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Spreads and Ovoids Translation with Respect to Disjoint Flags

ALAN OFFER aoffer@cage.ugent.be
Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Gent, Belgium

HENDRIK VAN MALDEGHEM hvm@cage.ugent.be
Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Gent, Belgium

Abstract. It is shown that if a spread of a finite split Cayley hexagon is translation with respect to two disjoint flags then it is either a hermitian spread or a Ree–Tits spread. Analogously, if an ovoid of a classical generalized quadrangle $Q(4, q)$ is translation with respect to two disjoint flags then it is either an elliptic quadric or a Suzuki–Tits ovoid. In the course of obtaining these results, we introduce the notion of local polarity for ovoid-spread pairings and show that if an ovoid-spread pairing is locally polar at each of its elements then it arises from a polarity.

Keywords: translation spread, translation ovoid, generalized polygon, split Cayley hexagon, polarity

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1. Overview

Generalized polygons, introduced by Tits [21], are a generalization of projective planes which are then just the generalized triangles. Details can be found in Thas [20] and Van Maldeghem [24]. In this work, we have been concerned foremost with the finite classical generalized hexagon $H(q)$, which is known as the split Cayley hexagon. When considering a problem in $H(q)$, it is frequently natural to consider a similar problem in the finite classical generalized quadrangle $Q(4, q)$ as there are strong relationships between $H(q)$ and $Q(4, q)$. This is what we have done.

Spreads and ovoids are certain structures of lines and points, respectively, and they draw much attention for their connections with such things as m-systems, perfect codes, strongly regular graphs, and constructions of various geometries. Both $H(q)$ and $Q(4, q)$ have "classical" versions—the hermitian spread in $H(q)$ and the elliptic quadric in $Q(4, q)$. Also, the absolute elements of a polarity form a spread and an ovoid. These are the spreads and ovoids that appear in this paper.

Translation spreads and ovoids, as introduced by Bloemen et al. [2], are essentially those with a reasonable level of symmetry. At the weakest level, there is the notion of translation with respect to a flag. Little has been done with only this much symmetry being assumed; it doesn't seem to be enough for determining further properties. A stronger symmetry property is that of being translation with respect to an element, which corresponds to being translation with respect to every flag containing that element. Assuming this much symmetry has been more fruitful. Ovoids of $Q(4, q)$ that are translation with respect to a point are of particular interest for their correspondence to translation generalized quadrangles, semifield flocks and semifield planes (see, for example, Lunardon [12] and Thas [19]). The classical spreads and ovoids mentioned in the previous paragraph are translation with respect to every element; the spreads and ovoids of $H(q)$ and $Q(4, q)$ arising from polarities are translation with respect to exactly one flag on each element.

It should be noted that we are only interested in ovoids in the case of $Q(4, q)$ as this generalized quadrangle only admits spreads when it is self-dual anyway by Thas [17], and similarly, we are only interested in spreads of $H(q)$ as this generalized hexagon only admits translation ovoids (with respect to a flag) when it is self-dual by Offer [14], Theorem 2.

In the present work, we classify the spreads of $H(q)$ and the ovoids of $Q(4, q)$ that are translation with respect to two disjoint flags. Specifically, we prove the following theorem.

MAIN THEOREM. *Let $\mathscr{T}$ be either a spread of $H(q)$ or an ovoid of $Q(4, q)$. If $\mathscr{T}$ is translation with respect to two disjoint flags then either $\mathscr{T}$ is classical or $\mathscr{T}$ arises from a polarity.*

2. Background

This section contains the definitions, notations and other necessary background material for the sections that follow.

When in the context of a quadric $\mathscr{Q}$ in a projective space, the notation $\perp$ indicates conjugation with respect to that quadric and $X^\perp$ is called the perp of X. That is, if u is a point in the projective space and b is the bilinear form associated with $\mathscr{Q}$, then the perp $u^\perp$ of u is the set $\{v \mid b(u, v) = 0\}$ of points that are conjugate to u. For a set X of points, its perp $X^\perp = \bigcap_{u \in X} u^\perp$ is the intersection of the perps of its points. For details, the theory of quadrics and their associated forms can be found in Hirschfeld and Thas [8], Chapter 22.

2.1. Generalized Polygons

A generalized n-gon is a geometry Γ of points and lines (i) with at least three points on each line, (ii) with at least three lines through each point, and (iii) whose incidence graph has diameter n and girth $2n$. An ordinary n-gon in Γ, i.e., a cycle of length $2n$, is called an apartment. A generalized n-gon has order s if there are exactly $s + 1$ points on each line and $s + 1$ lines through each point.

The distance $\delta(u, v)$ between two elements u and v of Γ is the distance between them in the incidence graph. In particular, $\delta(u, v) \leq n$. When $\delta(u, v) = n$, the elements u and v are opposite. When $d = \delta(u, v) < n$, there is a unique path γ of length d from u to v. For $0 \leq i \leq d$, the distance-i projection $v \rhd_i u$ of v onto u is the unique element of γ at distance i from u. When $i = 1$, we also refer more briefly to the projection of v onto u and write $v \rhd u$. Notice that $v \rhd u$ appears as $\mathrm{proj}_u v$ elsewhere, but we use the present notation to facilitate referencing all elements of the path γ.

A flag is an incident point-line pair. Two flags $\{p, L\}$ and $\{q, M\}$ in a generalized $2m$-gon are opposite if the points p and q are opposite and the lines L and M are opposite.

The generalized polygons of main interest here are the split Cayley hexagon $H(q)$ and the generalized quadrangle $Q(4, q)$, both of which have order $q = p^h$, with p a prime. There is a rather strong relationship between these geometries in the form of an analogy that is dual in nature; that is, for each fact about $H(q)$ there is often an analogous fact about $Q(4, q)$ in which the roles of points and lines have been interchanged. It is for this reason that we consider these geometries together.

2.2. *The Split Cayley Hexagon $H(q)$*

The split Cayley hexagon $H(q)$ has a natural embedding in a nondegenerate quadric $\mathscr{P}_6$ in $\mathrm{PG}(6, q)$. Its points are all the points of $\mathscr{P}_6$ and its lines are a certain subset of the lines on $\mathscr{P}_6$. Here it will suffice to describe some of the geometric consequences of the choice of subset; details can be found in Thas [20], Tits [21] and Van Maldeghem [24].

The set of points collinear with a given point u of $H(q)$ is a generator π_u of $\mathscr{P}_6$, i.e., a plane that is a maximal subspace contained in $\mathscr{P}_6$. Two points of $H(q)$ are opposite if and only if they are not collinear in $\mathscr{P}_6$. Equivalently, the points not opposite a point u are precisely those in the quadric cone $u^\perp \cap \mathscr{P}_6$. A consequence of these facts is that the set A of points collinear with a point x and at distance 4 from a point y opposite x is completely determined by two of its points, for it is the line $\pi_x \cap y^\perp$ in the quadric $\mathscr{P}_6$. This property is called point distance-2 regularity in Van Maldeghem [24] and the set A is denoted x^y.

A line regulus in $H(q)$ is a set $\mathscr{R}$ of $q+1$ pairwise opposite lines of $H(q)$ for which there are two opposite points, say u and v, that lie at distance 3 from each of them; that is, such that $\delta(L, u) = \delta(L, v) = 3$ for every $L \in \mathscr{R}$. The name derives from the fact that the lines of $\mathscr{R}$ all lie in a common three-dimensional space Π that meets $\mathscr{P}_6$ in a hyperbolic quadric (see Bloemen et al. [2], Section 2.2), and as such they form a regulus in the usual sense. It follows that $\mathscr{R}$ is uniquely determined by any two of its lines since the space Π is uniquely determined by them.

Notice that not every regulus in $\mathscr{P}_6$ is a line regulus in $H(q)$. Indeed, any regulus containing a line of $\mathscr{P}_6$ that is not a line of $H(q)$ is not a line regulus. Thus, the line reguli form a proper subset of all reguli on $\mathscr{P}_6$.

2.3. *The Generalized Quadrangle* $Q(4,q)$

The generalized quadrangle $Q(4,q)$ is the geometry of points and lines on a nondegenerate quadric $\mathscr{P}_4$ in $\mathrm{PG}(4,q)$.

Two points of $Q(4,q)$ are opposite if and only if they are not collinear in $\mathscr{P}_4$. Equivalently, the points collinear with a point u are precisely those in the conic cone $u^\perp \cap \mathscr{P}_4$.

A trace in $Q(4,q)$ is a set $\mathscr{C}$ of $q+1$ pairwise opposite points of $Q(4,q)$ for which there are two opposite points, say u and v, that are collinear with each of them; that is, such that $\delta(w,u) = \delta(w,v) = 2$ for every $w \in \mathscr{C}$. The trace is then $\mathscr{C} = \langle u, v \rangle^\perp \cap \mathscr{P}_4$, which is a conic in $\mathscr{P}_4$. It follows that $\mathscr{C}$ is uniquely determined by any three of its points since the plane of the conic is uniquely determined by them. The two points u and v are unique for $\mathscr{C}$ when q is odd as $\mathscr{C}^\perp$ is a secant line of $\mathscr{P}_4$.

Notice that not every conic in $\mathscr{P}_4$ is a trace in $Q(4,q)$. Indeed, any conic whose plane is internal or non-nuclear, i.e., whose perp does not contain two points of $\mathscr{P}_4$ (see Hirschfeld and Thas [8], Section 22.9), is not a trace. Thus the traces form a proper subset of all conics on $\mathscr{P}_4$.

Traces in $Q(4,q)$ and line reguli in $H(q)$ should be thought of as analogous structures, and then by extension, conics in $\mathscr{P}_4$ and reguli in $\mathscr{P}_6$.

2.4. *Spreads and Ovoids*

Let Γ be a generalized $2m$-gon. A spread of Γ is a set $\mathscr{S}$ of pairwise opposite lines such that for each element u of Γ there is a line $L \in \mathscr{S}$ whose distance from u is at most m. Dually, an ovoid of Γ is a set $\mathscr{O}$ of pairwise opposite points such that for each element u of Γ there is a point $w \in \mathscr{O}$ whose distance from u is at most m. If Γ has order s then a spread or an ovoid contains exactly $s^m + 1$ elements.

An ovoid-spread pairing of Γ is a union $\mathscr{O} \cup \mathscr{S}$ of an ovoid $\mathscr{O}$ and a spread $\mathscr{S}$ with the property that every point of $\mathscr{O}$ is incident with a line of $\mathscr{S}$ and vice versa. One source of ovoids–spread pairings is polarities. A polarity of Γ is an involutory incidence preserving map that interchanges points and lines. Given a polarity ρ, an element u of Γ is absolute if it is incident with its image u^ρ. Since $(u^\rho)^\rho = u$, the element u^ρ is absolute precisely when u is absolute. Thus absolute elements occur in incident pairs $\{u, u^\rho\}$ and, by Van Maldeghem [24], Proposition 7.2.5, the set of all absolute elements is an ovoid-spread pairing.

2.5. *Spreads and Ovoids of* $H(q)$ *and* $Q(4,q)$

In this section, we describe the spreads and ovoids of $H(q)$ and $Q(4,q)$ that are essential for the rest of this paper.

2.5.1. *Spreads and Ovoids Arising from Polarities*

As mentioned in the previous section, polarities give rise to spreads and ovoids. The split Cayley hexagon $H(q)$ can admit polarities only when $q = 3^{2e+1}$ (for proof, see De Smet and Van Maldeghem [3], Section 4.3.1, Ott [15] or Van Maldeghem [24], Proposition 7.2.7), and such polarities are all conjugate, so the resulting ovoid-spread pairings are isomorphic. The spreads and ovoids arising this way are the Ree–Tits spreads and ovoids; the Ree groups act on them as automorphism groups and the existence of polarities when $q = 3^{2e+1}$ was shown by Tits [22], Section 5.

The Ree–Tits ovoid-spread pairings are the only known ovoid-spread pairings in $H(q)$ when $q > 3$. For the case $q = 3$, others have recently been discovered by De Wispelaere; these involve taking a Ree–Tits spread and one of the exceptional ovoids described in De Wispelaere et al. [4], or dually, a Ree–Tits ovoid and an exceptional spread.

The generalized quadrangle $Q(4, q)$ can admit polarities only when $q = 2^{2e+1}$ (see Tits [23], Théorème 3.6, together with Thas [16]), and the resulting ovoid-spread pairings are isomorphic. The spreads and ovoids arising this way are the Suzuki–Tits spreads and ovoids; the Suzuki groups act as automorphism groups and the existence of polarities when $q = 2^{2e+1}$ was shown by Tits [22], Section 4.2.

Any spread and ovoid of a generalized quadrangle together form an ovoid-spread pairing, but the Suzuki–Tits ovoid-spread pairings are special for the intimate relationship between the constituent spread and ovoid.

2.5.2. *Classical Spreads and Ovoids*

The "classical" spreads of $H(q)$ and ovoids of $Q(4, q)$ are related by their similar constructions.

Let Π_5 be a hyperplane in $\mathrm{PG}(6, q)$ that meets $\mathscr{P}_6$ in an elliptic quadric. Then, by Thas [18], the lines of $H(q)$ that are contained in Π_5 form a spread. This spread is commonly known as a hermitian spread, a name earned by its relationship with hermitian unitals.

Let Π_3 be a hyperplane in $\mathrm{PG}(4, q)$ that meets $\mathscr{P}_4$ in an elliptic quadric $\mathscr{E}_3$. Then the points of the elliptic quadric $\mathscr{E}_3$ form an ovoid of $Q(4, q)$.

2.5.3. *Locally Classical Spreads and Ovoids*

Let us think of how the classical spreads of $H(q)$ and the classical ovoids of $Q(4, q)$ appear from the viewpoint of one of their elements, i.e., locally.

Let $\mathscr{S}_H$ be a hermitian spread of $H(q)$ determined by a hyperplane Π_5 and let $L \in \mathscr{S}_H$ be one of its lines. Consider another line $K \in \mathscr{S}_H$. Since L and K are in Π_5, the three-dimensional space Π_3 that they generate is contained in Π_5. It follows that the line regulus $\mathscr{R}$ determined by L and K is contained in Π_5 and so all of its lines belong to the spread $\mathscr{S}_H$. Thus, $\mathscr{S}_H$ is a union of reguli on L, disjoint except for the common line L. A spread $\mathscr{S}$ of $H(q)$ that is a union of reguli meeting pairwise in a common base line L is said to be locally hermitian at L. This notion was introduced in Bloemen et al. [2].

Now let $\mathscr{E}_3$ be an elliptic quadric in $Q(4,q)$ determined by a hyperplane Π_3 and let $u \in \mathscr{E}_3$ be one of its points. Then $\mathscr{E}_3$ is a union of conics through u, disjoint except for the common point u; just take sections by planes in Π_3 on a common tangent line through u. Thus an ovoid $\mathscr{O}$ of $Q(4,q)$ that is a union of conics meeting pairwise in a common base point u can be thought of as being locally classical at u.

Having established the concept of being locally classical, we now state a rather strong "local to global" result.

LEMMA 2.1. *Let $\mathscr{T}$ be a spread of $H(q)$ or an ovoid of $Q(4,q)$. If $\mathscr{T}$ is locally classical at two of its elements then $\mathscr{T}$ is classical.*

Proof. In the case that $\mathscr{T}$ is a spread of $H(q)$, this is Bloemen et al. [2], Theorem 9. If $\mathscr{T}$ is an ovoid of $Q(4,q)$, this follows from Gevaert et al. [6], Theorem 3.1, and the Klein correspondence. ■

All locally classical ovoids of $Q(4,q)$ are known, the only non-classical ones being the Kantor–Knuth ovoids, so named for their discovery by Kantor [10] and their relationship with the semifields of Knuth [11]. That these are indeed the only ones follows from the Klein correspondence and Gevaert et al. [6], Theorem 2.2, which establish a link with flocks of quadratic cones, and then Thas [19], Section 1.5.6, which applies since the resulting flock planes contain the common point that is the perp of $Q(4,q)$ with respect to the Klein quadric. See also Bloemen et al. [2], Lemma 15.

In $H(q)$, other locally hermitian spreads are known apart from the hermitian spreads themselves, but classification does not seem to be forthcoming.

As a remark, if one considers similar structures in the quadric $\mathscr{P}_6$ with reguli that need not be line reguli, i.e., whose lines need not be lines of $H(q)$, then instead of locally hermitian spreads of $H(q)$, one is led to consider locally hermitian 1-systems of the quadric $\mathscr{P}_6$. This is done, for instance, in Luyckx and Thas [13].

2.6. Translation Spreads and Ovoids

In this section, we introduce translation spreads and ovoids of generalized $2m$-gons following the paper [2].

Let Γ be a generalized $2m$-gon. For a flag $\{u,v\}$ in Γ, let $G^{\{u,v\}}$ be the group of collineations that fix both u and v elementwise. By Van Maldeghem [24], Theorem 4.4.2(v), the group $G^{\{u,v\}}$ acts semi-regularly on the elements opposite u.

Let $\mathscr{S}$ be a spread of Γ and let L be a line of $\mathscr{S}$. We will write $\mathscr{S}^+ = \mathscr{S} \setminus \{L\}$. Let u be a point incident with L. For each line $M \neq L$ incident with u, let V_M be the set of lines of $\mathscr{S}^+$ at distance $2m-2$ from M. These sets V_M partition $\mathscr{S}^+$. The spread $\mathscr{S}$ is a translation spread with respect to the flag $\{L,u\}$ if there exists a subgroup $G_{\{L,u\}} \leq$

$G^{\{L,u\}}$ fixing the spread $\mathscr{S}$ and acting transitively on each of the sets V_M. The group $G_{\{L,u\}}$ is the group associated with $\mathscr{S}$ with respect to the flag $\{L, u\}$ and it is uniquely determined since $G^{\{L,u\}}$ is semi-regular on $\mathscr{S}^+$.

We say that $\mathscr{S}$ is a translation spread with respect to the line L if it is a translation spread with respect to the flag $\{L, u\}$ for every point u incident with L.

Translation ovoids with respect to flags and points are defined dually.

Suppose that Γ is either $H(q)$ or $Q(4, q)$. Then $G^{\{L,u\}}$ is generated by elations (see Van Maldeghem [24], Lemma 5.2.3(i)), which are collineations of Γ that fix the elements of some path $x_0, x_1, \ldots, x_{2m-2}$ elementwise. As a consequence, the associated groups $G_{\{L,u\}} \leq G^{\{L,u\}}$ and any groups generated by them correspond to groups of projective special linear transformations of the ambient space of the quadric $\mathscr{P}_6$ or $\mathscr{P}_4$ where Γ lives (see Van Maldeghem [24], Appendix D, for explicit forms of elations). Thus, we may casually move our viewpoint from that of collineations of Γ to that of projective transformations. This is especially pertinent in the proof of Theorem 3.2.

Now let's overview the translation properties of the spreads and ovoids of $H(q)$ and $Q(4, q)$ introduced in the previous sections. First of all, the classical spreads and ovoids are very symmetric, being translation with respect to every one of their elements. Next, the spreads and ovoids arising from polarities are translation with respect to exactly one flag on each element u, namely the flag $\{u, u^\rho\}$ where ρ is the associated polarity. Furthermore, the associated groups $G_{\{u,u^\rho\}}$ are the same for both the ovoid and the spread determined by ρ. Finally, there are the locally classical spreads and ovoids. The Kantor–Knuth ovoids are translation with respect to the base point. Since these are the only non-classical locally classical ovoids of $Q(4, q)$, it follows that if an ovoid $\mathcal{O}$ of $Q(4, q)$ is locally classical at a point u then it is translation with respect to u. We cannot make the analogous claim for spreads of $H(q)$, but rather, we have the converse result that if a spread $\mathscr{S}$ of $H(q)$ is translation with respect to a line L then it is locally hermitian at L; in fact, by Bloemen et al. [2], Theorem 6, it is sufficient that $\mathscr{S}$ be translation with respect to two flags on the common line L.

3. Non-Opposite Flags

To prove the Main Theorem, we divide it into the two cases: that the disjoint flags are opposite and that they are not. In this section, we address the latter case.

THEOREM 3.1. *If $\mathscr{S}$ is a spread of $H(q)$ that is translation with respect to two disjoint non-opposite flags then $\mathscr{S}$ is a hermitian spread.*

Proof. Let the two flags with respect to which $\mathscr{S}$ is translation be $\{x, X\}$ and $\{y, Y\}$, with $X, Y \in \mathscr{S}$. As these flags are not opposite, there is a point $k = x \rhd_2 y$ collinear with x and y. Let K be the line kx.

Let $g \in G_{\{x,X\}}$ be such that $w = k^g \neq k$. Then $\mathscr{S}$ is translation with respect to the flag $\{u, U\}$, where $u = y^g$ and $U = Y^g$. Let $h \in G_{\{u,U\}}$ be such that $t = w^h \neq w$.

Putting $v = x^h$ and $V = X^h$, the spread $\mathscr{S}$ is then also translation with respect to the flag $\{v, V\}$.

Let $v' = k \triangleright V$ be the unique point on V at distance 4 from k. As k, K, w, wt, t, tv, v is a path of length 6, the points k and v are opposite and so $v' \neq v$. Let $L = v' \triangleright k$. Notice that $L \neq K$ as this would result in there being a pentagon. Since $L \notin \mathscr{S}$ (for $\delta(L, X) = 4$), there is a unique line $Z \in \mathscr{S}$ concurrent with L. Let z be the point in which Z and L meet.

Since $z \triangleright x = y \triangleright x = K$, there is an element of $G_{\{x,X\}}$ that maps $\{y, Y\}$ onto $\{z, Z\}$ and so $\mathscr{S}$ is translation with respect to the flag $\{z, Z\}$. Since $x \triangleright z = v' \triangleright z = L$, there is an element of $G_{\{z,Z\}}$ that maps $\{x, X\}$ onto $\{v', V\}$.

Thus, $\mathscr{S}$ is translation with respect to two flags on a common line, namely $\{v, V\}$ and $\{v', V\}$, and so $\mathscr{S}$ is locally hermitian at V by Bloemen et al. [2], Theorem 6. Since $G_{\{x,X\}}$ does not leave V fixed, the spread $\mathscr{S}$ is locally hermitian at other lines as well and so, by Lemma 2.1, it is a hermitian spread. ■

THEOREM 3.2. *If $\mathcal{O}$ is an ovoid of $Q(4, q)$ that is translation with respect to two disjoint non-opposite flags then $\mathcal{O}$ is an elliptic quadric.*

Proof. There are some values of q that we can disregard as it is already known that $Q(4, q)$ only admits classical ovoids for those values of q. In particular, it has recently been shown in Ball et al. [1] that $Q(4, q)$ only admits classical ovoids when q is a prime. However, it will be sufficient to suppose here that $q \notin \{3, 7\}$, values for which the nonexistence of non-classical ovoids has already been ''known'' for some time.

Let the two flags with respect to which $\mathcal{O}$ is translation be $\{x_0, L_0\}$ and $\{x_1, L_1\}$, with $x_0, x_1 \in \mathcal{O}$. Since the flags $\{x_0, L_0\}$ and $\{x_1, L_1\}$ are not opposite, the lines L_0 and L_1 meet in a point y. Let G be the group generated by the associated groups $G_{\{x_0,L_0\}}$ and $G_{\{x_1,L_1\}}$. Then G leaves y fixed and acts 2-transitively on the set $S = \mathcal{O} \cap y^{\perp}$ of points of $\mathcal{O}$ collinear with y, or equivalently, 2-transitively on the set of lines incident with y.

Our objective is to show that S is a conic, for then the result will follow from Lemmas 6.1 and 2.1.

Let $U = y^{\perp}$ be the three-dimensional space that contains the conic cone $yS = y^{\perp} \cap \mathscr{P}_4$ and let Π be the projective plane that is the residue of y in U. As noted in Section 2.6, we may consider G as a group of projective transformations. Thus, in addition to fixing y, the group G also fixes U and it induces on the plane Π a subgroup $\bar{G}$ of $\mathrm{PGL}(3, q)$. Furthermore, the cone yS corresponds to a conic Σ in Π, and $\bar{G}$ stabilizes Σ while acting 2-transitively on it, so $\mathrm{PSL}(2, q) \leq \bar{G} \leq \mathrm{PGL}(2, q)$, where the former containment follows from Dembowski [5], 1.4.33, and the latter from Hughes and Piper [9], Theorem 2.37.

Let λ be a line in Π external to the conic Σ. In the extension $\Pi' = \mathrm{PG}(2, q^2)$ of Π, the extension λ' of λ meets the extension Σ' of Σ in two conjugate points α and $\bar{\alpha}$. The subgroup of $\mathrm{PGL}(2, q)$ (acting on Σ) that leaves α and $\bar{\alpha}$ fixed is a Singer group, which is a cyclic group acting regularly on Σ (this construction of Singer groups is discussed in Hughes and Piper [9], II.10, in the context of projective planes). Taking $\bar{\sigma}$ to be the square of a generator of this Singer group, we then have $\bar{\sigma} \in \bar{G}$ since $\mathrm{PSL}(2, q)$ has index 1 or 2 in $\mathrm{PGL}(2,q)$, according as q is even or odd. Let $\sigma \in G$ be an element that induces $\bar{\sigma} \in \bar{G}$. Then as the action of σ on S corresponds to the action of $\bar{\sigma}$ on Σ, the element σ has exactly $d = (q+1, 2)$ orbits on S, each of size $|\bar{\sigma}| = (q+1)/d$. Notice that $|\bar{\sigma}| > 2$ since $q \neq 3$.

Consider a point μ on the line λ in Π. Let $v = \lambda^{\perp}$, which is the pole of λ when q is odd and the nucleus of Σ when q is even. Notice that v is not on λ since λ is external to Σ. In Π', the extension of the line μv meets Σ' in points β and γ, where $\beta = \gamma$ when q is even, and since μ is on the external line λ, these points β and γ are not on λ' and so are distinct from α and $\bar{\alpha}$. As $\mathrm{PGL}(2,q^2)$ is sharply triply transitive on Σ' (see Hirschfeld [7], Corollaries 7.14–15), the fact that $\bar{\sigma}$ already fixes both α and $\bar{\alpha}$ on Σ' implies that $\bar{\sigma}$ fixes neither β nor γ, and since $|\bar{\sigma}| > 2$, nor does it interchange β and γ. Hence the line μv is not fixed. But v is fixed since λ is, and so μ is not. As μ was chosen arbitrarily, it follows that no point of λ is left fixed by $\bar{\sigma}$.

The fact that $\bar{\sigma}$ fixes the line λ in Π means that σ fixes a plane π in U through y. Consider the action of σ on this plane π. Since σ fixes a point y in π, it also fixes a line L in π by Hughes and Piper [9], Theorem 13.3. As no point of λ is fixed by $\bar{\sigma}$, no line of π through y is fixed by σ and so L is a line not through y.

Let Ω be the set of planes in U through L and let $\pi' \in \Omega$ be the plane through L containing the arbitrary point $x' \in S$. The group $\mathrm{PGL}(2,q)$ has a natural action on Ω that is sharply triply transitive (see Hirschfeld [7], Section 6.3), so its subgroup K stabilizing π has order $q(q-1)$. As σ is a projective collineation of U that fixes L and π, the permutation $\hat{\sigma}$ that it induces on Ω belongs to K. Thus, the size of the σ-orbit of π' divides $|\hat{\sigma}|$ which in turn divides $|K| = q(q-1)$. Also, as the action of σ on π' is determined by its action on x', the size of the σ-orbit of π' divides the size of the σ-orbit of x', which we have seen is $(q+1)/d$. Thus the orbit of π' under σ has order dividing both $q(q-1)$ and $(q+1)/d$, and which is therefore either 1 or 2.

If q is even then $q(q-1)$ and $q+1$ are coprime, so σ fixes π'. In addition, σ is transitive on S, so the points of S all lie in a common plane and S is a conic. This completes the proof of the theorem in this case. From here on, suppose that q is odd, in which case $|\sigma| = (q+1)/2$.

Each of the two σ-orbits in S lies in 1 or 2 planes through L, so in total, the points of S lie in 1, 2, 3 or 4 planes through L, with each plane containing $q+1$, $(q+1)/2$ or $(q+1)/4$ points of S. Let π_0 be the plane through L containing x_0 and let $y' \neq y$ be the unique second point of $Q(4,q)$ that is collinear with all the points of $\pi_0 \cap \mathscr{P}_4$. Put $S_0 = S \cap \pi_0$, $S_0^+ = S_0 \backslash \{x_0\}$ and $L_0' = x_0 y'$. Let H be the stabilizer of π_0 in $G_{\{x_0, L_0\}}$. We will show that H is regular on S_0^+ and so $|S_0| = |H| + 1$.

Let $x, x' \in S_0^+$ and let g be the unique element of $G_{\{x_0, L_0\}}$ that maps x to x'. The group U' fixing L_0 and L_0' pointwise and fixing x_0 linewise acts transitively on $(\pi_0 \cap \mathscr{P}_4) \backslash \{x_0\}$ (this is the Moufang property; see Van Maldeghem [24], Section 4.4.4), so there is a $g' \in U'$ that maps x to x'. Notice that g' fixes π_0 since it fixes y and y'. Both U' and $G_{\{x_0, L_0\}}$ are contained in the group $G^{\{x_0, L_0\}}$ which is semi-regular on the points opposite x_0, hence $g = g'$. It now follows that g fixes π_0, so $g \in H$ and H acts regularly on S_0^+.

As $|G_{\{x_0, L_0\}}| = q = p^h$, where p is a prime, we have $|S_0^+| = |H| = p^k$ for some integer k. Hence the number of points of S in π_0 is $|S_0| = p^k + 1$. But we have already seen that $|S_0| = q+1, (q+1)/2$ or $(q+1)/4$. If $|S_0| = (q+1)/2$ then we find that $q = 3$, and if $|S_0| = (q+1)/4$ then we are led to $q = 7$, both of which are cases that we eliminated from the outset. Thus, we are left with $|S_0| = q+1$, so the points of S all lie in a common plane and S is a conic, as required. ∎

4. Local Polarity

Before proceeding to the case of opposite flags in the Main Theorem, we discuss ovoid-spread pairings whose local structure is like that of ovoid-spread pairings that arise from polarities. In particular, we prove a "local to global" theorem: if an ovoid-spread pairing has this local structure everywhere then it does in fact arise from a polarity. This result is used in the next section to complete the proof of the Main Theorem.

To introduce these concepts in generality, let Γ be any generalized $2m$-gon, not necessarily finite. Let $\mathcal{O} \cup \mathcal{S}$ be an ovoid-spread pairing in Γ. The associated function of $\mathcal{O} \cup \mathcal{S}$ is the involutory function ρ that maps each $x \in \mathcal{O} \cup \mathcal{S}$ to the unique element $x^\rho \in \mathcal{O} \cup \mathcal{S}$ incident with x.

Let's think about how an ovoid-spread pairing looks locally when it arises from a polarity. Suppose that the ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ arises from a polarity, which we may unambiguously denote by ρ as the associated function of $\mathcal{O} \cup \mathcal{S}$ is just the restriction of the polarity. Let $x, y, z \in \mathcal{O} \cup \mathcal{S}$ be three elements of the same type. Suppose that for some $i < 2m$, the elements y^ρ and z^ρ have the same distance-i projection onto x, i.e., $y^\rho \rhd_i x = z^\rho \rhd_i x$. Then applying the polarity ρ, the elements y and z have the same distance-i projection onto x^ρ.

Motivated by this, a general ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at x if

$$y^\rho \rhd_i x = z^\rho \rhd_i x \Rightarrow y \rhd_i x^\rho = z \rhd_i x^\rho,$$

for all $y, z \in \mathcal{O} \cup \mathcal{S}$ of the same type as x. The ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ is locally polar at x if it is distance-i polar at x for all $i < 2m$. In fact, it is sufficient to consider $1 \leq i < m$ as an ovoid-spread pairing is automatically distance-i polar everywhere for $i = 0$ and for $i \geq m$. We now give the promised "local to global" theorem.

THEOREM 4.1. *If an ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ of a generalized $2m$-gon Γ is locally polar at every element then it arises from a polarity.*

Proof. The proof of this involves extending the associated function ρ to the rest of Γ in such a way that the result is a polarity.

Let w be an arbitrary element of Γ and let x be an element of $\mathcal{O} \cup \mathcal{S}$ nearest to w, i.e., with $d = \delta(w, x)$ minimal. Then $d < m$. If $x' \in \mathcal{O} \cup \mathcal{S}$ is such that $\delta(w, x') = d$ then $\delta(x, x') < 2m$ from which we conclude that $x' = x$. Thus x is uniquely determined. Let y be another element of $\mathcal{O} \cup \mathcal{S}$ of the same type as x such that $y^\rho \rhd_d x = w$. Such a y certainly exists as we may start by choosing an arbitrary element y' of Γ at distance m from x and at distance $m - d$ from w, and then take y to be such that y^ρ is within distance m of y', as guaranteed to exist by the definition of ovoids and spreads. Now define w^ρ to be $y \rhd_d x^\rho$, as this is exactly what it should be if ρ is to be a polarity.

We must now check that ρ is well-defined, that it has order two and that it preserves incidence.

Let $z \in \mathcal{O} \cup \mathcal{S}$ be another element of the same type as x such that $z^\rho \rhd_d x = w$. Then the local polarity of $\mathcal{O} \cup \mathcal{S}$ at x implies that $z \rhd_d x^\rho = y \rhd_d x^\rho = w^\rho$. Thus w^ρ is independent of the choice of y and so ρ is well-defined.

Next we confirm that $(w^\rho)^\rho = w$. Since $y \rhd_d x^\rho = w^\rho$, the definition of ρ immediately gives $(w^\rho)^\rho = y^\rho \rhd_d x = w$ so long as x^ρ is the unique element of $\mathcal{O} \cup \mathcal{S}$ nearest to w^ρ. Certainly $\delta(w^\rho, x^\rho) = d < m$ and this implies that the nearest element of $\mathcal{O} \cup \mathcal{S}$ to w^ρ is either x^ρ or x. But $w^\rho \rhd x^\rho = y \rhd x^\rho$ and this is not equal to x since x and y are opposite. Thus x^ρ is the nearest element of $\mathcal{O} \cup \mathcal{S}$ to w^ρ and therefore $(w^\rho)^\rho = w$, as required.

Finally, we verify that ρ preserves incidence. Let v be an element incident with w. Suppose that the nearest element of $\mathcal{O} \cup \mathcal{S}$ to v is also x. Swapping v and w if necessary, we may suppose that $\delta(v, x) = d - 1$. Then $v = y^\rho \rhd_{d-1} x$ and so $v^\rho = y \rhd_{d-1} x^\rho$ which certainly is incident with $w^\rho = y \rhd_d x^\rho$.

Suppose then that nearest element z of $\mathcal{O} \cup \mathcal{S}$ to v is distinct from x. Then $w = z \rhd_{m-1} x$ and $v = x \rhd_{m-1} z = z \rhd_m x$, and from here $v^\rho = x^\rho \rhd_{m-1} z^\rho = z^\rho \rhd_m x^\rho$, which is incident with $w^\rho = z^\rho \rhd_{m-1} x^\rho$. ■

If the generalized $2m$-gon Γ has an order s then it is sufficient in Theorem 4.1 that the ovoid-spread pairing be locally polar at every element of just one type. This follows from the next theorem.

THEOREM 4.2. *Let $\mathcal{O} \cup \mathcal{S}$ be an ovoid-spread pairing of a generalized $2m$-gon Γ of order s and let ρ be its associated function. If $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at an element x then it is also distance-i polar at x^ρ.*

Proof. Without loss of generality, suppose that x is a point. Also, as remarked earlier, we may suppose that $1 \leq i < m$.

Let y be any other point of $\mathcal{O}$ and put $z = y^\rho \rhd_i x$. Let $A = \{w \mid \delta(w, x) = m$ and $\delta(w, z) = m - i\}$. Starting at z, there are s choices for an incident element that is one step further away from x, and then from there, there are another s choices, and so on. Hence $|A| = s^{m-i}$.

Let $B = \{L \in \mathcal{S} \mid L \rhd_i x = z\}$ be the set of lines in the spread $\mathcal{S}$ that have the same distance-i projection onto x as y^ρ. For each line $L \in B$, the element $L \rhd_m x$ is in A, and conversely, for each element $w \in A$ there is a unique line $L \in \mathcal{S}$ which then belongs to B. Thus $|B| = |A| = s^{m-i}$.

Similar to B, define $C = \{u \in \mathcal{O} \mid u \rhd_i x^\rho = y \rhd_i x^\rho\}$. Then $|C| = s^{m-i}$ by repeating essentially the same argument. As $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at x, for each line $L \in B$ we have $L^\rho \in C$. Thus $B^\rho \subseteq C$. But $|B^\rho| = |B| = |C|$ and so $B^\rho = C$, or equivalently, $B = C^\rho$. As y was arbitrary, it follows that $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at x^ρ. ■

5. Opposite Flags

In this section, we prove the case of opposite flags in the Main Theorem. First we address the matter for $H(q)$.

THEOREM 5.1. *Let Γ be a generalized hexagon of order s and let $\mathcal{S}$ be a spread of Γ that is translation with respect to two opposite flags. Then either (i) $\mathcal{S}$ is translation with respect to two non-opposite flags, or (ii) $\mathcal{S}$ is translation with respect to exactly one flag on each line. In the latter case, the union of these flags is an ovoid-spread pairing that is distance-1 polar everywhere.*

Proof. Let the two opposite flags with respect to which $\mathcal{S}$ is translation be $\{X, x\}$ and $\{Y, y\}$, with $X, Y \in \mathcal{S}$.

Let the s^2 flags in the orbit of $\{Y, y\}$ under $G_{\{X,x\}}$ be $\{Y_i, y_i\}, i = 1, 2, \ldots, s^2$, with the $Y_i \in \mathcal{S}$ and $Y_1 = Y$. The spread $\mathcal{S}$ is then translation with respect to each of the flags $\{Y_i, y_i\}$. Also, the points $y_i \rhd_3 X$ all belong to the set A of points collinear with $t = y \rhd X$ but not on X.

If two distinct points, y_i and y_j, have the same distance-3 projection onto X, then the spread $\mathcal{S}$ is translation with respect to the two non-opposite flags $\{Y_i, y_i\}$ and $\{Y_j, y_j\}$ and so we arrive at the conclusion (i) of the theorem.

Suppose then that the s^2 points $y_i \rhd_3 X$ are all distinct. As $|A| = s^2$, each point of A is then equal to $y_i \rhd_3 X$ for some unique choice of i. Let Z be a line of $\mathcal{S}$ that is not equal to any of the Y_i. For some i, the point $y_i \rhd_3 X$ is equal to the point $Z \rhd_2 t$ and so $Z = X^h$ for some $h \in G_{\{Y_i, y_i\}}$. Also, the orbit of Y_2 under $G_{\{Y,y\}}$ has order s^2 and does not contain the line Y, thus it includes some line Z that is not equal to any of the Y_i.

Hence $G = \langle G_{\{X,x\}}, G_{\{Y,y\}} \rangle$ is transitive on $\mathcal{S}$. Consequently, $\mathcal{S}$ is translation with respect to at least one flag on each of its lines.

Let $\mathcal{O}$ be the set of points w that are incident with a line $W \in \mathcal{S}$ such that $\mathcal{S}$ is translation with respect to the flag $\{W, w\}$.

If $\mathcal{S}$ is translation with respect to more than one flag on any of its lines then it is translation with respect to at least two flags on each line. The number of points of $\mathcal{O}$ not incident with X is then at least $2(|\mathcal{S}| - 1) = 2s^3$. As the number of points at distance 3 from X is only $(s+1)s^2 < 2s^3$, it follows that there are two points $u, v \in \mathcal{O}$ for which $u \rhd_3 X = v \rhd_3 X$. Letting U and V be the lines of $\mathcal{S}$ incident with u and v, respectively, the spread $\mathcal{S}$ is then translation with respect to the two non-opposite flags $\{U, u\}$ and $\{V, v\}$. Thus, we arrive at conclusion (i) of the theorem.

Suppose then that $\mathcal{S}$ is translation with respect to exactly one flag on each of its lines. If any two of these flags are not opposite then we arrive again at conclusion (i). Thus we suppose that these flags are pairwise opposite. As $|\mathcal{O}| = |\mathcal{S}|$ is the size of an ovoid and the points of $\mathcal{O}$ are pairwise opposite, the set $\mathcal{O}$ is an ovoid and $\mathcal{O} \cup \mathcal{S}$ is an ovoid-spread pairing. All that remains is to show that $\mathcal{O} \cup \mathcal{S}$ is distance-1 polar everywhere.

Let $U, V \in \mathcal{S}$ be such that $U \rhd x = V \rhd x$. Then there is a collineation $g \in G_{\{X,x\}}$ such that $V = U^g$. Let u and v be the unique points of $\mathcal{O}$ that are on U and V, respectively. Then also $v = u^g$. As g leaves X fixed pointwise, we then have $u \rhd X = (u \rhd X)^g = v \rhd X$ and so $\mathcal{O} \cup \mathcal{S}$ is distance-1 polar at x. By the transitivity of G, the ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ is distance-1 polar at every point, and finally, by Theorem 4.2, it is distance-1 polar at every line as well. ■

THEOREM 5.2. *Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to two opposite flags. Then $\mathcal{S}$ is either a hermitian spread or a Ree–Tits spread.*

Proof. By Theorem 5.1, either (i) $\mathcal{S}$ is translation with respect to two non-opposite flags, in which case $\mathcal{S}$ is a hermitian spread by Theorem 3.1, or (ii) $\mathcal{S}$ is translation with respect to exactly one flag on each line and these flags form an ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ that is distance-1 polar everywhere. We have only to show that in the latter situation, the spread $\mathcal{S}$ is a Ree–Tits spread. By Theorem 4.1, this will be achieved once we have shown that $\mathcal{O} \cup \mathcal{S}$ is distance-2 polar everywhere.

Suppose then that we are in the case (ii). Let $x, y, z \in \mathcal{O}$ be three points such that $y^\rho \rhd_2 x = z^\rho \rhd_2 x = u$. We must show that $y \rhd_2 x^\rho = z \rhd_2 x^\rho$.

Let v be a point collinear with y and at distance 4 from both x and z. Such a point v certainly exists, for y^x is a line in $\mathcal{P}_6$ and $z^\perp$ is a hyperplane in $\mathrm{PG}(6, q)$, so $y^x \cap z^\perp$ is nonempty and v can be taken to be any point from this set. Let w be the point collinear with both x and v. Notice that $w \in x^y$.

Since $\mathcal{O} \cup \mathcal{S}$ is distance-1 polar at x, the points y and z project onto the same point $y \rhd x^\rho = z \rhd x^\rho$ on x^ρ, which is then a point of $x^y \cap x^z$. Notice that also $u \in x^y \cap x^z$. It now follows from the point distance-2 regularity of $H(q)$ that $x^y = x^z$. Therefore, w also belongs to x^z and so, in particular, $\delta(z, w) = 4$.

Let L and K be the lines vw and xw, respectively. Then $\delta(y, L) = 3$ and, since $\delta(z, v) = \delta(z, w) = 4$, also $\delta(z, L) = 3$. Let g be the unique collineation in $G_{\{x^\rho, x\}}$ that maps y^ρ to z^ρ. This collineation g fixes K since it fixes the point x linewise, and it also fixes L since $L^g = (y \rhd_2 K)^g = y^g \rhd_2 K = z \rhd_2 K = L$. It follows that g fixes the unique line M in $\mathcal{S}$ that is concurrent with L. But $G_{\{x^\rho, x\}}$ is semi-regular on $\mathcal{S} \setminus \{x^\rho\}$ and so $M = x^\rho = K$. Thus $y \rhd_2 x^\rho = z \rhd_2 x^\rho = L$, as required. ■

Theorems 3.1 and 5.2 together give the Main Theorem in the case of $H(q)$. Now we complete the proof of the Main Theorem by addressing the opposite flags case in $Q(4, q)$.

THEOREM 5.3. *Let Γ be a generalized quadrangle of order s and let $\mathcal{O}$ be an ovoid of Γ that is translation with respect to two opposite flags. Then either (i) $\mathcal{O}$ is translation with respect to two non-opposite flags, or (ii) $\mathcal{O}$ is translation with respect to exactly one flag on each point. In the latter case, the union of these flags is an ovoid-spread pairing arising from a polarity.*

Proof. Let the two opposite flags with respect to which $\mathcal{O}$ is translation be $\{x, X\}$ and $\{y, Y\}$, with $x, y \in \mathcal{O}$.

Let the s flags in the orbit of $\{y, Y\}$ under $G_{\{x,X\}}$ be $\{y_i, Y_i\}, i = 1, 2, \ldots, s$, with the $y_i \in \mathcal{O}$ and $y_1 = y$. The ovoid $\mathcal{O}$ is then translation with respect to each of the flags

$\{y_i, Y_i\}$. Also, the lines Y_i are all concurrent with the line $T = Y \rhd x$ as this line is fixed by $G_{\{x,X\}}$.

If two distinct lines, Y_i and Y_j, meet T in a common point, then the ovoid $\mathcal{O}$ is translation with respect to the two non-opposite flags $\{y_i, Y_i\}$ and $\{y_j, Y_j\}$ and so we arrive at the conclusion (i) of the theorem.

Suppose then that the lines Y_i all meet T in distinct points. The points on T distinct from x are then precisely the s points y_i. Let z be a point of $\mathcal{O}$ that is not equal to any of the y_i. For some i, the line Y_i is incident with the point $z \rhd T$ and so $z = x^h$ for some $h \in G_{\{y_i, Y_i\}}$. Also, the orbit of y_2 under $G_{\{y,Y\}}$ has order s and does not contain the point y, thus it includes some point z that is not equal to any of the y_i.

Hence $G = \langle G_{\{x,X\}}, G_{\{y,Y\}} \rangle$ is transitive on $\mathcal{O}$. Consequently, $\mathcal{O}$ is translation with respect to at least one flag on each of its points.

Let $\mathcal{S}$ be the set of lines W for which $\mathcal{O}$ is translation with respect to the flag $\{w, W\}$, where w is the unique point of $\mathcal{O}$ incident with W.

If $\mathcal{O}$ is translation with respect to more than one flag on any of its points then it is translation with respect to at least two flags on each point. The number of lines of $\mathcal{S}$ not incident with x is then at least $2(|\mathcal{O}| - 1) = 2s^2$. As the number of points at distance 2 from x is only $s(s+1) < 2s^2$, it follows that there are two lines $U, V \in \mathcal{S}$ not on x for which $x \rhd U = x \rhd V$. Letting u and v be the points of $\mathcal{O}$ incident with U and V, respectively, the ovoid $\mathcal{O}$ is then translation with respect to the two non-opposite flags $\{u, U\}$ and $\{v, V\}$. Thus we arrive at conclusion (i) of the theorem.

Suppose then that $\mathcal{O}$ is translation with respect to exactly one flag on each of its points. If any two of these flags are not opposite then we arrive again at conclusion (i). Thus we suppose that these flags are pairwise opposite. As $|\mathcal{S}| = |\mathcal{O}|$ is the size of a spread and the lines of $\mathcal{S}$ are pairwise opposite, the set $\mathcal{S}$ is indeed a spread and $\mathcal{O} \cup \mathcal{S}$ is an ovoid-spread pairing. All that remains is to show that $\mathcal{O} \cup \mathcal{S}$ arises from a polarity.

Let $u, v \in \mathcal{O}$ be such that $u \rhd X = v \rhd X$. Then there is a collineation $g \in G_{\{x,X\}}$ such that $v = u^g$. Let U and V be the unique lines of $\mathcal{S}$ that are incident with u and v, respectively. Then also $V = U^g$. As g leaves x fixed linewise, we then have $U \rhd x = (U \rhd x)^g = V \rhd x$ and so $\mathcal{O} \cup \mathcal{S}$ is locally polar at X. By the transitivity of G, the ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ is locally polar at every line, and by Theorem 4.2, it is then locally polar at every point as well. Finally, by Theorem 4.1, $\mathcal{O} \cup \mathcal{S}$ arises from a polarity. ■

THEOREM 5.4. *Let $\mathcal{O}$ be an ovoid of $Q(4, q)$ that is translation with respect to two opposite flags. Then $\mathcal{O}$ is either an elliptic quadric or a Suzuki–Tits ovoid.*

Proof. By Theorem 5.3, either (i) $\mathcal{O}$ is translation with respect to two non-opposite flags, in which case it is an elliptic quadric by Theorem 3.2, or (ii) $\mathcal{O}$ is translation with respect to exactly one flag on each point and these flags form an ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ that arises from a polarity, which in $Q(4, q)$ means that $\mathcal{O}$ is a Suzuki–Tits ovoid. ■

Theorems 3.2 and 5.4 give the Main Theorem in the case of $Q(4,q)$. Together with Theorems 3.1 and 5.2 for the case of $H(q)$, this completes the proof of the Main Theorem.

6. Additional Results

In this section, we record a lemma that is used in the proof of Theorem 3.2. In Bloemen et al. [2], Corollary 16, a similar result was proved for an ovoid $\mathcal{O}$ that is translation with respect to a point; this lemma strengthens that result by only requiring that $\mathcal{O}$ be translation with respect to a flag. Also, in keeping with the theme of emphasizing the analogy between $H(q)$ and $Q(4,q)$, we prove an analogous result for $H(q)$, although it is not used in this paper.

LEMMA 6.1. *Let $\mathcal{O}$ be an ovoid of $Q(4,q)$ that is translation with respect to a flag $\{x, L\}$. If $\mathcal{O}$ contains a conic through x then $\mathcal{O}$ is either an elliptic quadric or a Kantor–Knuth ovoid with x as base point.*

Proof. Let C be a conic through x contained in $\mathcal{O}$ and put $C^+ = C\setminus\{x\}$ and $\mathcal{O}^+ = \mathcal{O}\setminus\{x\}$. We aim to show that $\mathcal{O}$ is a union of q conics through x; the result then follows from Bloemen et al. [2], Lemma 15 (see Section 2.5.3 of this paper).

Suppose that every point $p \neq x$ on L is collinear with a point of C^+. Then the images of C^+ under $G_{\{x,L\}}$ partition $\mathcal{O}^+$, so the ovoid $\mathcal{O}$ is a union of q conics through x, as required.

Suppose then that not every point $p \neq x$ on L is collinear with a point of C^+. Then there is a point p_1 on L that is collinear with two points $y, z \in C^+$. As p_1 is also collinear with the point $x \in C$, we have $C \subset \langle x, y, z\rangle \subset p_1^{\perp}$ and so p_1 is collinear with every point of C.

The generalized quadrangle $Q(4,q)$ is a translation generalized quadrangle with base point x (see Hirschfeld and Thas [8], Section 26.4). The corresponding translation group G is an abelian group that acts regularly on the set of points opposite x. In fact, the group G is the set of all collineations that fix x linewise and that act semiregularly on the set of points opposite x (see Thas [20], Section 9.1, Theorem 3), and consequently, it contains the associated group $G_{\{x,L\}}$.

Let $w \in \mathcal{O}^+$ be arbitrary and let h be the unique element of G that maps y to w. Since h commutes with $G_{\{x,L\}}$, the orbit of w under $G_{\{x,L\}}$ is the image of C^+ under h, and this together with x is a conic since C is a conic and h corresponds to a projective collineation of PG$(4,q)$ (see comments in Section 2.6). Thus the partition of $\mathcal{O}^+$ into $G_{\{x,L\}}$-orbits is a representation of $\mathcal{O}$ as a union of q conics through x, as required. ∎

LEMMA 6.2. *Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to a flag $\{L, x\}$. If $\mathcal{S}$ includes a regulus that contains L then $\mathcal{S}$ is locally hermitian at L.*

Proof. Let $\mathcal{R}$ be a regulus in $\mathcal{S}$ containing L. We must show that for each $K \neq L$ in $\mathcal{S}$, the unique line regulus determined by L and K is contained in $\mathcal{S}$.

Let $M \neq L$ be a line of $\mathscr{R}$, let $y = M \rhd_2 x$, let $x' \neq x$ be a point on L and let $y' = M \rhd_2 x'$. Let H be the subgroup of $G_{\{L,x\}}$ that fixes y. Then H has order q, acts regularly on $\mathscr{R} \backslash \{L\}$ and fixes y'. Combining Weiss [25], Lemma 2, and Van Maldeghem [24], Proposition 4.4.3(i), the group $G^{\{L,x\}}$ induces a semiregular permutation action on the set of points $x'' \neq x$ on xy. As $H \leq G^{\{L,x\}}$ fixes the point y, it follows that H fixes the line xy pointwise. Similarly, as H fixes the point y', it fixes the point x' linewise and the line $x'y'$ pointwise. Hence H is a group of elations for the path xy, x, L, x', $x'y'$. In $H(q)$, all such elations are axial elations and this means that H fixes all elements within distance 3 of L (see Van Maldeghem [24], Appendix D, for explicit forms of elations).

Let $K \neq L$ be a line of $\mathscr{S}$. Let $z = K \rhd_2 x$ and $z' = K \rhd_2 x'$, both of which are fixed by H as they are at distance 3 from L. The q lines of the H-orbit of K are then all at distance 3 from z and z', and therefore, together with L they form the line regulus determined by L and K. As H stabilizes the spread $\mathscr{S}$, this line regulus is contained in $\mathscr{S}$ and this is what was required to be proved. ■

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