

# Characterizations of the finite quadric Veroneseans $\mathcal{V}_n^{2^n}$

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## Abstract

We generalize and complete several characterizations of the finite quadric Veroneseans surveyed in [3]. Our main result is a characterization of the quadric Veronesean  $\mathcal{V}_n^{2^n}$  in terms of the subspaces of the ambient projective space generated by the sub Veroneseans isomorphic to  $\mathcal{V}_{n-1}^{2^{n-1}}$ . We also obtain, as an application, a completely new characterization of  $\mathcal{V}_n^{2^n}$ .

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## 1 Introduction and Statement of the Main Result

Let  $q$  be a fixed prime power. For any integer  $k$ , denote by  $\mathbf{PG}(k, q)$  the  $k$ -dimensional Projective Geometry over the finite (Galois) field  $\mathbf{GF}(q)$  of  $q$  elements. Let  $n \geq 1$  be an arbitrary integer. We choose coordinates in  $\mathbf{PG}(n, q)$  and in  $\mathbf{PG}(n(n+3)/2, q)$ . The *Veronesean map* maps a point of  $\mathbf{PG}(n, q)$  with coordinates  $(x_0, x_1, \dots, x_n)$  onto the point of  $\mathbf{PG}(n(n+3)/2, q)$  with coordinates

$$(x_0^2, x_1^2, \dots, x_n^2, x_0x_1, x_0x_2, \dots, x_0x_n, x_1x_2, x_1x_3, \dots, x_1x_n, \dots, x_{n-1}x_n).$$

The *quadric Veronesean*  $\mathcal{V}_n^{2^n}$ , or, for short  $\mathcal{V}_n$ , is the image of the Veronesean map. The set  $\mathcal{V}_n^{2^n}$  is a cap of  $\mathbf{PG}(n(n+3)/2, q)$  and has a lot of other nice geometric and combinatorial properties, summarized in [3]. We also refer to [3] for characterizations of this cap, sometimes called the *Veronesean cap*. Essentially, there are three kinds of characterizations known. The first kind characterizes  $\mathcal{V}_2^4$  in terms of the intersection numbers of a hyperplane and is valid for  $q$  odd. It was first considered and proved by Ferri [2]; the proof in [3] is much shorter because Hirschfeld and Thas make use of the other characterizations. Also, the proof of Ferri did not work for  $q = 3$ ; see [2]. The second kind is a

characterization of  $\mathcal{V}_2^4$  by means of the intersection properties of the so-called *conic planes* (which are called  $\mathcal{V}_1$ -subspaces below), and again only was proved for  $q$  odd. Tallini [6] introduced and proved the first theorems of this type. In the present paper, we generalize this characterization to  $q$  even and to  $\mathcal{V}_n^{2^n}$  for any  $q$  and any  $n$ . Finally, the third kind of characterization is much geometric in nature and arises from axiomatising the properties of the conics entirely contained in  $\mathcal{V}_n^{2^n}$  and the tangent lines to these. This was originated by Mazzocca and Melone [5], who essentially proved the case  $q$  odd (although they “forgot” one axiom and so erroneously thought they proved a stronger result); it was corrected and generalized to arbitrary  $q$  by Hirschfeld and Thas [3], who also provided counterexamples to the original statement of Mazzocca and Melone. Finally, this characterization was recently further generalized by weakening the axioms by Thas and Van Maldeghem [8], who also produced a drastically shorter proof, and generalized the characterization so that it also comprises the aforementioned counterexamples (which are certain projections of quadric Veroneseans). Hence the present paper can be seen as a sequel to [8] aiming at making the characterizations in [3] as general as possible; in particular allowing general  $q$  in all of them, and generalizing to arbitrary dimension  $n$  whenever reasonable, including some suitable projections of quadric Veroneseans when possible. We will apply our results to obtain a new type of characterization of  $\mathcal{V}_n^{2^n}$ . The results of the present paper also permit to generalize Ferri’s result mentioned above to all  $q$ . This will be proved in a forthcoming paper.

We now prepare the statements of our main results.

For the rest of this paper, put  $N_n := \frac{n(n+3)}{2}$ , for all positive integers  $n$ .

Consider the quadric Veronesean  $\mathcal{V}_n^{2^n}$  and the corresponding Veronesean map. Then the image of an arbitrary hyperplane of  $\mathbf{PG}(n, q)$  under the Veronesean map is a quadric Veronesean  $\mathcal{V}_{n-1}^{2^{n-1}}$  and the subspace of  $\mathbf{PG}(N_n, q)$  generated by it has dimension  $N_{n-1} = (n-1)(n+2)/2$  (see [3]). Such a subspace will be called a  $\mathcal{V}_{n-1}^{2^{n-1}}$ -subspace, or, for short, a  $\mathcal{V}_{n-1}$ -subspace, of  $\mathcal{V}_n^{2^n}$  (by abuse of language, since the subspace does not lie in  $\mathcal{V}_n^{2^n}$ ), or of  $\mathbf{PG}(N_n, q)$ . The image of a line of  $\mathbf{PG}(n, q)$  is a plane conic. If  $q$  is even, then the intersection of all tangent lines to a conic is a single point — called the *nucleus* — and we claim that the set of nuclei of all such conics is the Grassmannian of the lines of  $\mathbf{PG}(n, q)$  and hence generates a subspace  $C$  of dimension  $N_{n-1}$ , which we call the *nucleus subspace* of  $\mathcal{V}_n^{2^n}$ . Indeed, suppose henceforth  $q$  even. Let  $L$  be the line of  $\mathbf{PG}(n, q)$  determined by the points  $(x_0, x_1, \dots, x_n)$  and  $(y_0, y_1, \dots, y_n)$ . Then the image of  $L$  (viewed as the set of points incident with it) under the Veronesean map is the set of points

$$(x_0^2k^2+y_0^2\ell^2, \dots, x_n^2k^2+y_n^2\ell^2, x_0x_1k^2+y_0y_1\ell^2+(x_0y_1+x_1y_0)k\ell, \dots, x_0x_nk^2+y_0y_n\ell^2+(x_0y_n+x_ny_0)k\ell, \\ \dots, x_{n-1}x_nk^2+y_{n-1}y_n\ell^2+(x_{n-1}y_n+x_ny_{n-1})k\ell),$$

with  $k, \ell \in \mathbf{GF}(q)$  and  $(k, \ell) \neq (0, 0)$ . This is a conic  $\mathcal{C}$  in the plane generated by the three points

$$(x_0^2, \dots, x_n^2, x_0x_1, \dots, x_0x_n, \dots, x_{n-1}x_n),$$

$$(y_0^2, \dots, y_n^2, y_0y_1, \dots, y_0y_n, \dots, y_{n-1}y_n),$$

and

$$(0, \dots, 0, x_0y_1 + x_1y_0, \dots, x_0y_n + x_ny_0, \dots, x_{n-1}y_n + x_ny_{n-1}).$$

It can be easily checked that the latter point is the nucleus of  $\mathcal{C}$  and our claim follows.

We now consider again the general case ( $q$  arbitrary). We claim that  $C$  is the intersection of all hyperplanes of  $\mathbf{PG}(N_n, q)$  containing a unique  $\mathcal{V}_{n-1}$ -subspace. Indeed, first we show that every such hyperplane contains  $C$ . By transitivity (see [3]) we may assume that the  $\mathcal{V}_{n-1}$ -subspace  $W$  is generated by the image of the hyperplane in  $\mathbf{PG}(n, q)$  with equation  $X_0 = 0$ . The hyperplane in  $\mathbf{PG}(N_n, q)$  generated by  $W$  and the image of the hyperplane in  $\mathbf{PG}(n, q)$  with equation  $a_0X_0 + a_1X_1 + \dots + a_nX_n = 0$  has itself equation (with obvious notation)  $a_0X_{0,0} + a_1X_{0,1} + \dots + a_nX_{0,n} = 0$ , with  $(a_1, a_2, \dots, a_n) \neq (0, 0, \dots, 0)$ . So it is clear that the unique hyperplane of  $\mathbf{PG}(N_n, q)$  through  $W$  not containing any other  $\mathcal{V}_{n-1}$ -subspace has equation  $X_{0,0} = 0$ , and hence contains  $C$ . Now it is easy to see (considering also the hyperplanes in  $\mathbf{PG}(N_n, q)$  related to the hyperplanes in  $\mathbf{PG}(n, q)$  with equation  $X_i = 0$ ,  $i \in \{1, 2, \dots, n\}$ ) that the claim is true. For later reference, we refer to this property as the *nucleus subspace property* of  $\mathcal{V}_n^{2^n}$ .

Now let  $\mathcal{S}_n$  be the set of all  $\mathcal{V}_{n-1}$ -subspaces of the quadric Veronesean  $\mathcal{V}_n$  in  $\mathbf{PG}(N_n, q)$ . We note the following properties of  $\mathcal{S}_n$ , which can easily be verified (using coordinates, for instance; or see also [3]).

- (VS1) Every two members of  $\mathcal{S}_n$  generate a hyperplane of  $\mathbf{PG}(N_n, q)$ .
- (VS2) Every three members of  $\mathcal{S}_n$  generate  $\mathbf{PG}(N_n, q)$ .
- (VS3) No point is contained in every member of  $\mathcal{S}_n$ .
- (VS4) The intersection of any nonempty collection of members of  $\mathcal{S}_n$  is a subspace of dimension  $N_i = i(i+3)/2$  for some  $i \in \{-1, 0, 1, \dots, n-1\}$ .
- (VS5) There exist three members  $S, S', S''$  of  $\mathcal{S}_n$  with  $S \cap S' = S' \cap S'' = S'' \cap S$ .

It is known that for  $q$  odd and  $n = 2$ , the properties (VS1), (VS2) and (VS3) characterize  $\mathcal{V}_2^4$ , see [3]. Moreover, for  $n = 2$  and arbitrary  $q$ , property (VS4) follows immediately from (VS1), (VS2) and (VS3). In the present paper we will prove the following.

**Theorem 1.1** *Let  $\mathcal{S}$  be a collection of  $q^n + q^{n-1} + \dots + q + 1$  subspaces of dimension  $N_{n-1}$  of the projective space  $\mathbf{PG}(N_n, q)$ , with  $n \geq 2$ , satisfying (VS1) up to (VS5). Then either  $\mathcal{S}$  is the set of  $\mathcal{V}_{n-1}$ -subspaces of a quadric Veronesean  $\mathcal{V}_n^{2^n}$  in  $\mathbf{PG}(N_n, q)$ , or  $q$  is even, there are two members  $S_1, S_2 \in \mathcal{S}$  with the property that no other member of  $\mathcal{S}$  contains  $S_1 \cap S_2$ , and there is a unique subspace  $S$  of dimension  $N_{n-1}$  such that  $\mathcal{S} \cup \{S\}$  is the*

set of  $\mathcal{V}_{n-1}$ -subspaces together with the nucleus subspace of a quadric Veronesean  $\mathcal{V}_n^{2^n}$  in  $\mathbf{PG}(N_n, q)$ . In particular, if  $n = 2$ , then the statement holds under the weaker hypothesis of  $\mathcal{S}$  satisfying (VS1), (VS2), (VS3) and (VS5). In both cases, but with  $(q, n) \neq (2, 2)$  in the latter case,  $\mathcal{V}_n^{2^n}$  is the set of points of  $\mathbf{PG}(N_n, q)$  contained in at least  $q^{n-1} + q^{n-2} + \dots + q$  members of  $\mathcal{S}$ ; in the exceptional case there are 13 points contained in at least 2 members of  $\mathcal{S}$ , where 6 are coplanar while the others form  $\mathcal{V}_2^4$ .

For  $q$  large enough we can reduce this set of axioms.

**Theorem 1.2** *Let  $\mathcal{S}$  be a collection of  $q^n + q^{n-1} + \dots + q + 1$  subspaces of dimension  $N_{n-1}$  of the projective space  $\mathbf{PG}(N_n, q)$ , with  $n \geq 2$ , satisfying (VS1) up to (VS3). If  $q \geq n$ , then  $\mathcal{S}$  also satisfies (VS4).*

We can also say something more in the case where  $\mathcal{S}$  does not satisfy (VS5).

**Theorem 1.3** *Let  $\mathcal{S}$  be a collection of  $q^n + q^{n-1} + \dots + q + 1$  subspaces of dimension  $N_{n-1}$  of the projective space  $\mathbf{PG}(N_n, q)$ , with  $n \geq 2$ , satisfying (VS1) up to (VS4) and not satisfying (VS5), or satisfying (VS1) up to (VS3), with  $q \geq n$ , but not satisfying (VS5). Then  $q$  is even and there exists a unique subspace  $S$  of dimension  $N_{n-1}$  such that  $\mathcal{S} \cup \{S\}$  also satisfies (VS1) up to (VS4), and not (VS5), or satisfies (VS1) up to (VS3), and not (VS5), respectively. Moreover, if  $n = 2$ , then  $q = 2$  or  $q = 4$  and  $\mathcal{S}$  is uniquely determined in both cases, up to isomorphism.*

Hence, for  $q$  odd we have a most satisfying characterization, since in this case axioms (VS1) up to (VS4) really characterize the collection of  $\mathcal{V}_{n-1}$ -subspaces of a quadric Veronesean  $\mathcal{V}_n^{2^n}$ , and for  $q \geq n$ , axioms (VS1) up to (VS3) do this job. For  $q$  even we additionally need (VS5), although for  $n = 2$  one can classify all examples that do not satisfy (VS5). This classification remains open for  $n \geq 3$ . There is an infinite class of examples for  $q = 2$ , which we describe later on. In fact, if  $\mathcal{S}$  does not satisfy (VS5), then Theorem 1.3 implies that it is contained in the dual of an  $n$ -dimensional dual hyperoval, and so the classification for  $n = 2$  follows from a result of Del Fra [1], after remarking that  $q = 2, 4$  are the only possibilities.

There are two corollaries.

**Corollary 1.4** *If  $\mathcal{S}^*$  is a set of  $q^n + q^{n-1} + \dots + q + 2$  subspaces of dimension  $N_{n-1}$  of  $\mathbf{PG}(N_n, q)$  such that (VS1), (VS2), (VS3) and (VS5) hold for  $\mathcal{S}^*$  and either also (VS4) holds, or  $q \geq n$ , then  $q$  is even and  $\mathcal{S}^*$  is the set of all  $\mathcal{V}_{n-1}$ -subspaces together with the nucleus subspace of a quadric Veronesean  $\mathcal{V}_n^{2^n}$  in  $\mathbf{PG}(N_n, q)$ . Also,  $\mathcal{V}_n^{2^n}$  is the set of points of  $\mathbf{PG}(N_n, q)$  contained in  $q^{n-1} + q^{n-2} + \dots + q + 1$  members of  $\mathcal{S}$ .*

**Corollary 1.5** *Let  $\mathcal{S}$  be a set of  $k \geq q^n + q^{n-1} + \cdots + q + 1$  subspaces of dimension  $m - n - 1$  of  $\mathbf{PG}(m, q)$ , with  $m \geq N_n$  and such that  $q \geq n$ . Suppose every pair of elements of  $\mathcal{S}$  is contained in some hyperplane of  $\mathbf{PG}(m, q)$ , no three elements of  $\mathcal{S}$  are contained in a hyperplane of  $\mathbf{PG}(m, q)$ , no point is contained in all members of  $\mathcal{S}$  and there exist three members  $S, S', S''$  of  $\mathcal{S}$  with  $S \cap S' = S' \cap S'' = S'' \cap S$ . Then  $m = N_n$  and either  $k = q^n + q^{n-1} + \cdots + q + 1$  and  $\mathcal{S}$  is the set of  $\mathcal{V}_{n-1}$ -subspaces of a quadric Veronesean  $\mathcal{V}_n^{2^n}$ , or  $q$  is even,  $k \in \{q^n + q^{n-1} + \cdots + q + 1, q^n + q^{n-1} + \cdots + q + 2\}$  and  $\mathcal{S}$  consists of  $k$  members of the set of  $\mathcal{V}_{n-1}$ -subspaces together with the nucleus subspace of a quadric Veronesean  $\mathcal{V}_n^{2^n}$ . In both cases, but with  $(q, n) \neq (2, 2)$  if  $\mathcal{S}$  contains the nucleus subspace of  $\mathcal{V}_2^4$ ,  $\mathcal{V}_n^{2^n}$  is the set of points of  $\mathbf{PG}(m, q)$  contained in at least  $q^{n-1} + q^{n-2} + \cdots + q$  members of  $\mathcal{S}$ ; in the exceptional case there are 13 points contained in at least 2 members of  $\mathcal{S}$ , where 6 are coplanar while the other 7 form  $\mathcal{V}_2^4$ .*

The reduction of the hypotheses of Corollary 1.5 to the hypotheses of Theorem 1.1 and Theorem 1.2 will be done just after Lemma 2.1 below. We only remark here that every two members of  $\mathcal{S}$  generate a hyperplane; this follows from an easy counting argument similar to the one in the proof of Theorem 25.2.14 of [3]. More explicitly, if two members  $\pi, \pi'$  of  $\mathcal{S}$  do not generate a hyperplane, then the number of hyperplanes containing  $\pi$  and one element of  $\mathcal{S} \setminus \{\pi\}$  is at least  $(q^n + q^{n-1} + \cdots + q - 1) + (q + 1)$ , clearly a contradiction as  $\pi$  is contained in exactly  $q^n + q^{n-1} + \cdots + q + 1$  hyperplanes of  $\mathbf{PG}(m, q)$ .

We present the following application of our main results. It provides a very simple characterization of finite quadric Veroneseans.

**Theorem 1.6** *Let  $\theta : \mathbf{PG}(n, q) \rightarrow \mathbf{PG}(m, q)$  be an injective map from the point set of  $\mathbf{PG}(n, q)$  to the point set of  $\mathbf{PG}(m, q)$ ,  $m \geq N_n$ , with  $n \geq 2$  and  $q > 2$ , such that the image of any line of  $\mathbf{PG}(n, q)$  under  $\theta$  is a plane oval in  $\mathbf{PG}(m, q)$ , and such that the image of  $\theta$  generates  $\mathbf{PG}(m, q)$ . Then this image of  $\theta$  is projectively equivalent to the quadric Veronesean  $\mathcal{V}_n$ .*

## 2 Proof of the Main Results

In this section, we give ourselves a projective space  $\mathbf{PG}(N_n, q)$  of dimension  $N_n = \frac{n(n+3)}{2}$ , with  $n \geq 2$ , and a collection  $\mathcal{S}$  of  $q^n + q^{n-1} + \cdots + q + 1$  subspaces of dimension  $N_{n-1} = \frac{(n-1)(n+2)}{2}$  such that the following hold.

(VS1) Every two members of  $\mathcal{S}$  generate a hyperplane of  $\mathbf{PG}(N_n, q)$ .

(VS2) Every three members of  $\mathcal{S}$  generate  $\mathbf{PG}(N_n, q)$ .

(VS3) No point is contained in every member of  $\mathcal{S}$ .

Note that  $|\mathcal{S}|$  is precisely the number of points of  $\mathbf{PG}(n, q)$ .

We call  $\mathcal{S}$  a *Veronesean set of subspaces*. In the special case where no three members of  $\mathcal{S}$  pairwise meet in the same subspace — necessarily of dimension  $N_{n-2}$  — we call this an *ovoidal Veronesean set of subspaces*; a Veronesean set of subspaces which is not ovoidal is called *proper*. If a collection  $\mathcal{T}$  of subspaces of dimension  $N_{n-1}$  satisfies (VS1), (VS2) and (VS3), but no three members of  $\mathcal{T}$  pairwise meet in the same subspace, and if  $\mathcal{T}$  contains  $q^n + q^{n-1} + \cdots + q^2 + q + 2$  elements, then we call  $\mathcal{T}$  a *hyperovoidal Veronesean set of subspaces*. We can now rephrase our main problems as follows: we want to classify the proper Veronesean sets of subspaces, and show that every ovoidal Veronesean set of subspaces is contained in a unique hyperovoidal Veronesean set of subspaces. In fact, we will prove that, if  $q \geq n$ , then every Veronesean set  $\mathcal{S}$  of subspaces satisfies

(VS4) The intersection of any nonempty collection of members of  $\mathcal{S}$  is a subspace of dimension  $N_i$  for some  $i \in \{0, 1, \dots, n\}$ .

Then we will consider a proper Veronesean set  $\mathcal{S}$  of subspaces satisfying additionally (VS4) and prove that it either must be the collection of  $\mathcal{V}_{n-1}$ -subspaces of a quadric Veronesean  $\mathcal{V}_n^{2^n}$  in  $\mathbf{PG}(N_n, q)$ , or  $q$  is even and there is a unique subspace  $S$  of dimension  $N_{n-1}$  such that  $\mathcal{S} \cup \{S\}$  is the set of  $\mathcal{V}_{n-1}$ -subspaces together with nucleus subspace of a quadric Veronesean  $\mathcal{V}_n^{2^n}$  in  $\mathbf{PG}(N_n, q)$ . From this classification, it easily follows that the points of  $\mathbf{PG}(N_n, q)$  that are contained in at least two members of  $\mathcal{S}$  constitute a quadric Veronesean  $\mathcal{V}_n^{2^n}$ . Also, it follows that, if  $\mathcal{S}^*$  is a set of  $q^n + q^{n-1} + \cdots + q + 2$  subspaces of dimension  $N_{n-1}$  of  $\mathbf{PG}(N_n, q)$  such that (VS1), (VS2) and (VS3) hold for  $\mathcal{S}^*$  and either also (VS4) holds, or  $q \geq n$ , then  $q$  is even and either  $\mathcal{S}^*$  is the set of all  $\mathcal{V}_{n-1}$ -subspaces together with the nucleus subspace of a quadric Veronesean  $\mathcal{V}_n^{2^n}$  in  $\mathbf{PG}(N_n, q)$ , or it is a hyperovoidal Veronesean set of subspaces (proving Corollary 1.4).

The proof proceeds by induction on  $n$ , but we will show the smallest case  $n = 2$  along the way of the proof of the general case.

It will be convenient in many situations to consider the dual projective space. We fix a duality and denote the dual of an object  $x$  of  $\mathbf{PG}(N_n, q)$  by  $\bar{x}$  (for instance, we can consider the natural duality mapping a point with coordinates  $(a_0, a_1, \dots, a_{N_n})$  onto the hyperplane with equation  $a_0X_0 + a_1X_1 + \cdots + a_{N_n}X_{N_n} = 0$ ). In particular, we denote the dual of  $\mathbf{PG}(N_n, q)$  by  $\overline{\mathbf{PG}(N_n, q)}$ . We obtain a set  $\overline{\mathcal{S}}$  of  $n$ -dimensional subspaces of  $\overline{\mathbf{PG}(N_n, q)}$  satisfying the following properties.

$\overline{(\text{VS1})}$  Every two members of  $\overline{\mathcal{S}}$  meet in a point of  $\overline{\mathbf{PG}(N_n, q)}$ .

(VS2) No three members of  $\mathcal{S}$  have a point in common.

(VS3) No hyperplane of  $\overline{\mathbf{PG}(N_n, q)}$  contains all members of  $\overline{\mathcal{S}}$ .

We start by fixing an arbitrary member  $H$  of  $\mathcal{S}$ . Although  $H$  is fixed, it is clear that everything we prove for it is also valid for any other member of  $\mathcal{S}$ . We will take advantage of this by sometimes interchanging the roles of  $H$  and any other suitable member of  $\mathcal{S}$ .

We will prove our results by a sequence of lemmas. Working towards induction, we start by collecting properties of the set  $\mathcal{S}_H$  of subspaces of  $H$  of dimension  $N_{n-2} = \frac{(n-2)(n+1)}{2}$  obtained by intersecting  $H$  with all members of  $\mathcal{S} \setminus \{H\}$ .

We notice that, for any member  $\overline{S}$  of  $\overline{\mathcal{S}}$ , every point of  $\overline{S}$  is incident with a unique member of  $\mathcal{S} \setminus \{S\}$  (by (VS1) and (VS2)), except for a unique point, which we call the *nucleus* of  $\overline{S}$ .

**Lemma 2.1** *If  $q \geq n$ , or if  $\mathcal{S}$  satisfies (VS4), then any two members of  $\mathcal{S}_H$  generate a hyperplane of  $H$ .*

**Proof.** The lemma is trivial if  $\mathcal{S}$  satisfies (VS4). So we may assume  $q \geq n$ .

Let  $S_1, S_2 \in \mathcal{S}$  with  $H \cap S_1 \neq H \cap S_2$ . We have  $\langle H \cap S_1, H \cap S_2 \rangle \subseteq (H \cap \langle S_1, S_2 \rangle)$ , and the latter is a hyperplane of  $H$  by (VS2). Hence, there remains to show that  $\langle H \cap S_1, H \cap S_2 \rangle$  has dimension  $\geq N_{n-1} - 1$ . This is equivalent to showing that  $\dim(H \cap S_1 \cap S_2) \leq 2N_{n-2} - N_{n-1} + 1 = \frac{(n-3)n}{2} = N_{n-3}$ . Suppose by way of contradiction that  $\dim(H \cap S_1 \cap S_2) > N_{n-3}$ . Then  $\dim\langle \overline{H}, \overline{S}_1, \overline{S}_2 \rangle \leq 3n - 2$ . Put  $\overline{R}_1 = \langle \overline{H}, \overline{S}_1 \rangle$  and  $\overline{R}_2 = \langle \overline{R}_1, \overline{S}_2 \rangle$ . Since  $\overline{H}$  and  $\overline{S}_1$  meet in a point,  $\dim \overline{R}_1 = 2n$ . If  $\overline{R}_2 = \overline{R}_1$ , then  $H \cap S_1 = H \cap S_2$ , contrary to our assumption. We now claim that there is a sequence  $(\overline{S}_3, \overline{S}_4, \dots, \overline{S}_{n+1})$  of members of  $\overline{\mathcal{S}}$  such that, for all  $i \in \{2, 3, \dots, n\}$ ,

- (i) the subspace  $\overline{R}_i$  defined inductively by  $\overline{R}_i = \langle \overline{R}_{i-1}, \overline{S}_i \rangle$  has at least one  $i$ -dimensional subspace in common with  $\overline{S}_{i+1}$ , and
- (ii)  $\overline{R}_i$  does not contain  $\overline{S}_{i+1}$ .

Putting  $i = n$  in (i) and (ii), these two conditions clearly yield a contradiction, in view of the fact that  $\dim \overline{S}_{n+1} = n$ .

We show (i) and (ii) with an inductive argument, adding  $\overline{S}_2$  to the sequence, putting  $\overline{H} = \overline{R}_0$ , and noting that  $\overline{S}_2$  has at least one plane — and hence at least one line — in common with  $\overline{R}_1$ . We will make advantage of the fact that, for this first step, we actually have a bigger intersection than asked for in (i). Suppose now that we already have a sequence  $(\overline{S}_2, \overline{S}_3, \dots, \overline{S}_k)$ , for some  $k \in \{2, 3, \dots, n\}$ , satisfying (i) and (ii) for all

$i \in \{1, 2, \dots, k-1\}$ . First we remark that the dimension of  $\overline{R}_k$  is bounded by  $\dim \overline{R}_2 + (n-2) + (n-3) + \dots + (n-(k-1)) \leq 3n-2 + (n-2)(n-1)/2 = \frac{n(n+3)}{2} - 1$ , hence  $\overline{R}_k$  is contained in a hyperplane of  $\overline{\mathbf{PG}}(N_n, q)$ . Condition  $(\overline{\text{VS3}})$  therefore guarantees the existence of a subspace  $\overline{S}'_{k+1}$  not contained in  $\overline{R}_k$ . Now, there are at least  $q^n - 1$  members of  $\overline{\mathcal{S}}$  meeting  $\overline{S}'_{k+1}$  in a point outside  $\overline{R}_k$ , and, for all  $i \in \{2, 3, \dots, k\}$ , we claim that there are at most  $\frac{(q^n-1)}{(q-1)} - i$  of these meeting  $\overline{S}_i$  in a point of  $\overline{R}_{i-1}$ . Indeed, the  $i$  subspaces  $\overline{H}, \overline{S}_1, \dots, \overline{S}_{i-1}$  meet  $\overline{S}_i$  in a point of  $\overline{R}_{i-1}$  and have no points outside  $\overline{R}_k$ ; hence there remain  $\frac{(q^n-1)}{(q-1)} - i$  points of  $\overline{S}_i \setminus \overline{R}_{i-1}$  that possibly could be contained in a (necessarily unique) member of  $\overline{\mathcal{S}}$  meeting  $\overline{S}'_{k+1}$  in a point outside  $\overline{R}_k$ . The claim follows. An easy counting argument (using  $k \leq n \leq q$ ) now shows that at least one member  $\overline{S}_{k+1}$  of  $\overline{\mathcal{S}}$  meets  $\overline{S}'_{k+1}$  in a point outside  $\overline{R}_k$  and meets  $\overline{S}_i$  in a point outside  $\overline{R}_{i-1}$ , for all  $i \in \{2, 3, \dots, k\}$ . Putting  $k = n$ , we obtain a subspace  $\overline{S}_{n+1}$  having an  $n$ -dimensional subspace in common with  $\overline{R}_n$ , but not contained in  $\overline{R}_n$ , a contradiction. The lemma is proved.  $\square$

If we assume the hypotheses of Corollary 1.5, then the dual space has dimension  $m \geq N_n$ . If  $m > N_n$ , then the previous proof shows that (with above notation) we can find a sequence  $(\overline{S}_3, \overline{S}_4, \dots, \overline{S}_{n+1})$  of members of  $\overline{\mathcal{S}}$  satisfying (i) and (ii), for all  $i \in \{2, 3, \dots, n\}$ . Hence the hypotheses of Corollary 1.5 imply  $m = N_n$ . Now it is easy to see that Corollary 1.5 will follow from Theorem 1.2 and Corollary 1.4.

From now on we assume that  $\mathcal{S}$  is a collection of  $q^n + q^{n-1} + \dots + q + 1$  subspaces of dimension  $\frac{(n-1)(n+2)}{2}$  of  $\mathbf{PG}(\frac{n(n+3)}{2}, q)$ , with  $n \geq 2$ , satisfying (VS1), (VS2), (VS3), and that  $\mathcal{S}$  either satisfies (VS4), or that  $q \geq n$ .

We now fix some more terminology. For  $S \in \mathcal{S} \setminus \{H\}$ , we define  $[H, S]$  as the set of members of  $\mathcal{S}$  containing  $H \cap S$ . The dual of  $[H, S]$  is denoted by  $[\overline{H}, \overline{S}]$ . The  $H$ -number of  $S$  is the cardinality of  $[H, S]$ . The *spectrum*  $\text{Spec}H$  of  $H$  is the set of all  $H$ -numbers of elements of  $\mathcal{S} \setminus \{H\}$ . We aim at proving that for  $\text{Spec}H$  there are a limited number of possibilities.

**Lemma 2.2** *We have  $\text{Spec}H \subseteq \{2, q, q+1\}$ . If  $q$  is odd, then  $\text{Spec}H \subseteq \{q, q+1\}$ .*

**Proof.** We show first that, if some  $S \in \mathcal{S}$  has  $H$ -number at least 3, then it must be either  $q$  or  $q+1$ . So suppose that  $S, S' \in \mathcal{S}$  meet  $H$  in the same subspace  $R$ . We dualize the situation. By  $(\overline{\text{VS3}})$ , there is a member  $\overline{T} \in \overline{\mathcal{S}}$  not contained in  $\overline{R}$ . By Lemma 2.1,  $\overline{T}$  meets  $\overline{R}$  in a line  $\overline{L}$ , which has the distinct points  $\overline{h}, \overline{s}$  and  $\overline{s}'$  in common with  $\overline{H}, \overline{S}$  and  $\overline{S}'$ , respectively. Since every member of  $\overline{\mathcal{S}}$  that is contained in  $\overline{R}$  must meet  $\overline{T}$  (by  $(\overline{\text{VS1}})$ ) necessarily in a point of  $\overline{L}$ , and since these points must all be distinct, we already have that the  $H$ -number of  $S$  is not bigger than  $q+1$ . Suppose now, by way of contradiction, that the  $H$ -number of  $S$  is strictly less than  $q$ . Then there are at least two points on  $\overline{L}$  that are not contained in a member of  $\overline{\mathcal{S}}$  that is entirely contained in  $\overline{R}$ . One of these



points could be the nucleus of  $\overline{T}$ , so there is at least one point  $\overline{x}$  of  $\overline{L}$  that is contained in a member  $\overline{T}_1$  of  $\overline{\mathcal{S}}$  that does not belong to  $\overline{R}$ . We seek a contradiction.

The subspace  $\overline{T}_1$  meets  $\overline{R}$  in a line  $\overline{L}_1$  intersecting  $\overline{H}, \overline{S}$  and  $\overline{S}'$  in  $\overline{h}_1, \overline{s}_1$  and  $\overline{s}'_1$ , respectively. Denote by  $\overline{t}$  the intersection point of  $\overline{H}$  and  $\overline{S}$ , and let  $\overline{P}$  be the plane spanned by  $\overline{t}$  and  $\overline{L}$ . Clearly, we have  $\overline{P} = \langle \overline{H}, \overline{L} \rangle \cap \langle \overline{S}, \overline{L} \rangle$ . But since  $\langle \overline{H}, \overline{L} \rangle = \langle \overline{H}, \overline{x} \rangle = \langle \overline{H}, \overline{L}_1 \rangle$ , and similarly  $\langle \overline{S}, \overline{L} \rangle = \langle \overline{S}, \overline{L}_1 \rangle$ , we deduce that  $\overline{L}_1$  is contained in  $\overline{P}$ . Hence  $\overline{S}'$  has the two distinct points  $\overline{s}'$  and  $\overline{s}'_1$  in common with  $\overline{P}$ , implying that  $\overline{S}'$  meets both  $\overline{H}$  and  $\overline{S}$  in points of  $\overline{P}$  (the respective intersections are on the lines  $\langle \overline{t}, \overline{h}, \overline{h}_1 \rangle$  and  $\langle \overline{t}, \overline{s}, \overline{s}_1 \rangle$ ). So all members of  $\overline{\mathcal{S}}$  that are contained in  $\overline{R}$  meet  $\overline{H}$  in points of  $\overline{P}$ .

We can now select a point  $\overline{y}$  of  $\overline{H}$ , distinct from the nucleus of  $\overline{H}$ , and not lying in  $\overline{P}$ . There is a unique member  $\overline{U} \in \overline{\mathcal{S}} \setminus \{H\}$  through  $\overline{y}$  and it is not contained in  $\overline{R}$ , by the previous paragraph. Interchanging the roles of  $T$  and  $U$  in the previous paragraph, we conclude that  $\overline{H} \cap \overline{S}'$  is contained in the line  $\langle \overline{t}, \overline{y} \rangle$ , a contradiction. So we do have  $\text{Spec}H \subseteq \{2, q, q+1\}$ .

Suppose now that  $q$  is odd. We show that the  $H$ -number of  $S \in \mathcal{S} \setminus \{H\}$  cannot be equal to 2. Assume on the contrary that the  $H$ -number of such an  $S$  is equal to 2. Consider an arbitrary  $T \in \mathcal{S} \setminus \{H, S\}$ . Then, putting  $\overline{R} = \langle \overline{H}, \overline{S} \rangle$  again,  $\overline{T}$  meets  $\overline{R}$  in just a line  $\overline{L}$ . We denote the intersection point of  $\overline{H}$  and  $\overline{S}$  by  $\overline{t}$ . As before, the intersection of any member  $\overline{T}_1$  of  $\overline{\mathcal{S}} \setminus \{\overline{H}, \overline{S}, \overline{T}\}$  containing some point of  $\overline{L}$  with  $\overline{R}$  is a line  $\overline{L}_1$  contained in the plane  $\overline{P} = \langle \overline{t}, \overline{L} \rangle$ . If the nucleus of  $\overline{T}$  were not on  $\overline{L}$ , then there would be  $q-1$  choices for  $\overline{T}_1$ , and since no three of the corresponding lines  $\overline{L}_1$ , together with  $\overline{L}, \overline{L}_H := \overline{P} \cap \overline{H}, \overline{L}_S := \overline{P} \cap \overline{S}$  meet in a common point, there arises a dual  $(q+2)$ -arc, a contradiction. Hence there is a unique point  $\overline{h}$  on the line  $\overline{L}_H$  not contained in a member of  $\overline{\mathcal{S}}$  that meets  $\overline{R}$  in a line of  $\overline{P}$ . We claim that  $\overline{h}$  is the nucleus of  $\overline{H}$ . Indeed, suppose it is not. Then there is a member  $\overline{T}' \in \overline{\mathcal{S}} \setminus \{\overline{H}\}$  containing  $\overline{h}$ . If we denote by  $\overline{L}'$  the intersection of  $\overline{T}'$  with  $\overline{R}$ , then our previous argument shows that there are  $q-2 > 0$  members of  $\overline{\mathcal{S}}$  different from  $\overline{T}', \overline{H}, \overline{S}$  meeting  $\overline{R}$  in a line of the plane  $\langle \overline{t}, \overline{L}' \rangle$ , and hence these  $q-2$  members would also contain points of the line  $\overline{t}\overline{h}$  different from  $\overline{t}$  and  $\overline{h}$ , and so their intersections with  $\overline{R}$  would actually be contained in  $\overline{P}$ , a contradiction. Hence  $\overline{h}$  is the nucleus of  $\overline{H}$ . But  $T$ , and therefore also the line  $\overline{t}\overline{h}$ , was arbitrary. This contradicts the uniqueness of the nucleus of  $\overline{H}$ .

The lemma is proved. □

We now classify the case  $q \neq 2$  where all spectra are contained in  $\{q, q+1\}$ . For  $q = 2$ , we treat the case where all spectra are  $\{3\}$ .

**Lemma 2.3** *Let  $q > 2$ . Suppose  $\text{Spec}H \subseteq \{q, q+1\}$ . Then  $\text{Spec}H = \{q+1\}$ . If this holds for every member of  $\mathcal{S}$ , then  $\mathcal{S}$  is the set of  $\mathcal{V}_{n-1}$ -subspaces of a quadric Veronesean  $\mathcal{V}_n^{2^n}$ . The latter also holds for  $q = 2$ .*

**Proof.** Suppose the  $H$ -number of some  $S \in \mathcal{S} \setminus \{H\}$  equals  $q$ . Let  $T \in \mathcal{S}$  be such that it does not contain  $H \cap S$ , and let  $T' \in [H, T]$  (remember  $q > 2$ ). This means that  $\langle \overline{H}, \overline{T} \rangle = \langle \overline{H}, \overline{T}' \rangle$ , which implies that the lines  $\langle \overline{H}, \overline{T} \rangle \cap \overline{S}$  and  $\langle \overline{H}, \overline{T}' \rangle \cap \overline{S}$  coincide. Denoting this line by  $\overline{L}_T = \overline{L}_{T'}$ , we deduce that either all points on  $\overline{L}_T$  are contained in a member of  $[H, T]$  (if the  $H$ -number of  $T$  is  $q + 1$ ), or all but exactly one are contained in a member of  $[H, T]$  (if the  $H$ -number of  $T$  is  $q$ ). Also, there are exactly  $q$  points on the line  $\overline{L} := \overline{R} \cap \overline{T}$ , with  $\overline{R} = \langle \overline{H}, \overline{S} \rangle$ , contained in members of  $[H, S]$ . So there remains a unique point  $\overline{x}$  on  $\overline{L}$  which is not contained in any member of  $[H, S]$ . We claim that  $\overline{x}$  is the nucleus of  $\overline{T}$ .

Indeed, suppose not. Then let  $\overline{U}$  be the unique element of  $\overline{\mathcal{S}} \setminus \{\overline{T}\}$  containing  $\overline{x}$ . As in previous proofs, this implies that the intersection of  $\overline{U}$  with  $\overline{R}$  is a line  $\overline{M}$  contained in the plane  $\overline{P}$  spanned by  $\overline{L}$  and the point  $\overline{H} \cap \overline{S}$ . Also, every element of  $[H, S]$  meets  $\overline{M}$  in a point not incident with  $\overline{L}$ , and hence has a line in common with  $\overline{P}$ , implying that every member of  $[H, S]$  has a point in common with  $\overline{L}_T$  (which is contained in  $\overline{P}$ ). This contradicts the observation made in the previous paragraph about the points on  $\overline{L}_T$ . The claim follows.

So we have shown, as  $\overline{T}$  was essentially arbitrary, that all nuclei are contained in  $\overline{R}$ . If the  $H$ -number of  $T$  were also equal to  $q$ , then similarly all nuclei would be contained in  $\langle \overline{H}, \overline{T} \rangle$ , which is clearly a contradiction considering a member  $\overline{U}'$  of  $\overline{\mathcal{S}}$  meeting  $\overline{H}$  in a point off  $\overline{L}_T$ . Hence the  $H$ -number of all members of  $\mathcal{S} \setminus [H, S]$  is equal to  $q + 1$ . Counting the number of sets  $[H, H']$ , with  $H' \in \mathcal{S} \setminus [H, S]$ , we obtain  $(q^n + q^{n-1} + \dots + q^2 + 1)/q$ , which is not an integer. This is a contradiction and we conclude that the spectrum of  $H$  is equal to  $\{q + 1\}$ .

Suppose now  $\text{Spec} \mathcal{S} = \{q + 1\}$  for all  $S \in \mathcal{S}$ , and allow  $q = 2$ . First suppose  $n = 2$ . Let  $\mathcal{V}$  be the set of points of  $\mathbf{PG}(5, q)$  that are contained in precisely  $q + 1$  members of  $\mathcal{S}$ . Note that  $|\mathcal{V}| = \frac{(q^2+q+1)(q^2+q)/2}{(q+1)q/2} = q^2 + q + 1$  by counting the ordered triples  $(S, S', x)$  with  $S, S' \in \mathcal{S}$ ,  $S \neq S'$  and  $x \in S \cap S'$ , in two ways. Also, there are precisely  $q + 1$  points of  $\mathcal{V}$  in each member of  $\mathcal{S}$ . We claim that  $\mathcal{V}$  is a cap. Indeed, we show that, whenever a point  $x \in \mathcal{V}$  is contained in a line of  $\mathbf{PG}(5, q)$  carrying at least three points of  $\mathcal{V}$ , then the set of  $q + 1$  points in any member of  $\mathcal{S}$  containing  $x$  is the point set of a line of  $\mathbf{PG}(5, q)$ . Let  $x, y, z$  be three distinct point of  $\mathcal{V}$  on a common line  $L$ . Let  $S_x \in \mathcal{S}$  contain  $x$ . First suppose that  $L$  is not contained in an element of  $\mathcal{S}$ . Let  $S_y$  and  $S_z$  be two members of  $\mathcal{S}$  containing  $y$  and  $z$ , respectively, and such that the intersections with  $S_x$  — say  $y'$  and  $z'$ , respectively — are distinct (these members exist because there are  $q + 1$  elements of  $\mathcal{S}$  through each point of  $\mathcal{V}$ ). If  $x, y', z'$  were not collinear, then the plane  $S_x$  would be generated by these three points; but then the planes  $S_y$  and  $S_z$  would generate a 4-space containing  $y', z', y$  and  $z$ , hence also containing  $x$  and thus containing  $S_x$ , contradicting (VS2). Fixing  $S_x$  and  $S_y$ , but not  $S_z$ , there arise at least  $q$  distinct points on  $\langle x, y' \rangle \setminus \{x, y'\}$ , a contradiction. So  $L$  is contained in an element  $S$  of  $\mathcal{S}$ . Assume that  $S_x$  does not contain

*L.* From the foregoing argument it follows that the points of  $\mathcal{V}$  in  $S_x$  are contained in a line. Similarly, interchanging the roles of  $y, z$  and two arbitrary points of  $S_x \cap \mathcal{V}$  different from  $x$ , we see that also  $S \cap \mathcal{V}$  is a line. But now every member of  $\mathcal{S}$  has that property, since every member of  $\mathcal{S}$  contains a point of  $S_x$  (with above notation, fixing such an  $S_x$ ). Hence  $\mathcal{V}$  consists of the union of  $q^2 + q + 1$  lines and is consequently the point set of a plane  $\rho$  of  $\mathbf{PG}(5, q)$ . Clearly, every member of  $\mathcal{S}$  meets  $\rho$  in a line, which implies that the  $(q^2 + q + 1)(q^2 + q)/2$  distinct hyperplanes containing two elements of  $\mathcal{S}$  all contain  $\rho$ . But there are only  $q^2 + q + 1$  hyperplanes in  $\mathbf{PG}(5, q)$  through  $\rho$ , a contradiction. The claim is proved.

It now follows that the set  $S \cap \mathcal{V}$  is an oval, for every  $S \in \mathcal{S}$ . Hence we have a set  $\mathcal{O}$  of  $q^2 + q + 1$  ovals, pairwise meeting in a unique point and such that every oval contains  $q + 1$  points, and every point is contained in  $q + 1$  ovals. This is a symmetric  $2-(q^2 + q + 1, q + 1, 1)$  design, hence a projective plane. We conclude that every two points are contained in a unique oval. In order to apply the main result of [8] to conclude that  $\mathcal{V}$  is isomorphic to the quadric Veronesean  $\mathcal{V}_2^4$ , we only have to prove that the tangent lines at any fixed point  $x \in \mathcal{V}$  to the ovals  $O \in \mathcal{O}$  containing  $x$  are coplanar. To that end, we consider an arbitrary plane  $S \in \mathcal{S}$  containing  $x$  and project  $\mathcal{V} \setminus S$  from  $S$  onto a plane  $\pi$  of  $\mathbf{PG}(5, q)$  skew to  $S$ . Let us denote by  $\theta$  the projection map. We show that  $\theta$  is injective on  $\mathcal{V} \setminus S$ . Let  $y, z \in \mathcal{V} \setminus S$  and suppose that  $y^\theta = z^\theta$ . Let  $S' \in \mathcal{S}$  contain  $y, z$ . Then  $\langle S, y, z \rangle = \langle S, S' \rangle$  is 3-dimensional, a contradiction. Now let  $S' \in \mathcal{S} \setminus \{S\}$  be arbitrary. Since  $\langle S, S' \rangle$  is 4-dimensional, the projection of  $(S' \setminus S) \cap \mathcal{V}$  consists of  $q$  points on a line  $L'$  of  $\pi$ . Let  $u'$  be the unique point on  $L'$  that is not an image under  $\theta$  of any point of  $S' \cap \mathcal{V}$ . Suppose by way of contradiction that  $u'$  is the image of a point  $u \in \mathcal{V} \setminus S$  (necessarily  $u \notin S'$ ). The  $q + 1$  planes of  $\mathcal{S}$  through  $u$ , minus their intersection points with  $S$ , are mapped under  $\theta$  into  $q + 1$  different lines of  $\pi$  through  $u'$ , since every three distinct members of  $\mathcal{S}$  generate  $\mathbf{PG}(5, q)$ . Hence there is a member  $S'' \in \mathcal{S}$  through  $u$  which yields  $L'$ . Hence  $S, S', S''$  are contained in the hyperplane  $\langle S, L' \rangle$  of  $\mathbf{PG}(5, q)$ , contradicting (VS2). It now follows easily that the set of planes of  $\mathcal{S} \setminus \{S\}$  through  $x$  corresponds under  $\theta$  with the set of  $q + 1$  lines of  $\pi$  containing a fixed point  $x'$  of  $\pi$ , and that the 3-dimensional subspace  $W = \langle S, x' \rangle$  meets every member  $S'$  of  $\mathcal{S} \setminus \{S\}$  containing  $x$  in a line  $L_{S'}$  through  $x$  disjoint from  $\mathcal{V} \setminus S$ . So  $L_{S'}$  is tangent to the oval  $S' \cap \mathcal{V}$  at  $x$ . Now fix  $S' \in \mathcal{S} \setminus \{S\}$  with  $x \in S'$ . Then similarly, there is a 3-dimensional space  $W'$  containing  $S'$  and the tangents at  $x$  to the members of  $\mathcal{O}$  containing  $x$ . Clearly  $W \neq W'$  and so all those tangents are contained in the plane  $W \cap W'$ . This shows the lemma for  $n = 2$ .

Next, suppose  $n > 2$ . Consider the set  $\mathcal{S}_H = \{H \cap S \mid S \in \mathcal{S}, S \neq H\}$ . One calculates  $|\mathcal{S}_H| = q^{n-1} + q^{n-2} + \dots + q + 1$ . Also, putting  $\mathbf{PG}(N_{n-1}, q)$  equal to  $H$ , the set  $\mathcal{S}_H$  satisfies (VS1) and (VS3) — for the parameter  $n - 1$  instead of for  $n$ . We show that it also satisfies (VS2). Indeed, Let  $[H, S] \neq [H, T]$ , and let  $\bar{L}_T$  be as above. Then, again as above, all points of  $\bar{L}_T$  are contained in members of  $[\bar{H}, \bar{T}]$ . Hence any  $\bar{U} \in \mathcal{S} \setminus ([\bar{H}, \bar{S}] \cup [\bar{H}, \bar{T}])$  meets  $\bar{S}$  outside  $\langle \bar{H}, \bar{T} \rangle$ . This means that  $\langle \bar{H}, \bar{S} \rangle \cap \langle \bar{H}, \bar{T} \rangle \cap \langle \bar{H}, \bar{U} \rangle = \bar{H}$ , and the dual of

this is exactly (VS2). Also, if  $\mathcal{S}$  satisfies (VS4), then clearly  $\mathcal{S}_H$  satisfies (VS4). And if  $q \geq n$ , then  $q \geq n - 1$ . Hence the induction hypothesis implies that  $\mathcal{S}_H$  is either the set of all  $\mathcal{V}_{n-2}$ -subspaces of a quadric Veronesean  $\mathcal{V}_{n-1}^{2^{n-1}}$ , or  $q$  is even and  $\mathcal{S}_H$  is the nucleus subspace together with all  $\mathcal{V}_{n-2}$ -subspaces but exactly one of a quadric Veronesean  $\mathcal{V}_{n-1}^{2^{n-1}}$ , or it is an ovoidal Veronesean set of subspaces. We claim that the two latter cannot occur.

Indeed, In both these cases, there is a member of  $\mathcal{S}_H$ , which we may take to be  $H \cap S$ , with the property that it contains  $q^{n-1} + q^{n-2} + \dots + q^2 + q$  subspaces of dimension  $N_{n-2}$  arising as intersections of  $H \cap S$  with other members of  $\mathcal{S}_H$ . Let  $V$  be such a subspace and let  $U \in \mathcal{S} \setminus [H, S]$  contain  $V$ . Then all  $q+1$  elements of  $[H, U]$  contain  $V$ , but they define  $q+1$  distinct members of  $\mathcal{S}_S$ , each of which is defined by  $q$  different members of  $\mathcal{S}$ . As we have  $q^{n-1} + q^{n-2} + \dots + q^2 + q$  possibilities for  $V$ , we count at least  $q^{n+1} + q^n + \dots + q^4 + q^3 + (q+1)$  (the last “ $(q+1)$ ” is to account for  $[H, S]$ ) members of  $\mathcal{S}$ , a contradiction. The claim follows.

We now consider the set  $\mathcal{V}$  of all points of  $\mathbf{PG}(N_n, q)$  that are contained in precisely  $q^{n-1} + q^{n-2} + \dots + q + 1$  members of  $\mathcal{S}$ . From the previous paragraph it immediately follows that for each member  $H$  of  $\mathcal{S}$ , the intersection  $H \cap \mathcal{V}$  is a quadric Veronesean  $\mathcal{V}_{n-1}^{2^{n-1}}$ . We endow  $\mathcal{V}$  with all plane conics contained in these intersections  $H \cap \mathcal{V}$ , and denote by  $\mathcal{C}$  the set of all such conics. Now let  $x, y \in \mathcal{V}$ ,  $x \neq y$ . Then there are  $S, T \in \mathcal{S}$  with  $x \in S$  and  $y \in T$ . Suppose  $S \neq T$ . The  $q^{n-1} + q^{n-2} + \dots + q + 1$  members of  $\mathcal{S}$  containing  $x$  meet  $T$  in distinct subspaces (by (VS1) and the fact that their intersection contains  $x$ ), and hence  $y$  is contained in at least one of them. We have shown that  $x$  and  $y$  are contained in a common member of  $\mathcal{S}$ , and hence  $x$  and  $y$  are contained in a conic of  $\mathcal{C}$ . Assume, by way of contradiction, that  $x, y \in \mathcal{V}$  with  $x \neq y$  are contained in distinct conics  $C$  and  $C'$  of  $\mathcal{C}$ . Let  $z \in C' \setminus C$ . As before, it is easy to see that, if  $n > 2$ , then  $C$  is contained in at least one of the  $q^{n-1} + q^{n-2} + \dots + q + 1$  members of  $\mathcal{S}$  containing  $z$ . So  $C$  and  $C'$  are distinct conics of a quadric Veronesean  $\mathcal{V}_{n-1}^{2^{n-1}}$  sharing two distinct points, a contradiction. So any two distinct points of  $\mathcal{V}$  are contained in exactly one conic of  $\mathcal{C}$ .

Now let  $C$  be any member of  $\mathcal{C}$  and assume that  $x \in \mathcal{V} \setminus C$ . As before, it is easy to see that, if  $n \geq 3$ , then  $C$  is contained in at least one of the  $q^{n-1} + q^{n-2} + \dots + q + 1$  members of  $\mathcal{S}$  containing  $x$ . So  $x$  and  $C$  are contained in a common member  $H$  of  $\mathcal{S}$ , and since  $H \cap \mathcal{V}$  is a Veronesean, the tangents at  $x$  of the conics through  $x$  which have a point in common with  $C$  all lie in a fixed plane. By a similar argument it follows that two distinct elements of  $\mathcal{C}$  always generate a 4-dimensional space. Now assume that  $C, C' \in \mathcal{C}$ , with  $C \neq C'$ , and that  $x \in \langle C \rangle \cap \langle C' \rangle$ . We claim that  $x \in C \cap C'$ .

If  $x \notin C \cap C'$  and  $x$  is not the nucleus of at least one of  $C, C'$ , say  $x$  is not the nucleus of  $C$ , then there is an element  $S$  of  $\mathcal{S}$  containing  $C'$  and two distinct points  $y$  and  $z$  of  $C$ . If  $S$  also contains  $C$ , then, as  $C$  and  $C'$  are conics of some  $\mathcal{V}_{n-1}^{2^{n-1}}$ , it follows that  $x \in C \cap C'$ , a contradiction. If  $S$  does not contain  $C$ , then  $y$  and  $z$  are contained in common distinct members of  $\mathcal{C}$ , again a contradiction.

Now assume that  $x$  is the common nucleus of  $C$  and  $C'$ . Let  $z \in C'$ . Then  $z$  and  $C$  are contained in a common member  $S$  of  $\mathcal{S}$ . As  $x$  is the common nucleus of  $C$  and  $C'$  the space  $S$  cannot contain  $C'$ . So  $S \cap \langle C' \rangle = xz$ . Let  $z' \in C' \setminus \{z\}$ . If  $n > 3$ , then by similar arguments  $z, z'$  and  $C$  are contained in a common element  $S'$  of  $\mathcal{S}$ . As  $S' \cap \mathcal{V}$  is a quadric Veronesean  $\mathcal{V}_{n-1}^{2n-1}$ , the conics  $C$  and  $C'$  cannot have a common nucleus. So we may assume  $n = 3$ . If  $z, z', z''$  are distinct points of  $C'$ , then  $\langle z, C \rangle, \langle z', C \rangle, \langle z'', C \rangle$  are contained in respective elements  $S, S', S''$  of  $\mathcal{S}$ . The 5-dimensional spaces  $S, S', S''$  are distinct and share the plane  $\langle C \rangle$ . Let  $y \in C$ . Then  $y$  and  $C'$  belong to a Veronesean  $\mathcal{V}_2^4$ , so the tangents at  $y$  of  $\mathcal{V}_2^4$  are coplanar. Let  $D, D', D''$  be the conics containing respectively  $\{z, y\}, \{z', y\}, \{z'', y\}$ . Since the tangents at  $y$  of  $D, D', D''$  are coplanar,  $D''$  belongs to  $\langle S, S' \rangle$ . Consequently  $S'' \subseteq \langle S, S' \rangle$ . So  $S, S', S''$  belong to a common hyperplane, clearly a contradiction. We conclude that always  $x \in C \cap C'$ .

By Hirschfeld and Thas [3], it now follows that  $\mathcal{C}$  is the set of all conics on a quadric Veronesean  $\mathcal{V}_n^{2n}$ . hence by Theorem 25.1.9 of [3],  $\mathcal{S}$  is the set of all  $\mathcal{V}_{n-1}$ -subspaces of  $\mathcal{V}_n^{2n}$ .  $\square$

From now on we may assume that there exists some member of  $\mathcal{S}$  whose spectrum contains 2. In particular,  $q$  is even (see Lemma 2.2). We now first treat the case where the spectrum of some member of  $\mathcal{S}$  contains 2 and has size at least 2.

**Lemma 2.4** *Let  $H \in \mathcal{S}$  be such that  $2 \in \text{Spec}H$  and  $|\text{Spec}H| \geq 2$ . Then  $\text{Spec}H = \{2, q, q+1\}$ . If  $q > 2$ , there exists a unique member  $S$  of  $\mathcal{S} \setminus \{H\}$  such that  $|\langle H, S \rangle| = 2$  and there are precisely  $q-1$  members  $T$  of  $\mathcal{S} \setminus \{H\}$  such that  $|\langle H, T \rangle| = q$ . Also,  $\text{Spec}S = \{2\}$  and the spectrum of any other element of  $\mathcal{S}$  is equal to  $\{2, q, q+1\}$ . If  $q = 2$ , then there are precisely two members of  $\mathcal{S} \setminus \{H\}$  with  $H$ -number 2, one of which has spectrum  $\{2\}$  and while any other element of  $\mathcal{S}$  has spectrum  $\{2, 3\}$ .*

**Proof.** Let  $H, S \in \mathcal{S}$  be such that  $|\langle H, S \rangle| = 2$  and let  $U \in \mathcal{S} \setminus \{H\}$  be such that  $\ell := |\langle H, U \rangle| > 2$ . As before, let  $\bar{R}$  be the space generated by  $\bar{H}$  and  $\bar{S}$ , and let  $\bar{L}$  be the intersection  $\bar{R}$  with  $\bar{U}$ . The elements of  $[\bar{H}, \bar{U}]$  have to meet  $\bar{S}$  in the line joining  $\bar{t} := \bar{H} \cap \bar{S}$  and  $\bar{U} \cap \bar{S}$ ; on the other hand, the members of  $\mathcal{S} \setminus \{\bar{H}, \bar{S}, \bar{U}\}$  containing a point of  $\bar{L}$  must intersect  $\bar{R}$  in lines which are contained in the plane  $\langle \bar{t}, \bar{L} \rangle$ . It follows that there are at least  $q-1$  points on the line  $\bar{M}$  joining  $\bar{t}$  with  $\bar{H} \cap \bar{U}$  contained in members of  $[\bar{H}, \bar{U}]$ , distinct from  $\bar{H}$ , if  $\ell = q+1$ , and at least  $q-2$  such points if  $\ell = q$ . A similar statement holds for the line  $\bar{M}'$  joining  $\bar{t}$  with  $\bar{H} \cap \bar{U}'$ , for every  $\bar{U}' \in [\bar{H}, \bar{U}] \setminus \{\bar{H}\}$ . It readily follows that, if  $q > 2$ , then every such line  $\bar{M}'$  coincides with  $\bar{M}$ . Suppose now  $q > 2$ . Then there are precisely  $q$  points on the line  $\bar{M}$  contained in members of  $[\bar{H}, \bar{U}]$ , distinct from  $\bar{H}$ , if  $\ell = q+1$ , and precisely  $q-1$  such points if  $\ell = q$ . Since the line  $\bar{M}$  is uniquely determined by  $[\bar{H}, \bar{U}]$ , every member  $\bar{S}'$  of  $\mathcal{S} \setminus \{\bar{H}\}$  with  $H$ -number 2 must intersect  $\bar{H}$  on the line  $\bar{M}$ . It follows that  $\bar{S} = \bar{S}'$  if  $\ell = q+1$ , and there is at most one

choice for  $\bar{S}' \neq \bar{S}$  if  $\ell = q$ . So suppose that such a space  $\bar{S}' \neq \bar{S}$  exists. Put  $\bar{t}' = \bar{H} \cap \bar{S}'$ . Clearly every member  $\bar{U}' \in \bar{\mathcal{S}} \setminus [\bar{H}, \bar{U}]$ , with  $\bar{S} \neq \bar{U}' \neq \bar{S}'$ , has  $H$ -number  $q$ , and so, by the previous arguments, all elements of  $[\bar{H}, \bar{U}'] \setminus \{\bar{H}\}$  have a point in common with  $\bar{t}'$ , clearly a contradiction.

Suppose now  $q = 2$  and put  $[H, U] = \{H, U, U'\}$ . The only possible reason that the line  $\bar{M}$  would not contain  $\bar{H} \cap \bar{U}'$  is that there is no member of  $\bar{\mathcal{S}} \setminus \{\bar{H}, \bar{S}, \bar{U}\}$  containing a point of  $\bar{L}$ ; in other words, the nucleus of  $\bar{U}$  is contained in  $\bar{R}$ . Assume by way of contradiction that there are at least three members of  $\bar{\mathcal{S}} \setminus \{\bar{H}\}$  with  $H$ -number 2. Then for at least two of them, say  $\bar{S}$  and  $\bar{S}'$ , the nucleus  $\bar{x}$  of  $\bar{U}$  is contained in  $\langle \bar{H}, \bar{S} \rangle \cap \langle \bar{H}, \bar{S}' \rangle$ . Hence  $\bar{H} \cap \bar{U}$ ,  $\bar{S} \cap \bar{U}$ ,  $\bar{S}' \cap \bar{U}$  and the nucleus of  $\bar{U}$  are distinct collinear points, a contradiction. So in the case  $q = 2$ , there are at most 2 members with  $H$ -number 2, say  $\bar{S}$  and  $\bar{S}'$ . Putting  $\bar{H} \cap \bar{S} = \bar{t}$  and  $\bar{H} \cap \bar{S}' = \bar{t}'$ , the same argument also shows that, if  $\bar{H} \cap \bar{U} = \bar{u}$  and  $\bar{H} \cap \bar{U}' = \bar{u}'$ , then the line  $\bar{u}\bar{u}'$  contains either  $\bar{t}$  or  $\bar{t}'$ , and if it contains, say,  $\bar{t}'$ , then the nuclei of  $\bar{U}$  and  $\bar{U}'$  are contained in  $\langle \bar{H}, \bar{S} \rangle$ . With this notation, we now show that the spectrum of  $\bar{S}'$  is equal to  $\{2\}$ . Indeed, if  $[S', U]$  contains a member  $T \notin \{S', U\}$ , then  $\bar{T}$  must meet  $\bar{H}$  in the point  $\bar{u}'$ , a contradiction. Similarly,  $|[S', U']| = 2$  and so there are at least 3 members of  $\mathcal{S} \setminus \{S'\}$  with  $S'$ -number 2 (namely,  $H, U, U'$ ), and so by the previous arguments, there cannot be a member with  $S'$ -number equal to  $q + 1 = 3$ . Hence  $\text{Spec} S' = \{2\}$ .

Now we show that  $\text{Spec} S = \{2, 3\}$ . Suppose by way of contradiction that  $\text{Spec} S = \{2\}$ . First note that the argument in the previous paragraph immediately implies that the nucleus of  $\bar{H}$  is on the line  $\bar{t}\bar{u}$ , as otherwise  $\bar{U}'$  contains the third point of  $\bar{t}\bar{u}$  as well as  $\bar{u}'$ . Analogously the nucleus of  $\bar{H}$  is on the line  $\bar{t}'\bar{u}'$ . This yields a contradiction. It also follows that  $|\bar{\mathcal{S}}| = |\bar{\mathcal{S}}'| = 3$ , and so the nucleus of  $\bar{H}$  is on  $\bar{t}\bar{t}'$ .

Taking into account all previous arguments, we conclude that  $S'$  is the unique member of  $\mathcal{S}$  with spectrum  $\{2\}$ , and the other members are divided in pairs  $\{X, Y\}$  with respect to the relation “ $X$  has  $Y$ -number 2”. Moreover, the nucleus of  $\bar{X}$  and the points  $\bar{X} \cap \bar{Y}$ ,  $\bar{X} \cap \bar{S}'$  are collinear, and the two intersection points of the elements of  $[\bar{X}, \bar{Z}] \setminus \{\bar{X}\}$ , where  $Z \notin \{X, Y, S'\}$ , with  $\bar{X}$  are collinear with  $\bar{X} \cap \bar{S}'$ . Also, the nucleus of any  $\bar{X}$  is contained in the space  $\langle \bar{H}, \bar{S} \rangle$ . Indeed, this is obvious if  $X \in \{H, S\}$ . Suppose now  $X \notin \{H, S\}$ , and also assume  $X \neq S'$ . If the nucleus of  $\bar{X}$  were not contained in  $\langle \bar{H}, \bar{S} \rangle$ , then the unique member  $\bar{X}'$  of  $[\bar{H}, \bar{X}]$  distinct from  $\bar{H}$  and from  $\bar{X}$  would meet  $\langle \bar{H}, \bar{S} \rangle$  in a line of the plane spanned by  $\bar{t}$  and  $\bar{X} \cap \langle \bar{H}, \bar{S} \rangle$ , implying that the intersection points  $\bar{H} \cap \bar{X}$  and  $\bar{H} \cap \bar{X}'$  would be collinear with  $\bar{t}$ , contradicting an earlier observation. Now we prove that the nucleus of  $\bar{S}'$  is contained in  $\langle \bar{H}, \bar{S} \rangle$ . Suppose not. Then the third point of the line joining  $\bar{t}'$  and  $\bar{S} \cap \bar{S}'$  is on an element  $\bar{Z}$  of  $\bar{\mathcal{S}}$  intersecting  $\bar{H}, \bar{S}, \bar{S}'$  in distinct points of the line  $\bar{Z} \cap \langle \bar{H}, \bar{S} \rangle$ . But the nucleus of  $\bar{Z}$  is also on that line, a contradiction. We now show a similar thing for the case  $q > 2$ .

So let  $q > 2$ . A simple counting argument now yields the existence of at least one member  $T \in \mathcal{S} \setminus \{H\}$  with  $H$ -number  $q$ . We claim that the nuclei of all members of  $\overline{\mathcal{S}}$  are contained in  $\overline{R}^* = \langle \overline{H}, \overline{T} \rangle$ . Note that, similarly as in the proof of Lemma 2.3, this immediately implies that the only members of  $\mathcal{S} \setminus \{H\}$  with  $H$ -number  $q$  are those of  $[H, T]$ .

We now prove the claim. Put  $\overline{a} = \overline{H} \cap \overline{T}$ . Let  $W$  be any member of  $\mathcal{S} \setminus [H, T]$ . There is a unique point  $\overline{x}$  on the line  $\overline{L}_W = \overline{R}^* \cap \overline{W}$  not contained in a member of  $[\overline{H}, \overline{T}]$ . If this point were not the nucleus of  $\overline{W}$ , then it is contained in a member  $\overline{W}' \neq \overline{W}$ , and similarly as arguments before, we conclude that the members of  $[\overline{H}, \overline{T}] \setminus \{\overline{T}\}$  meet  $\overline{T}$  in points of the line  $\overline{T} \cap \langle \overline{a}, \overline{L}_W \rangle$ . But this line contains the points of intersection of  $\overline{T}$  with any member of  $[\overline{H}, \overline{W}] \cup [\overline{H}, \overline{W}']$ . It follows that the  $H$ -number of both  $W$  and  $W'$  is equal to 2, a contradiction. The claim is proved.

We now show that all elements of  $\mathcal{S} \setminus \{S\}$  have  $S$ -number equal to 2. If not, then most of them have  $S$ -number  $q+1$ . More exactly, it is easy to see that we can find  $W \in \mathcal{S} \setminus \{H, S\}$  such that  $W$  has  $H$ -number equal to  $q+1$  and  $S$ -number equal to  $q+1$ . It follows easily from previous arguments that at least  $q-2$  members of  $[\overline{H}, \overline{W}] \setminus \{\overline{H}, \overline{W}\}$  meet  $\overline{R} = \langle \overline{H}, \overline{S} \rangle$  in lines belonging to the plane generated by  $\overline{R} \cap \overline{W}$  and  $\overline{H} \cap \overline{S}$ . By symmetry, this also holds for at least  $q-2$  members of  $[\overline{S}, \overline{W}] \setminus \{\overline{S}, \overline{W}\}$ , which is clearly a contradiction.

Now it also follows that the spectrum of any other element of  $\mathcal{S}$  is  $\{2, q, q+1\}$ , because otherwise if its spectrum would be equal to  $\{2\}$  there would be at least 2 members of  $\mathcal{S} \setminus \{H\}$  with  $H$ -number 2.

The lemma is completely proved.  $\square$

**Lemma 2.5** *If  $\mathcal{S}$  is a proper Veronesean set of subspaces with the property that the number 2 is contained in the spectrum of at least one member of  $\mathcal{S}$ , then  $\mathcal{S}$  is the set of all  $\mathcal{V}_{n-1}$ -subspaces but one, together with the nucleus subspace of a quadric Veronesean  $\mathcal{V}_n^{2^n}$ .*

**Proof.** By a standard connectivity argument, we may assume that there is a member of  $\mathcal{S}$  whose spectrum contains 2 and one of  $q$  or  $q+1$  (or both) (if  $q=2$ , then  $q+1$  must occur). Lemma 2.4 implies that there is a unique member  $S$  of  $\mathcal{S}$  with spectrum  $\{2\}$ , and all other members of  $\mathcal{S}$  have spectrum  $\{2, q, q+1\}$ . Moreover, for  $H \in \mathcal{S} \setminus \{S\}$ , there is a unique set  $[H, T]$  of cardinality  $q$  (with  $T \in \mathcal{S} \setminus \{H, S\}$ ), and all elements of  $\mathcal{S} \setminus ([H, T] \cup \{S\})$  have  $H$ -number  $q+1$ . Also, from the proof of Lemma 2.4 follows that, for each member  $U$  with  $H$ -number  $q+1$ , the set of points  $\overline{H} \cap \overline{U}'$  with  $U' \in [H, U] \setminus \{H\}$ , is contained in a line  $\overline{L}_U$ , which is also incident with the point  $\overline{s} := \overline{H} \cap \overline{S}$ . The unique line  $\overline{L}_T$  of  $\overline{H}$  through  $\overline{s}$  that we cannot obtain in this way, contains the  $q-1$  points of intersection of  $\overline{H}$  with the elements of  $[\overline{H}, \overline{T}] \setminus \{\overline{H}\}$ , and the nucleus of  $\overline{H}$ .

We now claim that the set of all nuclei forms an  $n$ -dimensional subspace  $\overline{C}$  of  $\overline{\mathbf{PG}}(N_n, q)$ . Indeed, in view of the fact that there are precisely  $q^n + q^{n-1} + \dots + q + 1$  nuclei, it suffices

to show that all points of the line joining two distinct arbitrarily given nuclei are nuclei. In other words, it is sufficient to show that (1) the nuclei of all elements of  $[\overline{H}, \overline{U}]$  are collinear, and that (2) the nuclei of all elements of  $[\overline{H}, \overline{T}] \cup \{\overline{S}\}$  are collinear.

We put  $\overline{R} = \langle \overline{H}, \overline{T} \rangle$ , and, for each member  $U'$  of  $[H, U] \setminus \{H\}$ , we set  $\overline{M}_{U'} = \overline{R} \cap \overline{U}'$ . Note that the unique point of such a line  $\overline{M}_{U'}$  which is not contained in any member of  $[\overline{H}, \overline{T}]$  is the nucleus  $\overline{u}'$  of  $\overline{U}'$  (by the proof of Lemma 2.4 all nuclei are contained in  $\langle \overline{H}, \overline{T} \rangle$ ). Previous arguments imply that, for each  $T' \in [H, T] \setminus \{H\}$ , the points  $\overline{H} \cap \overline{T}'$  and  $\overline{U}' \cap \overline{T}'$ , with  $U' \in [H, U] \setminus \{H\}$ , constitute a line  $\overline{N}_{T'} = \langle \overline{H}, \overline{U} \rangle \cap \overline{T}'$ . It consequently follows that the  $q$  pairwise disjoint lines  $\overline{L}_U, \overline{N}_{T'}$ , with  $T'$  varying over  $[H, T] \setminus \{H\}$ , all meet each of the  $q + 1$  pairwise disjoint lines  $\overline{L}_T, \overline{M}_{U'}$ , with  $U'$  varying over  $[H, U] \setminus \{H\}$ . Hence the nuclei of all members of  $[\overline{H}, \overline{U}]$  are contained in the unique “missing” line of the ruled non-degenerate quadric determined by the  $2q + 1$  mentioned lines. This shows (1).

Now let  $\overline{L}_S$  be the intersection of  $\overline{R}$  with  $\overline{S}$ . Clearly this line contains the point  $\overline{s}$ , hence  $\langle \overline{L}_S, \overline{L}_T \rangle$  is a plane  $\overline{P}$ . Since for every  $T' \in [H, T]$ , the  $T'$ -number of  $S$  is equal to 2, the plane  $\overline{P}$  contains, for every such  $T'$ , the line  $\overline{L}_{T'}^*$  consisting of all intersection points  $\overline{T}' \cap \overline{T}''$ , with  $T'' \in [H, T] \setminus \{T'\}$ , the point  $\overline{S} \cap \overline{T}'$ , and the nucleus of  $\overline{T}'$ . Note that  $\overline{L}_T = \overline{L}_H^*$ . So the set of lines  $\overline{\mathcal{H}} = \{\overline{L}_{T'}^* \mid T' \in [H, T]\} \cup \{\overline{L}_S\}$  is a dual oval in  $\overline{P}$ . As all nuclei are contained in  $\langle \overline{H}, \overline{T} \rangle$ , the nucleus of  $\overline{S}$  belongs to  $\overline{L}_S$ . Hence (remarking that  $q$  is even by Lemma 2.2) the  $q + 1$  nuclei of the elements in  $[\overline{H}, \overline{T}] \cup \{\overline{S}\}$  form the nucleus line of the dual oval  $\overline{\mathcal{H}}$ , proving (2). The claim follows.

We now claim that also the set  $(\mathcal{S} \cup \{C\}) \setminus \{S\}$  is a Veronesean set of subspaces. Indeed, (VS1) follows from the fact that  $\overline{C}$  meets every member of  $\overline{\mathcal{S}}$  in a unique point; (VS2) follows from the fact that  $\overline{C}$  does not contain any point that is contained in two distinct members of  $\overline{\mathcal{S}}$ ; (VS3) is obviously satisfied. We will now show that for this Veronesean set of subspaces no spectrum contains 2. Assume, by way of contradiction, that the  $C$ -number of  $U \in \mathcal{S} \setminus \{S\}$  equals 2. We can find a subspace  $U' \in \mathcal{S} \setminus \{S\}$ ,  $U' \neq U$ , for which  $||[U, U']|| = q$ . By the foregoing  $\overline{C}$  is a subspace of  $\langle \overline{U}, \overline{U}' \rangle$ , and so  $U' \in [C, U]$ , a contradiction. It is now clear that no spectrum contains 2. Moreover, a similar argument shows that, if  $\mathcal{S}$  satisfies (VS4), then so does  $(\mathcal{S} \cup \{C\}) \setminus \{S\}$ . By Lemma 2.3 the latter is the set of  $\mathcal{V}_{n-1}$ -subspaces of a quadric Veronesean  $\mathcal{V}_n^{2n}$ . It is now obvious that  $S$  is the nucleus subspace of  $\mathcal{V}_n^{2n}$ , in view of the nucleus subspace property of  $\mathcal{V}_n^{2n}$ .

The lemma is proved. □

This completes the classification of all proper Veronesean sets of subspaces in  $\mathbf{PG}(N_n, q)$ . In particular, we have shown Theorem 1.1 and Theorem 1.2.

We now show Theorem 1.3. So, we show that every ovoidal Veronesean set  $\mathcal{S}$  of subspaces (of  $\mathbf{PG}(N_n, q)$ ) is contained in a unique hyperovoidal Veronesean set of subspaces. Clearly, it suffices to show that the set of all nuclei of members of  $\overline{\mathcal{S}}$  constitutes an  $n$ -dimensional subspace of  $\mathbf{PG}(N_n, q)$ . Therefore, it is enough to show that all points of the line joining



two arbitrary nuclei are nuclei. So let  $H_1, H_2 \in \mathcal{S}$  and let  $\bar{h}_i$  be the nucleus of  $\bar{H}_i$ ,  $i = 1, 2$ . Let  $\bar{h}$  be the intersection point of  $\bar{H}_1$  and  $\bar{H}_2$ . Let  $\bar{x}_1$  be an arbitrary point on the line  $\langle \bar{h}, \bar{h}_1 \rangle$  different from  $\bar{h}$  and from  $\bar{h}_1$ . Let  $\bar{S} \in \mathcal{S} \setminus \{\bar{H}_1\}$  contain  $\bar{x}_1$ , and let  $\bar{L}$  be the intersection of  $\bar{S}$  with  $\langle \bar{H}_1, \bar{H}_2 \rangle$ . Put  $\bar{x}_2 := \bar{S} \cap \bar{H}_2$ . As before, any member of  $\bar{\mathcal{S}}$  meeting  $\bar{L}$  has a line in common with the plane  $\langle \bar{h}, \bar{L} \rangle$ , and the set of all such lines forms a dual oval (if the nucleus of  $\bar{S}$  is on  $\bar{L}$ ) or a dual hyperoval (if the nucleus of  $\bar{S}$  is not on  $\bar{L}$ ). In the latter case, the point  $\bar{h}_1$  is contained in a unique line of that dual hyperoval, different from the line  $\langle \bar{h}, \bar{h}_1 \rangle$ , contradicting the definition of nucleus. Hence we have a dual oval, and it is now easy to see (by interchanging the roles of  $H_2$  and  $S$  if necessary) that the unique line of  $\langle \bar{h}, \bar{L} \rangle$  extending that dual oval to a dual hyperoval contains  $q + 1$  nuclei, amongst which  $\bar{h}_1$  and  $\bar{h}_2$ .

Now, if  $n = 2$ , then a hyperovoidal Veronesean set of subspaces is a 2-dimensional dual hyperoval, as defined by Huybrechts and Pasini [4]. Also, every three distinct elements of the 2-dimensional dual hyperoval we have here, generate  $\mathbf{PG}(5, q)$ . It is easy to see that this yields a design with the same parameters as a one-point extension of a projective plane of order  $q$ . Hence  $q = 2, 4$ . Also, if  $q = 2$ , then it is again easy to see that all 2-dimensional hyperovals are isomorphic. For  $q = 4$ , these objects are classified by Del Fra [1] and only one example turns up, related to the simple Mathieu group  $M_{22}$ . The example for  $q = 2$  can be generalized to general  $n$  as follows. Let  $\mathbf{AG}(n, 2)$  be an affine space in  $\mathbf{PG}(n, 2)$ . Consider in the Grassmannian of the lines of  $\mathbf{PG}(n, 2)$  all subspaces corresponding to the full line pencils of lines with vertex in  $\mathbf{AG}(n, 2)$ . Then one verifies that this gives a dual hyperovoidal Veronesean set of subspaces.

The proof of Theorem 1.3 is complete.

### 3 Proof of the application

We now prove Theorem 1.6.

First we consider a slightly more general situation. Let  $\theta : \mathbf{PG}(n, q) \rightarrow \mathbf{PG}(m, q)$  be an injective map from the point set of  $\mathbf{PG}(n, q)$  to the point set of  $\mathbf{PG}(m, q)$ , with  $n \geq 1$  and  $q > 2$ , such that the image of any line of  $\mathbf{PG}(n, q)$  under  $\theta$  is a plane oval in  $\mathbf{PG}(m, q)$ , and such that the image of  $\theta$  generates  $\mathbf{PG}(m, q)$ . Let  $W$  be the subspace of  $\mathbf{PG}(m, q)$  generated by the image under  $\theta$  of any hyperplane  $H$  of  $\mathbf{PG}(n, q)$ . Then we claim that  $W$  has codimension at most  $n + 1$  in  $\mathbf{PG}(m, q)$ . Indeed, we prove the claim by induction on  $n$ . For  $n = 1$ , this is obvious since in this case  $m = 2$  and the dimension of the image of a point is 0 (hence the codimension is 2). Now let  $n > 1$ . Let  $W'$  be the subspace of  $\mathbf{PG}(m, q)$  generated by the image of a hyperplane  $H'$  of  $\mathbf{PG}(n, q)$  with  $H' \neq H$  and let  $L$  be a line of  $\mathbf{PG}(n, q)$  not contained in  $H \cup H'$ . Set  $P = \langle L^\theta \rangle$ . Since  $q > 2$ , it is easy to see that every point  $x$  of  $\mathbf{PG}(n, q)$  is contained in a line  $M$  of  $\mathbf{PG}(n, q)$  meeting

$L \cup H \cup H'$  in three distinct points. Since the image under  $\theta$  of these points generate  $M^\theta$ , the point  $x^\theta$  is contained in  $\langle W, W', P \rangle$ . Hence  $\mathbf{PG}(m, q) = \langle W, W', P \rangle$ . If  $\dim W = w$ ,  $\dim W' = w'$  and  $\dim(W \cap W') = u$ , then this implies that  $m \leq w + w' - u + 1$ . Put  $U = W \cap W'$ . By the induction hypothesis  $w' - u \leq n$ , hence  $m - w \leq w' - u + 1 \leq n + 1$  and the claim follows.

A direct consequence of our claim is that, if  $n \geq 2$  and  $m \geq N_n = \frac{n(n+3)}{2}$ , then  $m = N_n$ . Indeed, considering a maximal chain of nested subspaces of  $\mathbf{PG}(n, q)$ , we deduce that  $m \leq 2 + 3 + 4 + \dots + n + (n + 1) = N_n$ . Moreover, in this case we clearly have that the dimension of the subspace of  $\mathbf{PG}(m, q)$  generated by the image of a  $k$ -dimensional subspace  $U$  of  $\mathbf{PG}(n, q)$  is equal to  $N_k = \frac{k(k+3)}{2}$ , for  $k \in \{0, 1, \dots, n\}$ . The proof of the previous claim also implies that, with the notation of the previous paragraph, if  $n \geq 2$ , then  $W$  and  $W'$  meet in a space of dimension  $N_{n-2}$ , and  $W, W'$  and  $P$  generate  $\mathbf{PG}(N_n, q)$ . Since every hyperplane  $H'' \notin \{H, H'\}$  of  $\mathbf{PG}(n, q)$  either contains a line meeting  $H \cup H'$  in just two points, or else meets every such line in a unique point outside  $H \cup H'$ , we deduce that the images of three distinct arbitrary hyperplanes under  $\theta$  generate  $\mathbf{PG}(N_n, q)$ . Hence we have shown that the set

$$\mathcal{S} := \{\langle H^\theta \rangle \mid H \text{ is a hyperplane of } \mathbf{PG}(n, q)\}$$

satisfies (VS1), (VS2) and (VS4). Clearly also (VS3) is satisfied. As for (VS5), it follows that, since every subspace of  $\mathbf{PG}(n, q)$  of codimension 2 is contained in exactly  $q + 1$  hyperplanes of  $\mathbf{PG}(n, q)$ , the cardinality of  $[S, S']$  is  $q + 1$ , for all  $S, S' \in \mathcal{S}$ . So, with the terminology of the previous section, the spectrum of every member of  $\mathcal{S}$  is equal to  $\{q + 1\}$ , and it follows from Theorem 1.1 that the image of  $\theta$  (which is precisely the set of points of  $\mathbf{PG}(N_n, q)$  contained in  $q^{n-1} + q^{n-2} + \dots + q + 1$  members of  $\mathcal{S}$ ) is projectively equivalent with the quadric Veronesean  $\mathcal{V}_n^{2^n}$ .

This proves Theorem 1.6. Remark that for  $q = 2$ , every cap of size  $2^{n+1} - 1$ ,  $n \geq 2$ , in some projective space  $\mathbf{PG}(m, 2)$ , with  $m \geq \frac{n(n+3)}{2}$  and where the cap generates  $\mathbf{PG}(m, q)$ , can be seen as the image of a mapping  $\theta$  satisfying the condition of Theorem 1.6, and hence the condition  $q > 2$  is necessary.

**Remark 3.1** The conclusion of Theorem 1.6 remains valid if, for  $n = 2$ , one considers *any* finite projective plane of order  $q$  instead of  $\mathbf{PG}(2, q)$ . This immediately follows from the above proof. So we obtain a characterization of the Desarguesian planes as the only ones admitting a representation spanning projective 5-space where the lines are ovals.

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