcal{V}_n$ if and only if $\mathrm{rank}(y_{ij}) = 1$, with $y_{ij} = y_{ji}$ if $i > j$. Quadric Veroneseans have some nice geometric properties, and for finite fields $\mathbb{K} = \mathbf{GF}(q)$, they are very useful objects in finite geometry. In fact, mostly one does not recognize them via the above definition, but via some of their characteristic properties. Mazzocca & Melone [3] formulate three geometric properties ((Q1), (Q2) and (Q3) below, with the remark that they assume "conics" in (Q1) instead of ovals) that should characterize $\mathcal{V}_n$ (they call the objects satisfying these three axioms *Veronesean caps*), and they erroneously thought they indeed did (so they thought they proved that every Veronesean cap was a quadric Veronesean). Hirschfeld & Thas [2] pointed out some counterexamples and added a fourth axiom to make the characterization work; it should also be noted that Hirschfeld & Thas [2] modified the proof of Mazzocca & Melone [3] so as to hold also in the even case. That extra fourth axiom is just a bound on the dimension of the ambient projective space. In the present paper, we delete this condition again, weaken one of the other conditions, and prove that the resulting geometric object, which we also call a Veronesean cap, is projectively equivalent either to a quadric Veronesean, or to a proper projection of some quadric Veronesean. This in particular solves the original problem of Mazzocca & Melone completely in the finite case.

The proof of the characterization of the quadric Veronesean using the axioms (Q1), (Q2), (Q3) and the bound on the dimension is rather long, see Hirschfeld & Thas [2], and we include a much shorter proof in the present paper. Consequently, we will prove the entire classification of Veronesean caps independent of the literature, except that we will not prove that a quadric Veronesean indeed satisfies the given axioms — this can be found in Mazzocca & Melone [3] and Hirschfeld & Thas [2]. Moreover, since we weaken one of the conditions of Mazzocca & Melone, a new proof is not only justified, but also desirable.

More precisely, we will prove:

Theorem 1.1 *Let X be a set of points in $\Pi := \mathbf{PG}(M,q)$, $M>2$, spanning Π, and let $\mathcal{P}$ be a collection of planes of Π such that for any $\pi \in \mathcal{P}$, the intersection $X \cap \pi$ is an oval in π. For $\pi \in \mathcal{P}$ and $x \in X \cap \pi$, we denote by $T_x(\pi)$ the tangent line to $X \cap \pi$ at x in π. We assume the following three properties.*

(Q1) *Any two points $x, y \in X$ lie in a unique member of $\mathcal{P}$ which we denote by $[x, y]$;*

(Q2) *if $\pi_1, \pi_2 \in \mathcal{P}$ and $\pi_1 \cap \pi_2 \neq \emptyset$, then $\pi_1 \cap \pi_2 \subseteq X$;*

(Q3) *if $x \in X$ and $\pi \in \mathcal{P}$ with $x \notin \pi$, then each of the lines $T_x([x, y])$, $y \in X \cap \pi$, is contained in a fixed plane of Π, denoted by $T(x, \pi)$.*

Then there exists a natural number $n \geq 2$ (called the index of X), a projective space $\Pi' := \mathbf{PG}(n(n+3)/2, q)$ containing Π, a subspace R of Π' skew to Π, and a quadric Veronesean $\mathcal{V}_n$ of index n in Π', with $R \cap \mathcal{V}_n = \emptyset$, such that X is the (bijective) projection of $\mathcal{V}_n$ from R onto Π. The subspace R can be empty, in which case X is projectively equivalent to $\mathcal{V}_n$.

Also, note that the set of planes $\mathcal{P}$ is uniquely determined by X if $q > 2$. Indeed, since $q > 2$, any conic on X contains at least four different points x_1, x_2, x_3, x_4. Suppose that $[x_1, x_2] \neq \langle x_1, x_2, x_3, x_4 \rangle$. Since X is a cap, the lines x_1x_2 and x_3x_4 meet in a point $x \notin \{x_1, x_2, x_3, x_4\}$. It follows that $[x_1, x_2] \cap [x_3, x_4]$ contains a point $x \notin X$, contradicting (Q2). Hence $[x_1, x_2] = \langle x_1, x_2, x_3, x_4 \rangle$ and the conic must belong to $\mathcal{P}$.

Now suppose $q = 2$. It is easy to check that $\mathcal{V}_2$ consists of 7 points of $\mathbf{PG}(5, 2)$ no 6 of which lie in a hyperplane. Since the setwise stabilizer of $\mathcal{V}_2$ in $\mathbf{PGL}(6, 2)$ is isomorphic to $\mathbf{Sym}(7)$, we can take any 7 3-subsets of $\mathcal{V}_2$ forming a projective plane and obtain an isomorphic copy of $(\mathcal{V}_2, \mathcal{P})$. For $n > 2$, a rather tedious calculation shows that a set of 7 points in $\mathcal{V}_n$ is contained in a 5-space if and only if these 7 points correspond to the set of points of a plane of $\mathbf{PG}(n,q)$. Hence the conic planes of $\mathcal{V}_n$ can be reconstructed as intersections of such 5-spaces. It is not clear, however, that this is also true for the projections of $\mathcal{V}_n$.

Despite the fact that X does not necessarily uniquely determine $\mathcal{P}$ for $q = 2$, we will nevertheless denote a Veronesean cap by X, with the implicit understanding that there is some prescribed set $\mathcal{P}$ of planes satisfying (Q1), (Q2) and (Q3).

We will prove Theorem 1.1 in a sequence of lemmas and propositions. First, we prove some easy geometric properties of Veronesean caps, including the existence of the index n. Then we show that, if $\dim \Pi \geq n(n+3)/2$, then X is projectively equivalent with $\mathcal{V}_n$ (this is also contained in Hirschfeld & Thas [2], but under the stronger condition that we have conics instead of ovals). Finally, we show that, if $M < n(n+3)/2$, then X is the projection from a point of another Veronesean cap spanning an $(M+1)$-dimensional projective space containing Π.

Henceforth, we assume that X is a Veronesean cap in $\Pi = \mathbf{PG}(M,q)$, satisfying the axioms (Q1), (Q2) and (Q3) stated above. For a point $x \in X$ and a plane $\pi \in \mathcal{P}$ containing x, we will occasionally denote the line $T_x(\pi)$ by $T_x(\pi \cap X)$; similarly, we will sometimes write $T(x', \pi \cap X)$ for $T(x', \pi)$, $x' \in X \setminus \pi$. Note that it immediately follows from (Q1) that X is a *cap* (i.e. a set of points no three of which are collinear). Hence the terminology of *Veronesean cap* is consistent.

2 Some properties of Veronesean caps

Our first main aim is to show that $(X, \mathcal{X}, \in)$, with $\mathcal{X} = \{\pi \cap X \mid \pi \in \mathcal{P}\}$, is the point-line truncation of a projective space of order q and some dimension $n \geq 2$.

As an immediate consequence of (Q1), (Q2) and (Q3), we may state:

Lemma 2.1 *If $x \in X$ and $\pi \in \mathcal{P}$ with $x \notin \pi$, then $T(x,\pi) \setminus \{x\}$ is the disjoint union of $T_x([x,y]) \setminus \{x\}$, with y ranging over $X \cap \pi$.* □

Property 2.2 *The incidence structure $(X, \mathcal{X}, \in)$ defined above is the point-line geometry of a projective space of order q and some dimension $n \geq 2$.*

Proof: Clearly, by (Q1), the incidence structure $(X, \mathcal{X}, \in)$ is a linear space in which all lines have size $q+1$. It suffices to check the axiom of Pasch (or Veblen-Young). So let $O_1, O_2 \in \mathcal{X}$ be two ovals meeting in a point $x \in X$, and let $O_3, O_4 \in \mathcal{X}$, with $x \notin O_3 \cup O_4$, be such that they both meet O_1 and O_2 in distinct points $x_{ij} := O_i \cap O_j$, $i \in \{1,2\}$ and $j \in \{3,4\}$. We must show that O_3 and O_4 are not disjoint. Both planes $T(x_{13}, O_2)$ and $T(x_{13}, O_4)$ contain the distinct lines $T_{x_{13}}(O_1)$ and $T_{x_{13}}([x_{13}, x_{24}])$, hence they coincide. By the previous lemma, there is some point $x \in O_4$ such that $T_{x_{13}}([x_{13}, x]) = T_{x_{13}}(O_3)$. But then (Q1) implies that $O_3 = [x_{13}, x] \cap X$. Hence $x \in O_3 \cap O_4$. $\square$

We call n *the index of* X.

Corollary 2.3 *Let* $(Y, \mathcal{Y}, \in)$ *be a subspace of* $(X, \mathcal{X}, \in)$ *of dimension* $r < n$. *Put* $\mathcal{P}(Y) = \{\pi \in \mathcal{P} \mid \pi \cap X \in \mathcal{Y}\}$. *Then* Y *satisfies axioms* (Q1), (Q2) *and* (Q3) *and hence is a Veronesean sub-cap of* X *of index* r *(with* $\mathcal{P}(Y)$ *as prescribed set of planes).* $\square$

Next, we introduce the tangent space of X at a point $x \in X$.

Property 2.4 *For* $x \in X$, *let* $T(x)$ *be the union of all lines* $T_x([x,y])$, $y \in X \setminus \{x\}$. *Then* $T(x)$ *is a subspace of* Π *of projective dimension* n.

Proof: We proceed by induction on the index n. If $n = 2$, then this follows from Lemma 2.1 and from the fact that any two elements of $\mathcal{P}$ intersect. Now let $n > 2$. Select any subspace $(Y, \mathcal{Y}, \in)$ of $(X, \mathcal{P}, \in)$ of projective dimension $n-1$ with $x \in Y$. By the induction hypothesis the union of all lines $T_x([x,y])$, $y \in Y \setminus \{x\}$, is an $(n-1)$-dimensional subspace U of Π. Pick $z \in X \setminus Y$. Then the space T generated by U and $T_x([x,z])$ is n-dimensional. For any $w \in X \setminus Y$, $w \neq z$, there is a unique point $w' \in Y \cap [w,z]$ (because $(Y, \mathcal{Y}, \in)$ defines a hyperplane of $(X, \mathcal{P}, \in)$. If $w' = x$, then clearly $T_x([x,w]) = T_x([x,z]) \subseteq T$. If $w' \neq x$, then $T_x([x,w]) \subseteq T(x, [z,w']) \subseteq \langle T_x([x,w']), T_x([x,z]) \rangle \subseteq T$. Hence $T(x)$ is contained in T. It is now easy to see that $T = T(x)$. $\square$

If $n = 2$, then $(X, \mathcal{X}, \in)$ is a projective plane. In the next section we will show that this projective plane is actually Desarguesian and that all ovals in $\mathcal{X}$ are conics.

3 A characterization of quadric Veroneseans

Throughout, we denote the n-dimensional projective space defined by $(X, \mathcal{X}, \in)$ by $\Pi(X)$. We also denote a point, a point set, or a subspace of $\Pi(X)$ with a superscript "$*$", and the corresponding point, point set, or Veronesean sub-cap in X by the same symbol without the superscript "$*$". We first show a bound on the dimension M of Π.

Property 3.1 $M \leq n(n+3)/2$.

Proof: We proceed by induction on n, first assuming $n > 2$. Let Y^* be a hyperplane of $\Pi(X)$ and $x \in X \setminus Y$. Then $\dim\langle Y \rangle \leq (n-1)(n+2)/2$ by induction. Let $z \in X \setminus \{x\}$. Then either $z \in \langle Y \rangle$ or $[x, z] \cap Y = \{z'\}$ and $z \in \langle T_x([x, z]), z' \rangle \subseteq \langle T(x), Y \rangle$. Hence $\Pi = \langle T(x), Y \rangle$, implying $\dim \Pi \leq 1 + n + (n-1)(n+2)/2 = n(n+3)/2$. If $n = 2$, then we can write down the same arguments replacing Y by an element of $\mathcal{X}$. $\square$

Proposition 3.2 *If $n = 2$, then $M = 5$, the plane $\Pi(X)$ is Desarguesian, all members of $\mathcal{P}$ are conics, and X is projectively equivalent to a quadric Veronesean of index 2.*

Proof: Let π be a fixed member of $\mathcal{P}$ and put $O = X \cap \pi$. We first show that $M = 5$. Let $x \in X \setminus O$. From the proof of Property 3.1, it follows that $\Pi = \langle T(x), \pi \rangle$. Suppose by way of contradiction that there exists $u \in T(x) \cap \pi$. Lemma 2.1 implies that the line xu is tangent to a certain oval $X \cap [x, y]$, for some $y \in O$. But now $u \notin X$ and nevertheless $u \in [x, y] \cap \pi$, contradicting (Q2). It now follows that $M = 5$.

Now we consider a subspace U of dimension 2 skew to π and denote by ρ the projection from π onto U. We claim that ρ is injective on $X \setminus O$. Indeed, if $\rho(x) = \rho(y)$, for $x, y \in X \setminus O$, then $\langle x, \pi \rangle = \langle y, \pi \rangle \supseteq [x, y]$ and hence $\pi \cap [x, y]$ is a line; this contradicts (Q2) and shows the claim. Clearly, the points of $X \setminus O$ on an oval O' of X different from O are mapped onto q points of a line of U; the missing point is the projection of the tangent line (minus its point on O) of O' at the point $O \cap O'$. So we obtain a set V of q^2 points of U and $q^2 + q$ lines of U all containing exactly q points of V; it follows that the remaining line L_∞ of U does not contain any point of V (indeed, if L_∞ contains a point $v \in V$, then consider two distinct lines L_1, L_2 through v, distinct from L_∞. The unique points x_1, x_2 of L_1, L_2, respectively, not in V are distinct and not incident with L_∞. But now the line x_1x_2 contains at most $q-1$ points of V, a contradiction), hence it contains the projections of all tangent lines (minus their points on O) mentioned above. We now also deduce that the projective plane $\Pi(X)$ is isomorphic to U, and hence it is in particular Desarguesian. So we can denote it by $\mathbf{PG}(2, q) := \Pi(X)$. We also deduce the property that, given a point $p \in O$ and a point $x \in X \setminus O$, the space $\langle T(p), x \rangle$ meets X in $X \cap [x, p]$ (this is the inverse image in X of the space generated by $\rho(x)$ and $\rho(T(p) \setminus O)$); we refer to this property as (*).

Next we show that all the ovals are conics.

Consider two points $a, b \in O$. We project $X \setminus O$ from the line ab onto some 3-dimensional space W of Π skew to ab. We call the projection map ρ'. Then the image of the q points not on O of an oval $O' \neq O$ belonging to $\mathcal{P}$ and containing a, together with the image of its tangent line (minus its point on O) at $O \cap O'$ is a line of W; similarly for ovals on X through b. Also, the set of images of tangent lines at a to ovals different from O through a, together with the image of $\pi \setminus \langle a, b \rangle$ is also a line of W and similarly for b. So we obtain a set of $(q+1)^2$ points of W containing two sets of $q+1$ mutually skew lines, and lines of different sets intersect in exactly one point; this is a hyperbolic quadric H. It follows that the image D under ρ' of any member of $\mathcal{P}$ not containing a nor b is a conic section of H; hence the oval is, as the intersection of a cone $(ab)D$ with a plane, itself a conic.

Now we consider three points p_0, p_1, p_2 of X such that p_0^*, p_1^*, p_2^* form a triangle in $\mathbf{PG}(2,q)$. We let $\mathcal{V}_2$ be the quadric Veronesean in $\Pi = \mathbf{PG}(5,q)$ associated with $\mathbf{PG}(2,q)$, and denote for each point or subset a^* of $\mathbf{PG}(2,q^2)$ the corresponding point or subset on $\mathcal{V}_2$ by $a^\dagger$. Since $\mathcal{V}_2$ satisfies in particular (Q1), (Q2), (Q3), we may treat $\mathcal{V}_2$ as a Veronesean cap and thus use appropriate notation. The planes $[p_0,p_1]$, $[p_1,p_2]$, $[p_2,p_0]$ generate Π, because the space they generate contains both $T(p_0)$ and $[p_1,p_2]$. We now project X and $\mathcal{V}_2$ from $[p_0,p_1]$ and $[p_0^\dagger,p_1^\dagger]$, respectively, onto the planes U and $U^\dagger$, respectively. Let θ be the field automorphism associated to the isomorphism from U to $U^\dagger$ which maps the projection of a point $x \in X$ onto the projection of $x^\dagger$. Then it is clear that there is an isomorphism (of conics, i.e., a bijection preserving the cross-ratio, up to a field automorphism) between $[p_0,p_2] \cap X$ and $[p_0^\dagger,p_2^\dagger] \cap \mathcal{V}_2$, and between $[p_1,p_2] \cap X$ and $[p_1^\dagger,p_2^\dagger] \cap \mathcal{V}_2$, respectively, with associated field automorphism θ, and mapping x onto $x^\dagger$. By extending these isomorphisms to the planes $[p_0,p_2]$ and $[p_1,p_2]$, respectively, and permuting the indices, we conclude that there are collineations $\alpha_i : [p_j,p_k] \to [p_j^\dagger,p_k^\dagger]$, for all i,j,k, with $\{i,j,k\} = \{0,1,2\}$, with associated field automorphism θ, mapping x to $x^\dagger$, for all $x \in X \cap ([p_0,p_1] \cup [p_1,p_2] \cup [p_2,p_0])$.

Now α_0 and α_1 extend to a common collineation α' between $\langle [p_1,p_2],[p_0,p_2]\rangle$ and $\langle [p_1^\dagger,p_2^\dagger],[p_0^\dagger,p_2^\dagger]\rangle$. Consider any $\xi \in \mathcal{P}$, with $p_0,p_1,p_2 \notin \xi$. Let r be the intersection of the line p_0p_1 and the tangent line L_2 of $[p_0,p_1] \cap X$ at $\xi \cap [p_0,p_1]$. Consider the tangent lines L_0 and L_1 of $[p_1,p_2] \cap X$ and $[p_0,p_2] \cap X$ at $\xi \cap [p_1,p_2]$ and $\xi \cap [p_0,p_2]$, respectively. Letting ξ play the role of π in the first part of the proof, we immediately see that the subspace $\langle L_0,L_1,L_2\rangle$ is 4-dimensional and meets X in ξ. Hence $\langle L_0,L_1\rangle$ intersects the line p_0p_1 in r (choose the plane U in the first part of the proof so that it contains p_0p_1). It follows that the restriction of α' to p_0p_1 coincides with the restriction of α_2 to p_0p_1. So there exists a collineation $\alpha : \Pi \to \Pi$ such that $\alpha \equiv \alpha_i$ on $[p_j,p_k]$, for all i,j,k, with $\{i,j,k\} = \{0,1,2\}$.

Now let $x \in X$ be arbitrary, but not belonging to $[p_0,p_1] \cup [p_1,p_2] \cup [p_2,p_0]$. Put $x_i := [x,p_i] \cap [p_j,p_k]$, for $\{i,j,k\} = \{1,2,3\}$. Then, by $(*)$, we see that x is the intersection of X with $\langle T(p_0),x_0\rangle \cap \langle T(p_1),x_1\rangle \cap \langle T(p_2),x_2\rangle$. Since this implies $x^* = p_0^*x_0^* \cap p_1^*x_1^* \cap p_2^*x_2^*$, we also have that $x^\dagger = \mathcal{V}_2 \cap \langle T(p_0^\dagger),x_0^\dagger\rangle \cap \langle T(p_1^\dagger),x_1^\dagger\rangle \cap \langle T(p_2^\dagger),x_2^\dagger\rangle$. We now claim that $\{x\} = \langle T(p_0),x_0\rangle \cap \langle T(p_1),x_1\rangle \cap \langle T(p_2),x_2\rangle$. Indeed, clearly $[p_1,p_2] \subseteq \langle x_0,T(p_1)\rangle$, hence $\Pi = \langle T(p_0),[p_1,p_2]\rangle = \langle T(p_0),T(p_1),x_0,x_1\rangle$. Consequently $\langle T(p_0),x_0\rangle \cap \langle T(p_1),x_1\rangle$ is a line of $\mathbf{PG}(5,q)$, which clearly contains x and $x' := T_{p_0}([p_0,p_1]) \cap T_{p_1}([p_0,p_1])$. Since $x' \neq x_2$, the assumption $x' \in \langle T(p_2),x_2\rangle$ would lead to $\dim\langle T(p_2),x_2\rangle \geq 4$ (remembering that $T(p_2) \cap [p_0,p_1] = \emptyset$), a contradiction. The claim follows. But similarly we conclude that $\{x^\dagger\} = \langle T(p_0^\dagger),x_0^\dagger\rangle \cap \langle T(p_1^\dagger),x_1^\dagger\rangle \cap \langle T(p_2^\dagger),x_2^\dagger\rangle$. Hence we now see that $\alpha(x) = x^\dagger$, and the theorem is proved. $\square$

The projection map ρ in the first part of the proof shows the following:

Lemma 3.3 *If $n = 2$, and $O \in \mathcal{X}$, then the planes $T(x)$, with $x \in O$ generate a 4-space of $\mathbf{PG}(5, q)$ which meets X precisely in O.*

We now consider the general case.

Proposition 3.4 *If $n \geq 2$ and $M \geq n(n+3)/2$, then $M = n(n+3)/2$ and X is projectively equivalent to the quadric Veronesean $\mathcal{V}_n$ of index n.*

Proof: The proof proceeds by induction on n, the case $n = 2$ being proved in Proposition 3.2. So suppose now that $n > 2$. We select two distinct hyperplanes X_1^* and X_2^* in $\mathbf{PG}(n, q)$. These correspond with two Veronesean sub-caps X_1 and X_2 of X, respectively, of index $n - 1$. We claim that $\dim\langle X_1\rangle = \dim\langle X_2\rangle = (n-1)(n+2)/2$. Indeed, by Property 3.1, $n_i := \dim\langle X_i\rangle \leq (n-1)(n+2)/2$, $i = 1, 2$. By the proof of Property 3.1 we know, for any $x \in X \setminus X_i$, that $\Pi = \langle T(x), X_i\rangle$, and hence, by Property 2.4, $n(n+3)/2 \leq 1 + n + n_i \leq n(n+3)/2$, implying $n_i = (n-1)(n+2)/2$, for $i = 1, 2$. Our claim is proved. Put $\langle X_i\rangle = \Omega_i$, $i = 1, 2$. The caps X_1 and X_2 meet in a Hermitian cap X_3 of index $n - 2$ in a subspace Ω of Π. Similarly as before (considering X_3 as a Veronesean sub-cap of X_1), one shows that $\dim \Omega = (n-2)(n+1)/2$. We now consider a line L^* in $\mathbf{PG}(n, q)$ not meeting $X_1^* \cap X_2^*$. It is easy to see that Π is generated by L ,X_1 and X_2. This already shows $M = n(n+3)/2$. Moreover, by induction, the caps X_i, $i = 1, 2, 3$, are projectively equivalent to quadric Veroneseans and can be identified as such.

We now proceed very similar as in the proof of Proposition 3.2. Let $\mathcal{V}_n$ be the quadric Veronesean in Π associated with $\mathbf{PG}(n, q)$ and denote for any point, point set or subspace a^* of $\mathbf{PG}(r, q^2)$ the corresponding point, point set or Veronesean sub-cap on $\mathcal{V}_n$ by $a^\dagger$. We will show that X and $\mathcal{V}_n$ are projectively equivalent, and that the projectivity can be chosen such that it maps any point $a \in X$ to the point $a^\dagger \in \mathcal{V}_n$. We may include these assertions in the induction hypothesis as they are valid for the case $n = 2$ by Proposition 3.2. Hence there is a collineation $\alpha_0 : \langle L\rangle \rightarrow \langle L^\dagger\rangle$ with associated field automorphism θ_0, and collineations $\alpha_i : \Omega_i \rightarrow \langle X_i^\dagger\rangle$, with respective associated field automorphisms θ_i, $i = 1, 2$, mapping x to $x^\dagger$, for every x in L and X_i, $i = 1, 2$, respectively (α_0 is obtained by restriction to L, after considering a Veronesean cap of index 2 containing L). Now consider a plane $\mathbf{PG}(2, q)$ in $\mathbf{PG}(n, q)$ containing L^*. With $\mathbf{PG}(2, q)$ there corresponds a Veronesean sub-cap X' of X and a Veronesean $\mathcal{V}_2$ on $\mathcal{V}_n$. Considering the restriction of α_i to $X' \cap X_i$, $i = 1, 2$, we have, by Proposition 3.2, that $\theta_0 = \theta_1 = \theta_2$, that there exists a collineation α' from $\langle \Omega_1, \Omega_2\rangle$ onto $\langle X_1^\dagger, X_2^\dagger\rangle$ having as restriction to Ω_1 and Ω_2 the collineations α_1 and α_2, respectively, and that α_0 and α' coincide on $\langle L\rangle \cap \langle X_1, X_2\rangle$. Hence there exists a collineation $\alpha : \Pi \rightarrow \Pi$ such that $\alpha(x) = x^\dagger$, for all $x \in L \cup X_1 \cup X_2$.

Now let x be any other point of X. Then there is a unique Veronesean cap of index 2 on X containing L and x (determined by the plane generated by x^* and L^* in $\mathbf{PG}(n,q)$). It has a unique conic in common with each of X_1 and X_2, and hence, as in the proof of Proposition 3.2, it follows that $\alpha(x) = x^\dagger$.

The theorem is proved. □

4 Classification of Veronesean caps of index n in $\mathbf{PG}(M,q)$

We now prove the main result of this paper, keeping all the previous notation.

Lemma 4.1 *If $n = 3$, then either $M = 8$ or $M = 9$. In the latter case, X is projectively equivalent to a quadric Veronesean of index 3.*

Proof: Consider a quadric sub-Veronesean X_1 of index 2 on X. We claim that $\langle X_1\rangle$, of dimension 5, contains at most one point of $X\setminus X_1$. Indeed, suppose $x, x' \in (X\setminus X_1)\cap\langle X_1\rangle$, $x \neq x'$. The set of all conics of $\mathcal{X}$ contained in X_1 will be denoted by $\mathcal{X}_1$, and the corresponding set of planes by $\mathcal{P}_1$. The unique conic O of $\mathcal{X}$ through x, x' has some point y in common with X_1 and therefore is entirely contained in $\langle X_1\rangle$. It follows that $T(y)$ is completely contained in $\langle X_1\rangle$. Let $\pi \in \mathcal{P}_1$ be such that $y \notin \pi$. Then by comparing dimensions we see that there exists a point u of Π in $T(y)\cap\pi$. By Property 2.4 there is a conic $O' \in \mathcal{X}$ through y with tangent line yu at y. Hence the plane $\langle O'\rangle$ meets π in a point not belonging to X, contradicting (Q2). The claim is proved. Note that the last part of the argument shows that *no space $T(y)$, $y \in X_1$, is contained in $\langle X_1\rangle$.*

So there is at most one point $x \in X \setminus X_1$ which is contained in $\langle X_1\rangle$. We choose a second quadric sub-Veronesean X_2 of index 2 with the only restriction that $x \notin X_2$ (and $X_1 \neq X_2$ of course). The intersection $O'' = X_1 \cap X_2$ belongs to $\mathcal{X}$. We claim that $\langle X_1\rangle \cap \langle X_2\rangle = \langle O''\rangle$. Indeed, suppose by way of contradiction that $\langle X_1\rangle \cap \langle X_2\rangle$ contains a 3-dimensional subspace U containing $\langle O''\rangle$ of Π. Let $z \in X_2 \setminus X_1$ be arbitrary. The tangent plane at z of X_2 has at least one point $v \neq z$ in common with U (by comparing dimensions and noting that $z \neq x$ if x exists), hence by Lemma 2.1 there is a point $z' \in O''$ such that vz is tangent to $X \cap [z,z']$. Hence the line vz' is contained in $[z,z']$, implying that it must be a tangent line to $X \cap [z,z']$ because otherwise $\langle X_1\rangle$ contains a point of $(X\cap[z,z'])\setminus\{z'\}$ (which is different from x if x exists). Now $T(z')$ is generated by the tangent plane of X_1 at z' and the line vz'. Consequently $T(z')$ is contained in $\langle X_1\rangle$, contradicting our last note in the previous paragraph. Our claim is proved. It is now clear that $M \geq 8$.

The assertion for $M = 9$ follows from Proposition 3.4. □

Proposition 4.2 *If $n = 3$ and $M = 8$, then there exists a projective space $\Pi' := \mathbf{PG}(9,q)$ containing Π, a point c of Π' and a quadric Veronesean Y of index 3 in Π', with $c \notin Y$, such that X is the projection of Y from c onto Π. Moreover, Y can be chosen in such a way that $Y \cap X$ is the union of two quadric sub-Veroneseans of index 2 of both X and Y, and Y is uniquely determined by this intersection, by the point c and by one point $x' \in Y$ with x' not belonging to $X \cap Y$ (and of course $cx' \cap X$ nonempty).*

Proof: The proof of Lemma 4.1 easily yields the existence of two quadric sub-Veroneseans X_1 and X_2 of index 2 of X such that $\langle X_1, X_2\rangle = \Pi$ and $\pi := \langle X_1\rangle \cap \langle X_2\rangle \in \mathcal{P}$. Now embed Π as a hyperplane in some 9-dimensional space $\Pi' = \mathbf{PG}(9,q)$ and let c be any point of $\Pi' \setminus \Pi$. Let $x \in X \setminus (X_1 \cup X_2)$ and choose arbitrarily $x' \in cx$, $x \neq x' \neq c$. Let $y \in X \setminus (X_1 \cup X_2)$ be arbitrary, $y \neq x$. The conic $X \cap [x,y]$ either has different points x_1, x_2 in common with X_1, X_2, respectively, or has a point z in common with $X_1 \cap X_2$. In the first case we define the point y^θ as the intersection of the plane $\langle x_1, x_2, x'\rangle$ with the line cy (which is well defined since both objects live in the 3-space $\langle x_1, x_2, x, c\rangle$); in the second case we define y^θ as the intersection of the plane $\langle T_z([x,z]), x'\rangle$ with the line cy. If $u \in X_1 \cup X_2$, then we set $u^\theta = u$. Also, $x^\theta := x'$. We define Y as the set of points y^θ such that $y \in X$, and θ is a well defined map from X to Y. It is clear that θ is bijective and its inverse is the restriction of a projection mapping with center c and image Π. Note that $\langle Y\rangle = \Pi'$. We now show that for every conic $O \in \mathcal{X}$, the set O^θ is a conic of Y.

If $O \subseteq X_1 \cup X_2$, then this is trivial. Also, if O contains x, then this is clear from the construction. Now suppose $x \notin O$ and O is not contained in $X_1 \cup X_2$. Then x and O are contained in a unique 5-dimensional space W containing the unique quadric sub-Veronesean Z of index 2 which contains both x and O. Now, Z either has distinct conics O_1 and O_2 in common with X_1 and X_2, respectively, or Z contains the conic $X \cap \pi$.

Assume we are in the first case and let $u \in O$ be arbitrary. If the conic determined by u and x contains distinct points of O_1 and O_2, then it is clear that u^θ is contained in the 5-space $\langle O_1, O_2, x'\rangle$. If the conic O' determined by u and x contains the unique common point v of O_1 and O_2, then the tangents at v of O', O_1, O_2 are coplanar by (Q3), and so, again, u^θ is contained in $\langle O_1, O_2, x'\rangle$. If $c \in \langle O_1, O_2, x'\rangle$, then $x \in \langle O_1, O_2\rangle$, so $x \in O_1 \cup O_2$, a contradiction. Hence in this first case O^θ is the intersection of the cone cO with $\langle O_1, O_2, x'\rangle$, implying that O^θ is a conic.

Now assume we are in the second case. If $u \in O$, then u^θ is contained in the 5-space Π'' generated by x' and all tangent planes of Z at points of $X \cap \pi$. If c were contained in Π'', then x would be in Π'', so $x \in \Pi'' \cap X = \pi \cap X$ (see Lemma 3.3), a contradiction. Hence in this second case O^θ is the intersection of the cone cO with Π'', implying as before that O^θ is a conic.

Hence it follows that every two points of Y are contained in a unique conic which is the image under θ of some conic of X. If we let $\mathcal{Q}$ be the set of all planes of all such conics, then we have shown that Y satisfies (Q1) for $\mathcal{Q}$. Let π'_1 and π'_2 be two planes of Π' containing the images under θ of distinct conics O_1 and O_2, respectively, with $\langle O_i \rangle = \pi_i \in \mathcal{P}$. Suppose $\pi'_1 \cap \pi'_2 \neq \emptyset$. As $\langle O_1, O_2 \rangle$ is at least 4-dimensional, we have $c \notin \langle \pi'_1, \pi'_2 \rangle$ and $|\pi'_1 \cap \pi'_2| = 1$. So $\pi_1 \cap \pi_2 \neq \emptyset$, and consequently $O_1 \cap O_2$ is a point by (Q2). It follows that $\pi'_1 \cap \pi'_2$ is a point of Y. This shows that Y satisfies (Q2). Finally, we show (Q3).

Therefore, let $w \in Y$ and let C be a conic of Y which is the image under θ of an element of $\mathcal{X}$; assume also that $w \notin C$. By (Q3) applied to X, the tangents at w of the conics on Y which contain w and a point of C, and which are images under θ of elements of $\mathcal{X}$, are contained in a 3-space ζ containing c.

First let $q > 2$. It is easy to show that all conics on Y which contain w and a point of C, and which are images under θ of elements of $\mathcal{X}$, generate a 5-space η (as the images under θ^{-1} of any of these conics intersect, any two of the considered conics on Y intersect). This 5-space η does not contain c, as otherwise there arises (by applying θ^{-1}) a Veronesean sub-cap of index 2 on X contained in a 4-space. It now follows that the tangents at w of the conics on Y which contain w and a point of C, and which are images of elements of $\mathcal{X}$ under θ, are contained in the plane $\zeta \cap \eta$. So we conclude that Y is a Veronesean cap of index 3 in a 9-dimensional space Π'. By Proposition 3.4, Y is projectively equivalent to a quadric Veronesean of index 3.

Now let $q = 2$. If we show that the image under θ of the point set of any Veronesean cap of index 2 on X is contained in a 5-space, then we can argue as in the previous paragraph and the proposition will be proved. Obviously, this is true for X_1 and X_2. Now consider the set $X_3 = (X \setminus (X_1 \cup X_2)) \cup (X_1 \cap X_2)$. In the present situation, $X \cap [x, y]$ contains a point z_y of $X_1 \cap X_2$, for all $y \in X \setminus (X_1 \cup X_2)$, $y \neq x$. All lines $T_{z_y}([x, z_y])$ belong to a common 4-dimensional space which intersects the cap of index 2 defined by $X_1 \cap X_2$ and x in $X_1 \cap X_2$. Hence all planes $\langle T_{z_y}([x, z_y]), x' \rangle$ belong to a common 5-dimensional space which does not contain c (otherwise the previous 4-dimensional space contains x, a contradiction). By construction, all corresponding points y^θ are contained in that 5-dimensional space, for all $y \in X \setminus (X_1 \cup X_2)$, $x \neq y$. These points y^θ together with x' and $X_1 \cap X_2$ form the Veronesean that is the image under θ of X_3. Finally, consider a Veronesean cap X_4 of index 2 on X different from X_1, X_2, X_3, and hence not containing $X_1 \cap X_2$. Put $\{z\} = X_1 \cap X_2 \cap X_4$ and let y_1, y_2 be the other points of $X_3 \cap X_4$. Put $C_i = X_4 \cap X_i$, $i = 1, 2, 3$. The tangent T to C_3 at z is contained in $\langle C_1, C_2 \rangle$ (by (Q3)). Now the conic C_3^θ is equal to $\{z, y_1^\theta, y_2^\theta\}$, and T is tangent to C_3^θ at z (as follows from the construction of θ). Hence $\langle X_4^\theta \rangle = \langle C_1, C_2, C_3^\theta \rangle$ is 5-dimensional (and does not contain c as otherwise X_4 is in a 4-dimensional space).

The proposition is proved. □

Lemma 4.3 *If $M < n(n+3)/2$, then there exist two distinct Veronesean sub-caps X_1, X_2 both of index $n-1$ such that $\langle X_1, X_2 \rangle = \Pi$.*

Proof: Suppose $M < n(n+3)/2$. We coordinatize the projective space $\mathbf{PG}(n,q)$ with respect to a basis $e_0, e_1, \ldots, e_n$ of the underlying vector space and we consider the points p_i^* corresponding to the vectors e_i, and the points $p_{i,j}^*$ corresponding to the vectors $e_i - e_j$, $i,j \in \{0,1,\ldots,n\}$, $i \neq j$. Note that $p_{i,j}^* = p_{j,i}^*$, so we may assume $i < j$. We denote the corresponding points on X by p_i and $p_{i,j}$, respectively. Let $x^* \in \mathbf{PG}(n,q)$ and let $\ell(x^*)$ be the minimal number of points of $\{p_0^*, p_1^*, \ldots, p_n^*\}$ needed to generate a subspace containing x^*. Put $S = \{p_k, p_{i,j} \,|\, 0 \leq k \leq n \text{ and } 0 \leq i < j \leq n\}$. If $\ell(x^*) = 1$, then clearly $x \in \langle S \rangle$ (recall that x is the point of X corresponding with x^* in $\mathbf{PG}(n,q)$). If $\ell(x^*) = 2$, then x belongs to some plane $\langle p_i, p_j, p_{ij} \rangle$, $i < j$, so belongs again to $\langle S \rangle$. Now assume $\ell(x^*) > 2$. If $q > 2$, then it is easy to see that there is a line L^* of $\mathbf{PG}(n,q)$ containing x^* and three distinct points x_1^*, x_2^*, x_3^* with $\ell(x_i^*) \leq \ell(x^*) - 1$, $i = 1,2,3$ (indeed, without loss of generality we may assume that x^* corresponds to the vector $v = \sum r_i e_i$, with $r_i \in \mathbf{GF}(q)$, and $r_i \neq 0$ for $i \in \{0,1,\ldots,\ell(x^*)-1\}$, and $r_i = 0$ otherwise; then one can take L^* through x^* and the point corresponding with the following vector w: if not all r_i are equal, say $r_0 \neq r_1$, then we choose $w = e_0 + e_1$; if all r_i are equal, then we choose $w = e_0 + ae_1$, with $a \notin \{0,1\}$. By induction on $\ell(x^*)$, we have $x_1, x_2, x_3 \in \langle S \rangle$, and hence $x \in \langle x_1, x_2, x_3 \rangle \subseteq \langle S \rangle$. If $q = 2$, then x^* corresponds to $v = \sum_{0 \leq i \leq \ell(x^*)-1} e_i$, and hence x is contained in the 5-space of the Veronesean sub-cap of index 2 determined by $e_0, e_1, e_0 + e_1 + v, e_0 + e_1, e_0 + v, e_1 + v$ (the 6 corresponding points of Π generate the 5-space). We again conclude that $x \in \langle S \rangle$. Hence $\langle S \rangle = \Pi$.

Since $M = \dim \Pi < |S| - 1$, some element p of S must satisfy $p \in \langle S \setminus \{p\} \rangle$. Without loss of generality we may assume that either $p = p_0$ or $p = p_{0,1}$. In the first case we can choose the two Veronesean sub-caps X_1 and X_2 of index $n-1$ as being determined by the hyperplanes of $\mathbf{PG}(n,q)$ with equation $X_0 = 0$ and $X_0 + X_1 + \cdots + X_n = 0$, respectively, while in the second case we can choose them being determined by the hyperplanes with equation $X_0 = 0$ and $X_1 = 0$, respectively. □

We can now finish off the proof of our Main Result.

Proposition 4.4 *If $M < n(n+3)/2$, then there is a projective space $\Pi' := \mathbf{PG}(M+1,q)$ containing Π, a point c of Π' not contained in Π, and a Veronesean cap Y of index n in Π', with $c \notin Y$ and $\langle Y \rangle = \Pi'$, such that X is the (bijective) projection of Y from c onto Π.*

Proof: Suppose $M < n(n+3)/2$. Let X_1, X_2 be as in the previous lemma. Embed Π as a hyperplane in a projective space Π' of dimension $M+1$ and let c be any point of $\Pi' \setminus \Pi$. Further, let x be any point of $X \setminus (X_1 \cup X_2)$, and choose arbitrarily a point $x' = x^\theta$ different from x and c on the line cx. Also, we choose a conic O of $\mathcal{X}$ through x which has different points x_1 and x_2 in common with X_1 and X_2, respectively. As in the proof of Proposition 4.2, we define y^θ for $y \in O$. Now let $y \in X$ be arbitrary, but not belonging to $X_1 \cup X_2 \cup O$. Then there is a quadric sub-Veronesean Z of index 2 of X containing O

and y, and Z has different conics $O_1, O_2 \in \mathcal{X}$ in common with X_1, X_2, respectively. We define y^θ as the intersection of the spaces $\langle O_1, O_2, x^\theta \rangle$ and cy. Let O' be any conic in $\mathcal{X}$. As in the proof of Proposition 4.2, the assertion will follow if we show that O'^θ is again a conic (checking (Q2) and (Q3) will then be completely similar, except for (Q3) in the case $q = 2$; we will do that below). If O and O' have a nonempty intersection, then this follows immediately from our construction. Hence we may assume that $O \cap O' = \emptyset$. We consider the unique Veronesean sub-cap Z' of index 3 containing both O and O'. It is easy to see that Z' meets X_1 and X_2 in different quadric sub-Veroneseans Z_1 and Z_2 of index 2, respectively, meeting in a conic O'' of $\mathcal{X}$. If $x \notin \langle Z_1, Z_2 \rangle$, then O'^θ is the intersection of the space $\langle Z_1, Z_2, x^\theta \rangle$ with the cone cO', hence it is a conic itself. If $x \in \langle Z_1, Z_2 \rangle$, then this follows from Proposition 4.2, its proof and in particular the uniqueness part of the statement.

Now let $q = 2$. We show (Q3). With the above notation, the sub-Veronesean Z contains the points x, y, x_1, x_2, $z = O_1 \cap O_2$, the third point u_1 of O_1 and the third point u_2 of O_2. hence $z \in [x, y]$. So $T_z([x, z])$ is in the plane $\langle T_z(O_1), T_z(O_2) \rangle$. It follows that $\langle T_z([x, z]), x^\theta \rangle \cap cy = \langle O_1, O_2, x^\theta \rangle \cap cy$. Now consider any Veronesean cap U of index 2 on X and let U' be a Veronesean cap of index 3 on X containing U and x. Relying on the foregoing and the last part (the case $q = 2$) of the proof of Proposition 4.2, we see that the corresponding set U^θ belongs to a 5-dimensional space (not containing c). Now it is clear that (Q3) holds.

This completes the proof of the proposition. □

An obvious induction on M now completes the proof of our main result Theorem 1.1.

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