Hermitian Veroneseans Over Finite Fields

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Abstract

The variety of $(n+1) \times (n+1)$ rank one Hermitian matrices over a finite field $\mathbb{F}_q^2$, which is naturally in one-to-one correspondence with the points of a projective space $\mathbf{PG}(n, q^2)$ and which gives rise to a cap in the projective space $\mathbf{PG}(n^2+2n, q)$ on which the group $\mathbf{PGL}(n+1, q^2)$ acts 2-transitively is studied. Our main result is a geometric characterization of this cap and some of its projections along the lines of the characterization of quadric Veroneseans by Mazzocca and Melone.

Key words: Hermitian matrix, projective space, elliptic quadric, cap, Veronesean.

1 Introduction

Set $k = \mathbb{F}_q$ and $K = \mathbb{F}_{q^2}$. Let γ be the generator of the Galois group of K/k so that for $a \in K, a^\gamma = a^q$. For convenience we will often denote the image of a under γ by $\bar{a}$. Recall that an $(n+1) \times (n+1)$ matrix m is *Hermitian* if $m^T = \bar{m}$ where T denotes the transpose map and by $\bar{m}$ we mean the result of applying γ to each of the elements of m. We shall denote the space of all $(n+1) \times (n+1)$ Hermitian matrices over K by $H(n+1, q^2)$. This is a $k-$linear space of dimension $(n+1)^2$. The group $G = \mathbf{GL}(n+1, q^2)$ acts on $H(n+1, q^2)$ with the action given by

$$g \circ m = gm\bar{g}^T.$$

For $1 \leq i \leq n+1$ let $H_i(n+1, q^2)$ be the collection of matrices in $H(n+1, q^2)$ with rank i and $PH_i(n+1, q^2)$, or simply PH_i, the set of 1-spaces spanned by the matrices in $H_i(n+1, q^2)$. Then each PH_i is an orbit for G under the induced action on $\mathbf{PG}(H(n+1, q^2))$ considered as a projective space of dimension $n^2 + 2n$ over k.

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In [3], Cossidente & Siciliano prove that PH_1 is isomorphic to the variety $\mathcal{V}_{n+1,2}$ introduced by Lunardon [7]. Hence PH_1 is canonically in one-to-one correspondence with the projective space $\mathbf{PG}(n, K)$ by the second part of Theorem 1 in [7]. For later reference, we need to make this more explicit. Let $V = K^{n+1}$ consist of column vectors. For $\langle v \rangle \in \mathbf{PG}(V)$ set $\pi(\langle v \rangle) = \langle v\bar{v}^T \rangle$. Clearly $v\bar{v}^T$ is a rank one matrix in $H(n+1, K)$ and so $\pi(\langle v \rangle)$ is in PH_1. The linear group G preserves this action:

$$\pi(\langle gv \rangle) = \langle (gv)(\bar{g}\bar{v})^T \rangle = \langle g(v\bar{v}^T)\bar{g}^T \rangle = g \circ \pi(\langle v \rangle).$$

Since G is transitive on PH_1 it follows that $\pi(\mathbf{PG}(V)) = PH_1$. In fact, π is one-to-one from $\mathbf{PG}(V)$ onto PH_1.

Next note that PH_1 is a *cap* in $PH = \mathbf{PG}(H(n+1, q^2))$, that is, no three points are collinear. We refer to the first part of Theorem 1 in [7].

We now present an alternative explicit construction of PH_1 in $\mathbf{PG}(n^2+2n, k)$. In fact, this amounts to choosing an explicit k-base in $H(n+1, q^2)$, and then applying the previous construction, in particular, the map π. We skip the computational details and give the result. Let $r \in K \setminus k$ be arbitrary. Then the map π above can be given as (where $x_i \in K$, for all $i \in \{0, 1, \ldots, n\}$)

$$\pi(\langle (x_0, x_1, \ldots, x_n) \rangle) = \langle (y_{i,j})_{0 \le i,j \le n} \rangle,$$

with $y_{i,i} = x_i\bar{x}_i$, $y_{i,j} = x_i\bar{x}_j + \bar{x}_i x_j$ (for $i < j$), and $y_{i,j} = r x_i\bar{x}_j + \bar{r}\bar{x}_i x_j$ (for $i > j$). From this representation, it is clear that the inverse image with respect to π of the intersection of PH_1 with a hyperplane of $\mathbf{PG}(n^2+2n, k)$ is a (not necessarily nonsingular) Hermitian variety, and conversely every Hermitian variety of $\mathbf{PG}(V)$ arises in this way. It follows that PH_1 is not contained in a hyperplane of $\mathbf{PG}(n^2+2n, k)$. We refer to this representation as the *r-representation*, $r \in K \setminus k$.

For $n = 2$, a slightly modified version of this representation can be found in [7].

We point out that the lines of $\mathbf{PG}(V)$ have a natural interpretation in terms of the geometry of PH : the span in $H(n+1, q^2)$ of the image $\pi(L)$, L a line of $\mathbf{PG}(V)$, is a 4-dimensional space and we shall denote by $\xi(L)$ the subspace $\langle \pi(L) \rangle$ of PH. Since $\pi(L)$ is a cap of size q^2+1 in the 3-dimensional projective space $\xi(L)$ it is an ovoid for $q > 2$ and it is easy to see that it is always an elliptic quadric, and that $\xi(L) \cap PH_1 = \pi(L)$, see Corollary 2 of [7].

Thus, the lines of $\mathbf{PG}(V)$ can be interpreted as certain 3-dimensional projective subspaces of PH in which the points of PH_1 form an elliptic quadric. We will denote by Ξ the collection of all such subspaces. Further, for a point $p \in PH_1$ and $\xi \in \Xi$ with $p \in$

ξ we shall denote by $T_p(\xi)$ the tangent plane to $\xi \cap PH_1$ at p in ξ (it is the union of all lines through p in ξ which intersect PH_1 in precisely p). We now record some properties of $\mathcal{H}_{n,n^2+2n} = PH_1$ which we shall refer to as a *Hermitian Veronesean of index* n (respectively, *of* $\mathbf{PG}(V)$), for obvious reasons. We remark that some of the properties below have been proved by Cossidente & Siciliano in [3] for the case $n = 2$. Since we consider general index n, we have to provide full proofs, reproving some of the results in [3] when putting n equal to 2.

PROPERTY 1.1 *Let $\mathcal{H}_{n,n^2+2n}$ be a Hermitian Veronesean of index n in $\mathbf{PG}(n^2+2n, q)$. Then each elliptic quadric in some $\mathbf{PG}(3, q) \subseteq \mathbf{PG}(n^2 + 2n, q)$ contained in $\mathcal{H}_{n,n^2+2n}$ corresponds to a line of $\mathbf{PG}(V)$. Also, every n-dimensional subspace over $\mathbb{F}_q$ of $\mathbf{PG}(V)$ corresponds to a quadric Veronesean $\mathcal{V}_n$ over $\mathbb{F}_q$ on $\mathcal{H}_{n,n^2+2n}$ and we have $\langle\mathcal{V}_n\rangle \cap \mathcal{H}_{n,n^2+2n} = \mathcal{V}_n$.*

Proof: It is easy to see that the images in $\mathcal{H}_{n,n^2+2n}$ of the n-dimensional Baer subspaces of $\mathbf{PG}(V)$ are quadric Veroneseans $\mathcal{V}_n$ over $\mathbb{F}_q$. The last assertion follows from an easy calculation assuming, without loss of generality, that $\mathcal{V}_n$ contains the points $\pi(\langle(1, 0, 0, \ldots, 0)\rangle)$, $\pi(\langle(0, 1, 0, \ldots, 0)\rangle)$, $\pi(\langle(0, 0, 1, 0, \ldots, 0)\rangle)$,..., $\pi(\langle(0, 0, \ldots, 0, 1)\rangle)$ and $\pi(\langle(1, 1, 1 \ldots, 1)\rangle)$. Hence by Theorem 25.1.9 of [6], for $q \neq 2$ (for $q = 2$ Theorem 25.1.9 of [6] has to be revised), each conic on $\mathcal{H}_{n,n^2+2n}$ corresponds to a Baer subline in $\mathbf{PG}(V)$, and conversely. The first assertion now follows easily for $q \neq 2$.

Now let $q = 2$, and let E be an elliptic quadric in some $\mathbf{PG}(3, 2)$ and contained in $\mathcal{H}_{n,n^2+2n}$. Assume, by way of contradiction, that E does not correspond to a line of $\mathbf{PG}(V)$. Let $E = \{p_1, p_2, p_3, p_4, p_5\}$ and let x_1, x_2, x_3, x_4, x_5 be the corresponding points of $\mathbf{PG}(V)$. If at least four of these points, say x_1, x_2, x_3, x_4, are in a common plane of $\mathbf{PG}(V)$, then we may assume without loss of generality that $x_1 = \langle(1, 0, 0, \ldots, 0)\rangle$, $x_2 = \langle(0, 1, 0, \ldots, 0)\rangle$, $x_3 = \langle(0, 0, 1, 0, \ldots, 0)\rangle$ and $x_4 = \langle(1, 1, 1, 0, \ldots, 0)\rangle$. If x_1, x_2, x_3, x_4 generate a 3-dimensional space, then we may assume without loss of generality that x_1, x_2, x_3 are as above, and $x_4 = \langle(0, 0, 0, 1, 0, \ldots, 0)\rangle$. In both cases one easily calculates that $\mathcal{H}_{n,n^2+2n} \cap \langle p_1, p_2, p_3, p_4\rangle = \{p_1, p_2, p_3, p_4\}$, a contradiction. So we conclude that E corresponds to a line of $\mathbf{PG}(V)$. $\square$

A 3-dimensional subspace generated by an elliptic quadric on PH_1 will be called an *elliptic space* of PH_1. By the foregoing property, every elliptic space corresponds to a line of $\mathbf{PG}(V)$ and vice versa.

PROPERTY 1.2 *Let $PH_1 = \mathcal{H}_{n,n^2+2n}$ be a Hermitian Veronesean of index n in $\mathbf{PG}(n^2+2n, q)$.*

(*i*) *Any two points* p, q *of* PH_1 *lie in a unique member of* Ξ *which we will denote by* $\xi[p, q]$*;*

(*ii*) *Two subspaces in* Ξ *are either disjoint or else meet in a (unique) point of* HP_1*;*

(*iii*) *Assume* $\xi \in \Xi$, $p \in PH_1$, $p \notin \xi$ *and put* $E = \xi \cap PH_1$*. Then* $\cup_{p' \in E} T_p(\xi[p, p'])$ *is a projective subspace of dimension four.*

Proof: Property (*i*) is obvious by the foregoing paragraphs. We now prove (*ii*).

Assume $p \in \xi \cap \xi'$, with ξ and ξ' distinct elements of Ξ. We have to show that p is a point of PH_1. Assume that $p \notin PH_1$ and put $E = \xi \cap PH_1$ and $E' = \xi' \cap PH_1$. First we consider the case $n = 3$. Then E and E' have exactly one point $p' \neq p$ in common. Consider planes ζ and ζ' through pp' in ξ and ξ', respectively, with $C = \zeta \cap E$ and $C' = \zeta' \cap E'$ distinct nonsingular conics. Then $C \cup C'$ is contained in a quadric Veronesean $\mathcal{V}_2$, and so $\langle C, C' \rangle$ must be 4-dimensional, a contradiction. Now let $n > 3$. Consider planes ζ and ζ' through p in ξ and ξ', respectively, with $C = \zeta \cap E$ and $C' = \zeta' \cap E'$ distinct nonsingular conics. Then $C \cup C'$ is contained in a quadric Veronesean $\mathcal{V}_n$, and so $p = C \cap C'$, again a contradiction. So we conclude $p \in PH_1$.

We now show (*iii*). With p corresponds a point x in $\mathbf{PG}(V)$ and with ξ corresponds a line L in $\mathbf{PG}(V)$. With the spaces $\xi[p, p']$, $p' \in E = \xi \cap PH_1$, correspond the lines in $\mathbf{PG}(V)$ which join x to a point of L. Let p', p'' be distinct points of E. Then, by (*ii*), $\xi[p, p'] \cap \xi[p, p''] = \{p\}$, and so $\langle T_p(\xi[p, p']), T_p(\xi[p, p''])\rangle$ is 4-dimensional. Now we consider distinct points p', p'', p'' of E, corresponding to the distinct points x', x'', x'', respectively, of the line L. There are precisely $q + 1$ Baer subplanes in $\mathbf{PG}(V)$ containing x, x', x'', x'''. To these Baer subplanes correspond $q + 1$ quadric Veroneseans on PH_1. If $\mathcal{V}_2$ is one of these Veroneseans, then the tangent plane of $\mathcal{V}_2$ at p contains a line of each of the planes $T_p(\xi[p, p'])$, $T_p(\xi[p, p''])$, $T_p(\xi[p, p'''])$. If for two distinct such Veroneseans $\mathcal{V}_2$ and $\mathcal{V}_2'$ their tangent planes at p coincide, then $\langle \mathcal{V}_2 \rangle = \langle \mathcal{V}_2' \rangle$, from which it easily follows that $\mathcal{V}_2 = \mathcal{V}_2'$ (compare Property 1.1), a contradiction; if for $\mathcal{V}_2$ and $\mathcal{V}_2'$ the tangent planes at p contain a common line of e.g. $T_p(\xi[p, p'])$, then $\langle T_p(\xi[p, p'']), T_p([p, p'''])\rangle$ is 3-dimensional, again a contradiction. Now it easily follows that $T_p(\xi[p, p'''])$ belongs to $\langle T_p(\xi[p, p']), T_p([p, p''])\rangle$. So the $q^2 + 1$ planes $T_p(\xi[p, p'])$, $p' \in E$, which pairwise intersect in p, all belong to a common 4-dimensional space, hence their union is a 4-dimensional space. $\square$

Now let $\Pi \cong \mathbf{PG}(N, q)$ be a projective space. Let X be a subset of the point set of Π which spans Π and for which there exists a collection Ξ of 3-dimensional (projective) subspaces (hereafter 3-subspaces) of Π, called the *elliptic spaces of* X, such that for any $\xi \in \Xi$, $X(\xi) := X \cap \xi$ is an ovoid (not necessarily an elliptic quadric) in ξ. When $\xi \in \Xi$,

$x \in X(\xi)$ we will denote by $T_x(\xi)$ the tangent plane to x in ξ relative to the ovoid $X(\xi)$. We say that X is a *Hermitian set* if the following holds:

(H1) Any two points $x, y \in X$ lie in a unique member of Ξ which we denote by $[x, y]$;

(H2) If $\xi_1, \xi_2 \in \Xi$ and $\xi_1 \cap \xi_2 \neq \emptyset$ then $\xi_1 \cap \xi_2 \subseteq X$;

(H3) If $x \in X$ and $\xi \in \Xi$, $x \notin \xi$ then each of the planes $T_x([x, y])$, $y \in X(\xi)$ is contained in a fixed 4-subspace of Π which we denote by $T(x, \xi)$.

As we shall now show, a Hermitian set is a cap and consequently, we shall thereafter refer to a Hermitian set in the projective space Π as a *Hermitian cap*. The reader should compare this with the definition of a Veronesean cap in a projective space as defined by Mazzocca and Melone in [8] (see also Hirschfeld and Thas [6]). Clearly, a Hermitian Veronesean is a Hermitian cap. But one also obtains a Hermitian cap X from a Hermitian Veronesean $PH_1 \subseteq PH$ by setting $\Pi = PH/Z$, where Z is a subspace of PH which does not intersect any elliptic space, nor any 4-space $T(x, \xi)$ (with $x \in PH_1$ and ξ an elliptic subspace not containing x) and letting X be the image of PH_1. Such a Hermitian cap will be called a *quotient* of the Hermitian Veronesean PH_1. It is the purpose of this paper to classify Hermitian caps, yielding a characterization of the Hermitian Veroneseans. So we will prove:

THEOREM 1.3 *Let X be a Hermitian cap in the projective space Π. If Ξ is the corresponding set of elliptic spaces, then the incidence structure $(X, \mathcal{X})$, with $\mathcal{X} = \{X(\xi) \mid \xi \in \Xi\}$, is the point-line structure of a projective space over the field $\mathbb{F}_{q^2}$ and we refer to the dimension of this projective space as the index of the cap. If X has index r, then X is projectively equivalent to a quotient of the Hermitian Veronesean of index r.*

In order to obtain this result, one has to prove some particular cases and lemmas, some of which could be of independent interest. Specifically we will also prove:

THEOREM 1.4 *Let X be a Hermitian cap in the projective space $\Pi = \mathbf{PG}(N, q)$ and assume that Ξ is the corresponding set of elliptic spaces, where $|\Xi| > 1$. Denote $\mathcal{X} = \{X(\xi) \mid \xi \in \Xi\}$. Then the following hold:*

(i) If the index of X is r then $N \leq (r+1)^2 - 1$.

(ii) If $N = (r+1)^2 - 1$, with r the index of X, then X is projectively equivalent to the Hermitian Veronesean of index r in $\mathbf{PG}(r^2 + 2r, q)$.

(*iii*) *If the index $r = 2$ or 3 then X is projectively equivalent to the Hermitian Veronesean of index r.*

(*iv*) *If X is a Hermitian cap of index r and every hyperplane Y of the r-dimensional projective space $(X, \mathcal{X})$ has the property that $X(\langle Y \rangle) := X \cap \langle Y \rangle = Y$, then X is projectively equivalent to the Hermitian Veronesean of index r.*

Note that Theorem 1.3 is similar to the recent characterization of Veronesean caps by Thas and Van Maldeghem [10], which completes the partial classification by Mazzocca and Melone [8] (which was already improved by Hirschfeld and Thas [6]).

Also, note that the set of elliptic spaces of a Hermitian cap X in $\mathbf{PG}(N, q)$ is uniquely determined if $q > 2$. This follows immediately from (H2) by considering two coplanar bisecants, with no common point on X, of a hypothetical ovoid contained in X and not lying in an elliptic space of X. If $q = 2$, this is not clear. Nevertheless, we will denote a Hermitian cap by X, with the implicit understanding that there is some prescribed set Ξ of elliptic spaces.

The layout of the present paper is as follows. In Section 2 we prove that a Hermitian set is a cap and then go on to show that $(X, \mathcal{X})$ is the point-line incidence structure of a projective space over $\mathbb{F}_{q^2}$. The dimension of this projective space is called the *index* of X. In Section 3 we define the tangent space at a point of X and show that for a Hermitian cap of index r the tangent space is a $2r$-dimensional subspace of Π and derive some properties. In Section 4 we prove that a Hermitian cap of index 2 is projectively equivalent to the Hermitian Veronesean of index 2. Section 5 is devoted to proving that, if X is a Hermitian cap of index r in the projective space $\Pi = \mathbf{PG}(N, q)$, then $N \leq (r+1)^2 - 1$. We also prove that if equality occurs, so if $N = (r+1)^2 - 1$, then X is projectively equivalent to the Hermitian Veronesean of index r. As a corollary we obtain our characterization of Hermitian caps of index 3. We also are able to show that if every hyperplane Y of $(X, \mathcal{X})$ has the property that $X(\langle Y \rangle) = Y$ then X is projectively equivalent to a Hermitian Veronesean of index r. Finally, in the last section (Section 6) we are able to prove our main result Theorem 1.3.

One might consider the papers by Tallini [9], Ferri [5] and Mazzocca and Melone [8] as a first series of characterizations of a "quadratic" embedding of an incidence geometry and this paper as a next step. Apart from the fact that the Veronesean and Hermitian caps are interesting (finite) algebraic varieties, such characterizations might play a role in obtaining classifications of projective embeddings of Lie incidence geometries, for example in determining the maximal projective embedding dimensions of dual polar spaces of symplectic and unitary type where such quadratic embeddings of projective spaces and the Grassmannians naturally arise, though this dimension can also be obtained by more elementary methods (cf. [1] and [2]).

2 The Index of a Hermitian Cap

In this section X is a Hermitian cap in the projective space $\Pi = \mathbf{PG}(N,q)$, with Ξ as set of elliptic spaces. Our main objective is to show that the incidence structure $(X, \mathcal{X})$, with $\mathcal{X}$ as defined above, is the point-line incidence structure of a projective space in which lines have q^2+1 points. Towards that end we prove some preliminary lemmas.

LEMMA 2.1 *(i) The incidence structure $(X, \mathcal{X})$ is a linear space, that is, every two points lie on a unique block.*

(ii) The set X is a cap in Π.

Proof: Clearly (i) follows immediately from (H1). We now prove (ii). For $x, y \in X$, the elliptic space $[x,y]$ contains the line $\langle x,y\rangle$. Since $X([x,y])$ is an ovoid it follows that $X(\langle x,y\rangle) := X \cap \langle x,y\rangle = \{x,y\}$ and X is a cap as claimed. $\square$

LEMMA 2.2 *Let $x \in X$, $\xi, \xi' \in \Xi$, $\xi \neq \xi'$, with $x \in \xi \cap \xi'$. Then $T_x(\xi) \cap T_x(\xi') = x$.*

Proof: We have $T_x(\xi) \cap T_x(\xi') \subset \xi \cap \xi' = x$ by (H2) and (H1). $\square$

LEMMA 2.3 *Let $x \in X$, $\xi \in \Xi$, $x \notin \xi$. Then $\{T_x([x,y])/x \mid y \in X(\xi)\}$ is a spread of $T(x,\xi)/x$.*

Proof: The spaces $T_x([x,y])/x$, with $y \in X(\xi)$, are lines of the 3-subspace $T(x,\xi)/x$ of Π/x. There are q^2+1 such lines, one for each point of $X(\xi)$. By (2.2) they are disjoint and consequently they are a spread of $T(x,\xi)/x$. $\square$

LEMMA 2.4 *Let $x \in X$, $\xi \in \Xi$, $x \notin \xi$. Let $y, x \in X(\xi)$, $y \neq z$, and assume that $\xi' \in \Xi$, $x \notin \xi'$ and ξ' intersects $[x,y]$ and $[x,z]$. Then ξ' intersects ξ.*

Proof: If either $y \in \xi'$ or $z \in \xi'$ then we are done. So we may assume that $\xi' \cap [x,y] = y' \neq y$ and $\xi' \cap [x,z] = z' \neq z$. Since $[y,x] = [y,y']$, we have $T_y([y,x]) \subset T(y,[x,z]) \cap T(y,\xi')$. The inclusion $T_y([y,z']) \subset T(y,[x,z]) \cap T(y,\xi')$ is trivial since $z' \in [x,z]$. However, $T_y([y,x])$ and $T_y([y,z'])$ meet only in y (by Lemma 2.2) and as $T(y,[x,z]), T(y,\xi')$ are 4-subspaces, we have $T(y,[x,z]) = \langle T_y([y,x]), T_y([y,z'])\rangle = T(y,\xi')$. Hence $T_y(\xi) \subset T(y,[x,z]) = T(y,\xi')$. It follows from Lemma 2.3 that there must exist $a \in X(\xi')$ such that $T_y(\xi) \cap T_y([y,a])$ contains a line on y. By Lemma 2.2 it follows that $\xi = [y,a]$. Consequently $a \in \xi \cap \xi'$. $\square$

PROPOSITION 2.5 *The pair $(X,\mathcal{X})$ with inclusion as incidence is the point-line geometry arising from the points and the lines of a projective space in which all lines have q^2+1 points.*

Proof: From Lemma 2.1 we deduce that $(X,\mathcal{X})$ is a linear space. From Lemma 2.4, Pasch's axiom holds and therefore $(X,\mathcal{X})$ is the point-line geometry of a projective space. The statement about lines follows from the fact that $|X(\xi)| = q^2+1$ for $\xi \in \Xi$. $\square$

Following the convention of Mazzocca and Melone [8], we will refer to the dimension of the projective space $(X,\mathcal{X})$ as the *index* of the Hermitian cap X. A subset Y of X will be said to be *a subspace of* $(X,\mathcal{X})$ if for every $y_1, y_2 \in Y$, we have $X([y_1,y_2]) \subset Y$. If we denote by $\Xi(Y)$ the collection of all $\xi \in \Xi$ such that $X(\xi) \subset Y$, and by $\mathcal{X}(Y)$ the collection of all $X(\xi)$ with $\xi \in \Xi(Y)$, then $(Y,\mathcal{X}(Y))$ defines a projective subspace of the space $(X,\mathcal{X})$. We complete this section with the following

LEMMA 2.6 *Assume that Y is an r-dimensional subspace of $(X,\mathcal{X})$. Then Y is a Hermitian cap of index r in $\langle Y\rangle$.*

Proof: This is immediate from the definition of a Hermitian cap since the properties (H1), (H2) and (H3) are inherited. $\square$

We end this section with a geometric property of Hermitian caps X of index 2 in $\mathbf{PG}(8,q)$, characterizing points of $\mathbf{PG}(8,q)$ on secant lines of X. We will need this result in the proof of Theorem 4.1 below. Since for a Hermitian cap of index 2 any two elliptic spaces intersect, a 4-subspace $T(x,\xi)$ will also be denoted by $T(x)$.

LEMMA 2.7 *Let X be a Hermitian cap of index 2 in $\mathbf{PG}(8,q)$, and let $\mathcal{X}$ be the set of ovoids on X in the elliptic spaces of X. Let $O_i \in \mathcal{X}$, $i = 1,2,3$, be such that $O_i \cap O_j = \{p_{ij}\}$, $1 \le i < j \le 3$, and suppose $p_{12} \neq p_{23} \neq p_{13} \neq p_{12}$. Let x be any point* *of* $\mathbf{PG}(8,q) \setminus X$ *on the line* $p_{12}p_{23}$*. Let* $Z = T_{p_{12}}(\langle O_2\rangle) \cap T_{p_{23}}(\langle O_2\rangle)$ *and let* p *be any point of* X *in the plane* $\langle Z, x\rangle$*. Further, let* p', p'' *be two points of* $O_1 \setminus \{p_{12}, p_{13}\}$ *and* $O_3 \setminus \{p_{13}, p_{23}\}$*, respectively, such that* $\{p\} = [p_{13}, p_0] \cap O_2$*, with* $p_0 = [p_{23}, p'] \cap [p_{12}, p'']$*. Then* $\langle p', T_{p_{23}}(\langle O_3\rangle)\rangle \cap \langle p'', T_{p_{12}}(\langle O_1\rangle)\rangle$ *is a point* x' *and* $\{x\} = \langle T(p_{13}), x'\rangle \cap p_{12}p_{23}$*.*

Proof: Clearly $\langle p', T_{p_{23}}(\langle O_3\rangle), p'', T_{p_{12}}(\langle O_1\rangle)\rangle = \langle O_1, O_3\rangle$ is 6-dimensional by (H2). Hence $\langle p', T_{p_{23}}(\langle O_3\rangle)\rangle \cap \langle p'', T_{p_{12}}(\langle O_1\rangle)\rangle$ is a point x'. As $\langle T(p_{13}), x',\ p_{12}p_{23}\rangle = \langle O_1, O_3\rangle$, it is also obvious that $\langle T(p_{13}), x'\rangle \cap p_{12}p_{23}$ is a point x_0. Now, clearly, x' is contained in the intersection $\langle p', T(p_{23})\rangle \cap \langle p'', T(p_{12})\rangle$, which contains p_0 and Z. Comparing dimensions, we see that (since obviously $\langle p', T(p_{23}), p'', T(p_{12})\rangle = \mathbf{PG}(8,q)$) $\langle p', T(p_{23})\rangle \cap \langle p'', T(p_{12})\rangle$ is 2-dimensional and hence is equal to $\langle p_0, Z\rangle$. Hence x' — and consequently also x_0 — is contained in $\langle T(p_{13}), p_0, Z\rangle$. Now, $\langle T(p_{13}), p_0, Z\rangle \cap \langle O_2\rangle$ is at most 2-dimensional, but since it contains Z and the intersection p of $[p_{13}, p_0]$ and O_2, it is 2-dimensional and equal to $\langle Z, p\rangle$. This implies $x_0 \in \langle Z, x\rangle$. On the other hand, x_0 is also contained in the 6-space generated by O_1 and O_3. Consequently x_0 lies on $p_{12}p_{23}$ and must therefore be equal to x. □

3 The Tangent Space at a Point

In this section we assume that X is a Hermitian cap of index $r \geq 2$ in the projective space $\Pi \cong \mathbf{PG}(N,q)$, with Ξ as set of elliptic spaces. For a point $x \in X$ we define the *tangent space at* x to be the union of all $T_x(\xi)$ such that $x \in X(\xi)$. We will denote the tangent space at x by $T(x)$. The purpose of this section is to study these tangent spaces. The next proposition justifies the use of the word "space" for the notion of "tangent space". Note also that, if X is the projection of a some Hermitian Veronesean PH_1, then this tangent space is the projection of a tangent space in the algebraic sense of PH_1 as an algebraic variety. Hence there can be no confusion with terminology.

PROPOSITION 3.1 *For* $x \in X$*,* $T(x)$ *is a subspace of* Π *of dimension* $2r$*. Moreover, the spaces* $T_x([x,y])/x$*, with* $y \in X \setminus \{x\}$*, form a line spread of* $T(x)/x$*.*

Proof: We first prove by induction on r that $T(x)$ is a subspace. When $r = 2$ then for any $\xi \in \Xi$, $x \notin \xi$, $T(x) = \cup_{y \in X \cap \xi} T_x([x,y])$ is a 4-space by (H3) and Lemma 2.3. Now let $x \in Y \subset X$ with Y corresponding to a hyperplane of $(X, \mathcal{X})$. Then Y is a Hermitian cap of index $r-1$ in $\langle Y\rangle$ by Lemma 2.6. Thus by induction $T_x(Y) := \cup_{y \in Y \setminus \{x\}} T_x([x,y])$ is a subspace of $\langle Y\rangle$ of dimension $2r-2$ and the collection $\{T_x([x,y])/x \mid y \in Y\setminus\{x\}\}$ is a line spread of $T_x(Y)/x$.

Now suppose $z \in X$, $z \notin Y$. We first claim that $T_x([x,z]) \cap T_x(Y) = x$. For suppose to the contrary that $\langle x,u\rangle$ is a line on x in $T_x([x,z]) \cap T_x(Y)$. Since $\langle x,u\rangle \subset T_x(Y)$, by the induction hypothesis there is a $y \in Y$, $y \neq x$, so that $\langle x,u\rangle \subset T_x([x,y])$. But then $\langle x,u\rangle \subset T_x([x,y]) \cap T_x([x,z])$, which contradicts Lemma 2.2. Thus the claim is proved. It follows from this that the space spanned by $T_x(Y)$ and $T_x([x,z])$ has dimension $2r$.

Now suppose $z' \in X$, $z' \notin Y$, $z' \neq z$. Let $\xi = [z,z']$, so $X(\xi)$ is a line of the projective space $(X,\mathcal{X})$ which meets the hyperplane Y in a unique point y. Assume that $y \neq x$. Now $T(x,[z,z']) = T(x,[z,y]) = \langle T_x([x,z]), T_x([x,y])\rangle \subset \langle T_x(Y), T_x([x,z])\rangle$. Thus $T_x([x,z']) \subset \langle T_x(Y), T_x([x,z])\rangle$. This implies that $T(x) \subset \langle T_x(Y), T_x([x,z])\rangle$. It remains to show that $T(x) = \langle T_x(Y), T_x([x,z])\rangle$ from which it will also follow that $T(x)/x$ is partitioned by the $T_x([x,y])$, $y \in X$, $y \neq x$. Suppose u is a point in $\langle T_x(Y), T_x([x,z])\rangle$, $u \notin T_x(Y) \cup T_x([x,z])$. Then there are unique lines ax in $T_x(Y)$ and bx in $T_x([x,z])$ such that u is on the plane $\langle a,b,x\rangle$. Note that, by our assumption on u, we have $a,b \neq x$. Then by our induction hypothesis there is a $y \in Y\setminus\{x\}$ such that $ax \subseteq T_x([x,y])$. But now $u \in \langle T_x([x,z]), T_x([x,y])\rangle = T(x,[y,z])$ and by Lemma 2.3 there is a $z' \in [y,z] \cap X$ such that $u \in T_x([x,z'])$. This completes the proof. $\square$

COROLLARY 3.2 *For $x \in X$, we have $X(T(x)) := X \cap T(x) = x$.*

Proof: This follows immediately from the fact that $X \cap T_x([x,y]) = x$ for $y \in X$, $y \neq x$ and the definition of $T(x)$. $\square$

LEMMA 3.3 *For $x,y \in X$, $x \neq y$, we have $T(x) \cap T(y) = T_x([x,y]) \cap T_y([x,y])$.*

Proof: Suppose $u,w \in X$ and $u,w \notin [x,y]$. Then $[x,u] \cap [y,w]$ is either empty or a point of X. Clearly if $[x,u] \cap [y,w] = \emptyset$ then $T_x([x,u]) \cap T_y([y,w]) = \emptyset$. Suppose $[x,u]$ and $[y,w]$ meet in a point. Then we can assume that this point is $u = w$. Then $T_x([x,u]) \cap T_y([y,u]) \subset [x,u] \cap [y,u] = u \in X$. As $T_x([x,u]) \cap X = x \neq u$, it follows that $T_x([x,u]) \cap T_x([y,u]) = \emptyset$. It is now also clear that for $u,w \in X$ with $u,w \notin [x,y]$, we have $T_x([x,u]) \cap T_y([y,w]) = \emptyset$. It therefore follows that $T(x) \cap T(y) \subset T_x([x,y]) \cap T_y([x,y])$. Since the opposite inclusion is obvious we get equality. $\square$

LEMMA 3.4 *Let $\xi \in \Xi$, $x \in X$, $x \notin \xi$. Then $\xi \cap T(x) = \emptyset$.*

Proof: Suppose $a \in T(x) \cap \xi$. Then $a \neq x$ since $x \notin \xi$. But then also $a \notin X$ since $X(T(x)) = \{x\}$. So, by Proposition 3.1, there is a $y \in X$, $y \neq x$, such that $a \in [x, y]$. Then $a \in \xi \cap [x, y]$ which is in X by (H2), a contradiction. $\square$

REMARK 3.5 Let Y be a hyperplane of $(X, \mathcal{X})$ and $x \in X$, $x \notin Y$. Set $\mathcal{P}_x = \{T_x([x, y]) \mid y \in Y\}$, $\mathcal{L}_x = \{T(x, \xi) \mid \xi \in \Xi(Y)\}$. Then it is immediate that the the pair $(\mathcal{P}_x, \mathcal{L}_x)$ considered as an incidence geometry is isomorphic to $(Y, \mathcal{X}(Y))$.

4 A Characterization of the Hermitian Veronesean of Index 2

The purpose of this section is to prove the following result classifying Hermitian caps of index 2 and characterizing the Hermitian Veronesean of index 2.

THEOREM 4.1 *If X is a Hermitian cap of index 2 in $\Pi = \mathbf{PG}(N, q)$, then $N = 8$ and X is projectively equivalent to the Hermitian Veronesean of index 2.*

Proof: Before proceeding to the proof we first note that for any $\xi \in \Xi$ (with Ξ the distinguished set of elliptic spaces) and distinct points $x, y \in X(\xi)$ we have $\xi = \langle y, T_x(\xi) \rangle$ which follows from the fact that $X(\xi)$ is a ovoid in ξ.

We now prove that $N = 8$. Toward this end let $\xi \in \Xi$ and $x \in X$, $x \notin X(\xi)$. We will show that $X \subseteq \langle \xi, T(x) \rangle$ from which it follows that $N = 8$ in view of the facts that $\xi \cap T(x) = \emptyset$ (see Lemma 3.4), that the (projective) dimensions of ξ and $T(x)$ are 3 and 4, respectively, and that $\langle X \rangle = \Pi$. Suppose that $y \in X$, $y \neq x$. If $y \in \xi$ then clearly $y \in \langle \xi, T(x) \rangle$. Otherwise, let $y' = \xi \cap [x, y]$ which is a point of X since $(X, \mathcal{X})$ is a projective plane. Now, $y \in [x, y] = [x, y'] = \langle y', T_x([x, y]) \rangle \subseteq \langle \xi, T(x) \rangle$ by the remark above. Thus $X \subseteq \langle \xi, T(x) \rangle$ and $N = 8$ as claimed.

Now let $O = X(\xi)$ be fixed, with $\xi \in \Xi$. We project X from ξ onto some 4-dimensional subspace U of Π, with $\langle \xi, U \rangle = \Pi$ and denote the projection map by ρ. Let $O = \{p_0, p_1, \ldots, p_{q^2}\}$, and denote by ξ_i an arbitrary element of Ξ distinct from ξ and containing p_i, $i \in \{0, 1, \ldots, q^2\}$. We now fix i arbitrarily. Clearly $O_i \setminus \{p_i\}$, with $O_i = X(\xi_i)$, is projected onto an affine plane $A_i \subseteq U$ (first project $O_i \setminus \{p_i\}$ from p_i onto a hyperplane of Π not containing p_i, then we obtain an affine plane and this also shows that the line L_i at infinity of A_i is the image under ρ of $T_{p_i}(\xi_i) \setminus \{p_i\}$) contained in some projective subplane π_i. Since $T(p_i)$ is generated by $T_{p_i}(\xi_i)$ and $T_{p_i}(\xi) \subseteq \xi$, the line L_i is also the image of $T(p_i) \setminus \{p_i\}$ under the projection ρ. Hence L_i is independent of the choice of ξ_i containing p_i. So, if $\xi'_i \in \Xi$ is a third solid containing p_i, then the projection of $O'_i \setminus \{p_i\}$, with $O'_i := X(\xi'_i)$, is an affine plane A'_i with L_i as line at infinity. Suppose by way of contradiction that $A_i = A'_i$. Then $U_i =: \langle \xi, \xi_i, \xi'_i \rangle$ is a 6-dimensional space. If $j \neq i$, with $j \in \{0, 1, \ldots, q^2\}$, then O_j intersects O, O_i and O'_i in distinct points p_j, a_i and a'_i, respectively. Since X is a cap, the space $\langle p_j, a_i, a'_i \rangle$ is a plane contained in U_i. Since $T(p_i)$ is contained in U_i and has dimension 4, it intersects $\langle p_j, a_i, a'_i \rangle$ nontrivially, contradicting the fact that $\xi_j \cap T(p_i) = \emptyset$, as proved above.

So, in U, we obtain a set of q^2 distinct projective planes containing the line L_i (each plane corresponds to a member of $\Xi \setminus \{\xi\}$ containing p_i). Let Y_i be the union of the remaining $q + 1$ planes of U through L_i. We claim that Y_i is a 3-dimensional space containing all L_j, $j \in \{0, 1, \ldots, q^2\}$. Indeed, pick such $j \neq i$ arbitrarily. Since $T(p_j)$ and O_i generate Π, the projections L_j and A_i generate U. Considering dimensions, this implies that L_j and π_i are disjoint. It follows that L_j is contained in Y_i. Since $\langle L_i, L_j \rangle$ has just L_i in common with each of the q^2 planes π_i, we have $\langle L_i, L_j \rangle \subseteq Y_i$, so $Y_i = \langle L_i, L_j \rangle$ and our claim follows. Hence we may put $Y = Y_i$, since it is independent of $i \in \{0, 1, \ldots, q^2\}$. Consequently, $\{L_0, L_1, \ldots, L_{q^2}\}$ is a spread of Y, and all $T(p_i)$ are contained in a common 7-dimensional space $\langle \xi, Y \rangle$. If we identify the points of the projective plane $(X, \mathcal{X})$ not on O with there image under ρ, and those on O with the image under ρ of their tangent space, then we clearly obtain an André-Bose-Bruck representation of $(X, \mathcal{X})$, implying that the latter is a translation plane with translation line O. Since O was chosen arbitrarily, every line of $(X, \mathcal{X})$ is a translation line, and so $(X, \mathcal{X})$ is a Desarguesian plane. We denote this plane by its standard notation $\mathbf{PG}(2, q^2)$. From now on, if x is some point or subset of X, then we denote by x^* the corresponding point or subset in $\mathbf{PG}(2, q^2)$.

From the André-Bose-Bruck representation above it also follows easily that the Baer sublines of $\mathbf{PG}(2, q^2)$ correspond with the plane sections of size $q + 1$ of the ovoids of $\mathcal{X}$. Hence these plane sections of every such ovoid induce on that ovoid the structure of a classical Möbius plane. Consequently every element of $\mathcal{X}$ is an elliptic quadric (see also Dembowski [4]) .

Let $p \in X$ not be contained in some $\xi \in \Xi$, and put $O = X(\xi)$. Consider the projection ρ of $X \setminus \{p\}$ from $T(p)$ onto ξ. If $y \in X \setminus \{p\}$, then first projecting y from $T_p([p, y])$ and noting that $\langle T_p([p, y]), y \rangle \cap \xi$ is the point $[p, y] \cap O$, we see that $\rho(y) \in O$. Moreover, this argument also shows that the inverse image of any point $x \in O$ with respect to ρ precisely corresponds to the line of $\mathbf{PG}(2, q^2)$ joining p^* with x^*. Hence every other point of $\mathbf{PG}(2, q^2)$ corresponds with a point of X lying outside the space generated by $T(p)$ and x.

Now we consider three points p_0, p_1, p_2 of X such that p_0^*, p_1^*, p_2^* form a triangle in $\mathbf{PG}(2, q^2)$.

Let Y be a Hermitian Veronesean in $\Pi = \mathbf{PG}(8, q)$ associated with $\mathbf{PG}(2, q^2)$, and denote for each point or subset a^* of $\mathbf{PG}(2, q^2)$ the corresponding point or subset on Y by $a^\dagger$. We will show that X and Y are projectively equivalent, and that the projectivity can be chosen such that it maps any point $a \in X$ to the point $a^\dagger \in Y$.

Clearly, the elliptic quadrics $X([p_0, p_1])$, $X([p_1, p_2])$, $X([p_2, p_0])$ generate Π, because the space they generate contains both $T(p_0)$ and $[p_1, p_2]$. If C is a conic on $X([p_0, p_1])$, then also $C^\dagger$ is a conic on $Y([p_0^\dagger, p_1^\dagger])$ (both corresponding to the same Baer subline in $\mathbf{PG}(2, q^2)$). It easily follows that there are projectivities $\alpha_i : [p_j, p_k] \to [p_j^\dagger, p_k^\dagger]$ with some associated field automorphism θ_i, for all i, j, k, with $\{i, j, k\} = \{0, 1, 2\}$, $j < k$, mapping x to $x^\dagger$, for all $x \in X([p_0, p_1]) \cup X([p_1, p_2]) \cup X([p_2, p_0])$. We claim that $\theta_1 = \theta_2$. Let C_j, $j = 1, 2$, be a conic on $X([p_0, p_j])$ containing the points p_0 and p_j. Let r be a generator of the multiplicative group of $\mathbb{F}_q$. Choose points $x_j, y_j \in C_j$ such that the cross ratio $(p_0, p_j; x_j, y_j)$ on C_j is equal to r. Projecting $X \setminus X([p_1, p_2])$ from $[p_1, p_2]$ onto some 4-dimensional subspace disjoint from $[p_1, p_2]$, denoting the projection map by ρ, and denoting the projection of the tangent line to C_j at p_j by q_j, we see that the 4-tuples $(\rho(p_0), q_1, \rho(x_1), \rho(y_1))$ and $(\rho(p_0), q_2, \rho(x_2), \rho(y_2))$ are in perspective. It is easy to see that now also the 4-tuples $(p_0^*, p_1^*, x_1^*, y_1^*)$ and $(p_0^*, p_2^*, x_2^*, y_2^*)$ are in perspective, and this in turn implies, by a similar but reciprocal reasoning,

$$\theta_1(r) = (p_0^\dagger, p_1^\dagger; x_1^\dagger, y_1^\dagger) = (p_0^\dagger, p_2^\dagger; x_2^\dagger, y_2^\dagger) = \theta_2(r),$$

which is only possible if $\theta_1 = \theta_2$. The claim is proved. Similarly $\theta_0 = \theta_1$, hence $\theta_0 = \theta_1 = \theta_2$. Since $[p_0, p_1]$ meets $[p_1, p_2]$ in a single point, there exists a unique projectivity $\alpha' : \langle [p_0, p_1], [p_1, p_2] \rangle \to \langle [p_0^\dagger, p_1^\dagger], [p_1^\dagger, p_2^\dagger] \rangle$ such that $\alpha' \equiv \alpha_i$ on $[p_j, p_k]$, for all i, j, k with $\{i, j, k\} = \{0, 1, 2\}$, $i \in \{0, 2\}$. It follows from Lemma 2.7 that α_1 and α' coincide on the line p_0p_2, which is the intersection of their domains of definition. Hence there exists a unique projectivity $\alpha : \Pi \to \Pi$ extending α_0, α_1 and α_2, and hence mapping x to $x^\dagger$ for $x \in [p_0, p_1] \cup [p_1, p_2] \cup [p_2, p_0]$.

Now let $x \in X$ be arbitrary, but not belonging to $[p_0, p_1] \cup [p_1, p_2] \cup [p_2, p_0]$. Denote by x_i^* the intersection (in $\mathbf{PG}(2, q^2)$) of the line $p_i^* x^*$ with the line $p_j^* p_k^*$, for all i, j, k, with $\{i, j, k\} = \{0, 1, 2\}$. Now we show that $\langle T(p_0), x_0 \rangle$, $T(p_1), x_1 \rangle$, $\langle T(p_2), x_2 \rangle$ intersect in just one point. The spaces $\langle T(p_0), x_0 \rangle$ and $T(p_1), x_1 \rangle$ intersect in a plane β. This plane contains the line $N = T_{p_0}([p_0, p_1]) \cap T_{p_1}([p_0, p_1])$. Assume, by way of contradiction, that $\beta \cap \langle T(p_2), x_2 \rangle$ contains a line. Then N contains a point c of $\langle T(p_2), x_2 \rangle$. This point c belongs to $[p_0, p_1]$. As $[p_0, p_1] \cap T(p_2)$ is empty, we necessarily have $x_2 = c$. So N contains a point of X, clearly a contradiction. Hence x and $x^\dagger$ are the unique intersections of $\langle T(p_0), x_0 \rangle$, $\langle T(p_1), x_1 \rangle$,

$\langle T(p_2), x_2\rangle$ and of $\langle T(p_0^\dagger), x_0^\dagger\rangle$, $\langle T(p_1^\dagger), x_1^\dagger\rangle$, $\langle T(p_2^\dagger), x_2^\dagger\rangle$, respectively. Consequently x is mapped onto $x^\dagger$ by α, and the theorem is proved. $\square$

5 Hermitian Caps of Index $r > 2$ in $\mathbf{PG}(r^2 + 2r, q)$

In this section we prove that, if X is a Hermitian cap of index r in $\Pi = \mathbf{PG}(N, q)$, then $N < (r+1)^2$. We further show that if $N = r^2 + 2r$ then X is projectively equivalent to the Hermitian Veronesean $\mathcal{H}_{r,r^2+2r} = PH_1$.

LEMMA 5.1 *If X is a Hermitian cap of index $r \geq 2$ in $\Pi = \mathbf{PG}(N, q)$, then $N < (r+1)^2$.*

Proof: We prove the result by induction on r. The case $r = 2$ was treated in Theorem 4.1, so we may assume $r > 2$. Let $Y \subset X$ be the point set of a hyperplane of $(X, \mathcal{X})$. By Lemma 2.6, Y is a Hermitian cap of index $r - 1$ in $\Pi(Y) = \langle Y\rangle$. Let $N(Y)$ be the dimension of $\Pi(Y)$. By induction $N(Y) \leq r^2 - 1$.

If $X \subseteq \Pi(Y)$, then $\Pi = \Pi(Y)$ and so $N = N(Y) < r^2 < (r+1)^2$. Therefore we may assume there is an $x \in X \setminus Y$ with $x \notin \Pi(Y)$. We claim that $\Pi = \langle \Pi(Y), T(x)\rangle$. It suffices to show, as we did in the proof of Theorem 4.1, that $X \subset \langle \Pi(Y), T(x)\rangle$. Assume that $z \in X$. Of course we may assume that $z \neq x$ and that $z \notin \Pi(Y)$ so that, in particular, $z \notin Y$. Now since Y is a hyperplane in $(X, \mathcal{X})$, $Y \cap X([x, z])$ is a unique point y. Then $z \in [x, z] = [x, y] = \langle y, T_x([x, y])\rangle \subseteq \langle \Pi(Y), T(x)\rangle$ as claimed. Now, by Proposition 3.1, $\dim \Pi \leq r^2 - 1 + 2r + 1 = r^2 + 2r < (r+1)^2$. $\square$

LEMMA 5.2 *If X is a Hermitian cap of index $r \geq 2$ in $\Pi = \mathbf{PG}(r^2 + 2r, q)$, if Y is a hyperplane of the projective space $(X, \mathcal{X})$, and if $x \in X \setminus Y$, then $T(x) \cap \langle Y\rangle = \emptyset$.*

Proof: As in the proof of the previous lemma, one shows that $\Pi = \langle T(x), Y\rangle$. As $\dim T(x) = 2r$, $\dim\langle Y\rangle \leq r^2 - 1$ and $\dim \Pi = r^2 + 2r$, the result follows. $\square$

We now prove the main result of Theorem 1.4:

THEOREM 5.3 *If X is a Hermitian cap of index r in $\Pi = \mathbf{PG}(r^2 + 2r, q)$, then X is projectively equivalent to the Hermitian Veronesean of index r.*

Proof: The proof proceeds by induction on r, the case $r = 2$ being proved in Theorem 4.1. So suppose now that $r > 2$. We select two distinct hyperplanes H_1 and H_2 in $\mathbf{PG}(r, q^2)$. As shown above, these correspond with two Hermitian caps X_1 and X_2, respectively, of index $r-1$. We claim that $\dim\langle X_1\rangle = \dim\langle X_2\rangle = r^2 - 1$. Indeed, by Lemma 5.1, $n_i := \dim\langle X_i\rangle \leq r^2 - 1$. Let $x \in X$, $x \notin \langle X_i\rangle$, $i \in \{1, 2\}$. It follows from the proof of Lemma 5.1 that $\Pi = \langle T(x), X_i\rangle$, and hence, by Proposition 3.1, $r^2 + 2r \leq 2r + n_i + 1 \leq 2r + r^2$, implying $n_i = r^2 - 1$. Our claim is proved. Put $\langle X_i\rangle = \Omega_i$, $i = 1, 2$. The caps X_1 and X_2 meet in a Hermitian cap X_3 of index $r - 2$ in a subspace Ω of Π. Similarly as before, one shows that $\dim \Omega = r^2 - 2r$. We consider a line L^* in $\mathbf{PG}(r, q^2)$ not meeting $H_1 \cap H_2$. If x is the common point of X_1 and L, then $T(x) \subseteq \langle L, X_1\rangle$, so $\langle T(x), X_2\rangle \subseteq \langle L, X_1, X_2\rangle$, hence $\Pi = \langle L, X_1, X_2\rangle$. Moreover, by induction, the caps X_i, $i = 1, 2$, are Hermitian Veroneseans.

We now proceed very similar to the proof of Theorem 4.1. We again use similar notation as before with regard to the superscript $*$. Let Y be a Hermitian Veronesean in $\Pi = \mathbf{PG}(r^2 + 2r, q)$ associated with $\mathbf{PG}(r, q^2)$, and denote for each point or subset a^* of $\mathbf{PG}(r, q^2)$ the corresponding point or subset on Y by $a^\dagger$. We will again show that X and Y are projectively equivalent, and that the projectivity can be chosen such that it maps any point $a \in X$ to the point $a^\dagger \in Y$. The mapping "$\dagger$" maps the elliptic quadrics on X (belonging to Ξ) onto the elliptic quadrics of Y; it also maps the conics of X on those quadrics onto conics of Y. This, together with the induction hypothesis, the results of the previous paragraph, and the fact that every automorphism of $\mathbf{PG}(r, q^2)$ uniquely defines an automorphism of $\mathbf{PG}(r^2 + 2r, q)$ stabilizing $\mathcal{H}_{r,r^2+2r}$, implies the existence of projectivities $\alpha_0 : \langle L\rangle \to \langle L^\dagger\rangle$, $\alpha_i : \Omega_i \to \langle X_i^\dagger\rangle$, $i = 1, 2$, with some associated field automorphisms θ_0, θ_i, $i = 1, 2$, respectively, mapping x to $x^\dagger$, for every x in L and X_i, $i = 1, 2$, respectively. By considering any subplane $\mathbf{PG}(2, q^2)$ of $\mathbf{PG}(r, q^2)$ containing L^*, we can use exactly the same arguments as in the proof of Theorem 4.1 to obtain $\theta_0 = \theta_1 = \theta_2$. So there exists a projectivity $\alpha' : \langle X_1, X_2\rangle \to \langle X_1^\dagger, X_2^\dagger\rangle$ such that $\alpha(x) = x^\dagger$ for all $x \in X_1 \cup X_2$. By considering a subplane $\mathbf{PG}(2, q^2)$ of $\mathbf{PG}(r, q^2)$ containing L^* and using Lemma 2.7 as in Theorem 4.1, we see that α' extends to a projectivity $\alpha : \Pi \to \Pi$ such that $\alpha(x) = x^\dagger$, for all $x \in L \cup X_1 \cup X_2$. Now let x be any other point of X. Then there is a unique Hermitian Veronesean of index 2 on X containing L and x (determined by the plane generated by x^* and L^* in $\mathbf{PG}(r, q^2)$). It has a unique elliptic quadric in common with each of X_1 and X_2, and hence, as in the proof of Theorem 4.1, it follows that $\alpha(x) = x^\dagger$.

The theorem is proved. $\square$

In the next proof, and in the proof of Lemma 5.6, we will use the fact that every Hermitian $m \times m$ matrix M over K can be written as the sum of at most m rank 1 Hermitian

$m \times m$ matrices over K. This is easily seen by induction considering the matrix $M' = M - a_{11}^{-1} M_1 \overline{M}_1^T$, where M_1 is the $m \times 1$ matrix arising from the first column of M, and where a_{11} is the element of M in the first column and first row (and which is for now assumed to be unequal 0). The matrix M' has as first row a zero row, and as first column a zero column, and so we may apply induction. If $a_{11} = 0$, then we first apply a suitable transformation belonging to $\mathbf{GL}(m, K)$ to obtain a matrix M'' with the desired property $a''_{11} \neq 0$.

LEMMA 5.4 *If X is a Hermitian cap of index r in Π and Y is the point set of a subplane of $(X, \mathcal{X})$, then $X(\langle Y \rangle) := X \cap \langle Y \rangle = Y$.*

Proof: Suppose the contrary. Let $p \in X(\langle Y \rangle) \backslash Y$. Now, as follows from our remark above, there exist three distinct points $a, b, c \in Y$ such that $p \in \langle a, b, c \rangle$ (if a, b, c were not distinct, then e.g. $p \in \langle a, b \rangle$ and this contradicts X being a cap) . But then $\langle a, b \rangle \cap \langle p, c \rangle \neq \emptyset$. However, $\langle a, b \rangle \cap \langle c, p \rangle \subseteq [a, b] \cap [c, p]$ and the latter is either empty or in X. Thus, $\langle a, b \rangle \cap \langle c, p \rangle \in X$. But this contradicts the fact that X is a cap. $\square$

THEOREM 5.5 *If X is a Hermitian cap of index 3 in $\Pi = \mathbf{PG}(N, q)$, then $N = 15$ and X is projectively equivalent to the Hermitian Veronesean of index 3.*

Proof: By the main theorem it suffices to prove that $N = 15$. Assume to the contrary that $N < 15$. Let Y be a hyperplane of $(X, \mathcal{X})$. Then, as we have previously shown, Y is a Hermitian cap of index 2 in $\langle Y \rangle$. Then by Theorem 4.1 we know that $\dim\langle Y \rangle = 8$ and Y is projectively equivalent to the Hermitian Veronesean of index 2. Let $x \in X \setminus Y$, so that, by Lemma 5.4, we have $x \notin \langle Y \rangle$. Since $N < 15$ we must have $T(x) \cap \langle Y \rangle \neq \emptyset$. By the definition of $T(x)$, there is some $z \in Y$ such that $T_x([x, z]) \cap \langle Y \rangle \neq \emptyset$. Let a be a point in $T_x([x, z]) \cap \langle Y \rangle$. Suppose $X(\langle z, a \rangle) := X \cap \langle z, a \rangle \neq \{z\}$. Then there is a unique second point $z' \in X(\langle z, a \rangle)$. But then $z' \in Y$. Since Y is a subspace of $(X, \mathcal{X})$ it follows that $X([z, z']) = X([z, x]) \subseteq Y$ which contradicts $x \notin Y$. Therefore $X(\langle z, a \rangle) = \{z\}$ and, consequently, $a \in T_z([z, x])$. It follows from this that $\dim(T(z) \cap \langle Y \rangle) > 4$. Now let $\xi \in \Xi(Y)$, $z \notin \xi$. Then by a dimension argument it follows that $T(z) \cap \xi \neq \emptyset$ which contradicts Lemma 3.4. Thus, we cannot have $N < 15$ and then by Theorem 5.3, X is projectively equivalent to the Hermitian Veronesean of index 3. $\square$

For a subspace Y of $(X, \mathcal{X})$, we say that Y is *full* if $X(\langle Y \rangle) = Y$.

LEMMA 5.6 *If X is a Hermitian cap of index r in $\mathbf{PG}(N, q)$ and $N < (r+1)^2 - 1$, then some hyperplane of X is not full.*

Proof: We prove the result by induction on r. Let Y be a hyperplane of $(X, \mathcal{X})$. If $X(\langle Y \rangle) \neq Y$, then we are done; so assume that Y is full. By Lemma 2.6, Y is a Hermitian cap in $\langle Y \rangle$ of index $r-1$. Suppose $\dim\langle Y \rangle < r^2 - 1$. Then by induction there is a hyperplane Z of Y and a point $y \in Y$, $y \notin Z$, such that $y \in \langle Z \rangle$. Now let $x \in X$, $x \notin Y$. Since Y is full, $x \notin \langle Y \rangle$. Now let Y' be the hyperplane of $(X, \mathcal{X})$ spanned by Z and x. Since $x \notin Y$, $Y' \neq Y$ and then $Y' \cap Y = Z$. Thus, $y \notin Y'$. However, $y \in \langle Z \rangle \subseteq \langle Y' \rangle$ and therefore Y' is not full.

Thus, we may assume that $\dim\langle Y \rangle = r^2 - 1$ and so by Theorem 5.3, Y is projectively equivalent to the Hermitian Veronesean in $\langle Y \rangle$. Let $x \in X$, $x \notin Y$. Since $N < (r+1)^2 - 1$ it must be the case that $T(x) \cap \langle Y \rangle \neq \emptyset$. Let a be a point in $T(x) \cap \langle Y \rangle$. Clearly $a \neq x$ since $x \notin \langle Y \rangle$. Then there is a unique $y \in Y$ such that $a \in T_x([x, y])$. As in the proof of Theorem 5.5 we can assume that $X(\langle y, a \rangle) = \{y\}$ and $a \in T_y([x, y])$. Then $\dim(T(y) \cap \langle Y \rangle) > 2r - 2$. Now let Z be a hyperplane of Y, with $y \notin Z$. It follows from Lemma 5.2 that $\langle Y \rangle$ is generated by $\langle Z \rangle$ and the tangent space U of the Hermitian cap Y at some $u \in Y \setminus Z$. Hence $\dim\langle Z \rangle + 2r - 2 + 1 \geq r^2 - 1$, and so $\dim\langle Z \rangle = (r-1)^2 - 1$ and again by Theorem 5.3, Z is projectively equivalent to the Hermitian Veronesean of index $r - 2$ in $\langle Z \rangle$. Now by a dimension argument it follows that $T(y) \cap \langle Z \rangle \neq \emptyset$. On the other hand $\langle Z \rangle \cap U = \emptyset$ for in the contrary case $\dim\langle Y \rangle$ will be less than $r^2 - 1$. Let b be a point in $T(y) \cap \langle Z \rangle$. By the definition of $T(y)$ there is a $z \in X \setminus Y$ such that $b \in [y, z]$. We obtain $X([y, z]) \cap Y = \{y\}$.

Now by the structure of an ovoid $X([y, z])$ there are points $a_1, a_2 \in X([y, z])$ such that b is on the line $\langle a_1, a_2 \rangle$. Also, as noted just before Lemma 5.4, there are (not necessarily distinct) points b_i, $1 \leq i \leq r-1$, such that $b \in \langle b_i \,|\, 1 \leq i \leq r-1 \rangle$. But then $b \in \langle a_1, a_2 \rangle \cap \langle b_i \,|\, 1 \leq i \leq r-1 \rangle$ which implies that $a_2 \in \langle a_1, b_1, \ldots, b_{r-1} \rangle$. We next prove that the subspace of $(X, \mathcal{X})$ spanned by $a_1, b_1, \ldots, b_{r-1}$ is contained in a hyperplane Y' containing Z for which $a_2 \notin Y'$. Clearly the subspace of $(X, \mathcal{X})$ spanned by $b_1, \ldots, b_{r-1}$ is contained in Z. Now the subspace spanned by Z and a_1, a_2 is the same as the subspace spanned by Z, y and z. Since $y \notin Z$, the subspace spanned by Z and y is Y and then the subspace of $(X, \mathcal{X})$ spanned by Y and z is all of X since $z \notin Y$. It now follows that the subspace of $(X, \mathcal{X})$ spanned by Z and a_1 is a hyperplane Y' of $(X, \mathcal{X})$ which does not contain a_2. However, $\langle Y' \rangle \supseteq \langle Z, a_1 \rangle \supseteq \langle b_1, \ldots, b_{r-1}, a_1 \rangle$ and therefore $a_2 \in \langle Y' \rangle$. We conclude that Y' is not full. □

COROLLARY 5.7 *If X is a Hermitian cap of index r in $\mathbf{PG}(N, q)$ and every hyperplane is full, then $N = (r+1)^2 - 1$ and consequently X is projectively equivalent to the Hermitian Veronesean of index r.*

Proof: By Lemma 5.6, if $N < (r+1)^2 - 1$, then there is a non-full hyperplane, contrary to the assumption. Thus, $N = (r+1)^2 - 1$ and then by Theorem 5.3, X is projectively equivalent to the Hermitian Veronesean of index r in $\mathbf{PG}(r^2+2r, q)$. $\square$

6 Hermitian Caps of index $r > 3$ in $\mathbf{PG}(N, q)$, $N < r^2 + 2r$

We are now ready to prove our main result Theorem 1.3.

So we are given some Hermitian cap X of index r in $\mathbf{PG}(N, q)$. We may assume that $r > 3$ and $N < r^2 + 2r$ by Theorem 4.1, Theorem 5.3 and Theorem 5.5. By Lemma 5.6, there is a hyperplane Y_0 of $(X, \mathcal{X})$ which is not full. Let $x \in (X \setminus Y_0) \cap \langle Y_0 \rangle$. We can now pick a set of q hyperplanes $Y_1, Y_2, \ldots, Y_q$ of $(X\mathcal{X})$ neither of which contains x, which all share the same subspace Y of index $r-2$ with Y_0, and such that the union $Y_0 \cup Y_1 \cup \ldots \cup Y_q$ is a degenerate Hermitian variety $\mathcal{H}$ of $(X, \mathcal{X})$.

We claim that $\langle \mathcal{H} \rangle = \mathbf{PG}(N, q)$. Indeed, let $y \in X \setminus \mathcal{H}$ be arbitrary. Choose any plane Z in $(X, \mathcal{X})$ containing x and y and such that Y and Z generate $(X, \mathcal{X})$. The corresponding Hermitian cap of index 2 is a Hermitian Veronesean; the intersection $\mathcal{H} \cap Z$ is a Hermitian variety in the subspace Z of $(X, \mathcal{X})$ and hence spans a 7-dimensional subspace of $\langle Z \rangle$. As we are in the case of index 2, we necessarily have $\langle \mathcal{H} \cap Z \rangle \cap Z = \mathcal{H} \cap Z$, and so we easily deduce that $\langle x, \mathcal{H} \cap Z \rangle = \langle Z \rangle = \langle x, y, \mathcal{H} \cap Z \rangle$. Hence $y \in \langle x, \mathcal{H} \rangle$ and the claim is proved.

We now embed $\mathbf{PG}(N, q)$ as a hyperplane in $\mathbf{PG}(N+1, q)$ and choose arbitrary but distinct points $c, x^\theta \in \mathbf{PG}(N+1, q) \setminus \mathbf{PG}(N, q)$ such that c, x, x^θ are collinear. For any point $y \in \mathcal{H}$, we define the point $y^\theta := y$. If $y \in X \setminus \mathcal{H}$, $y \neq x$, then we define y^θ as follows. It follows easily from Lemma 2.6 and the proof of Theorem 4.1 that the ovoid $O_y := X([x, y])$ is an elliptic quadric and that $C_y := O_y \cap \mathcal{H}$ is a plane intersection of O_y, not containing x. If $|C_y| = q + 1$, then we define y^θ as the intersection of cy with the space $\langle C_y, x^\theta \rangle$. If $|C_y| = 1$ — say $C_y = \{t\}$ — then we define y^θ as the intersection of cy with the space $\langle T_t([x, y]), x^\theta \rangle$. In both cases y^θ is well defined since we are taking intersections of the line cy with 3-spaces not containing c in a 4-space $\langle c, [x, y] \rangle$. We now define $X^\theta = \{y^\theta \mid y \in X\}$.

We first claim that $\langle X^\theta \rangle = \mathbf{PG}(N+1, q)$. Indeed, $\langle X^\theta \rangle \supseteq \langle x^\theta, \mathcal{H} \rangle = \mathbf{PG}(N+1, q)$. The claim follows.

Next we claim that X^θ has the natural structure of a Hermitian cap. Indeed, let L be an arbitrary line of $(X, \mathcal{X})$. We choose a solid S in $(X, \mathcal{X})$ containing L and x, and such that S and Y generate $(X, \mathcal{X})$. Then, by our construction, every point y^θ, with $y \in L$, is contained in the 15-dimensional space U of $\mathbf{PG}(N+1, q)$ spanned by $\langle S \cap \mathcal{H} \rangle$ and x^θ. Hence θ is the restriction of a projectivity (in fact a perspectivity) between the two 15-spaces $\langle S \rangle$ and U. Hence $L^\theta := \{y^\theta \,|\, y \in L\}$ is an elliptic quadric and Axiom (H1) is satisfied, with the obvious set $\mathcal{X}^\theta = \{X(\xi)^\theta \,|\, \xi \in \Xi\}$ as set of ovoids. As none of the spaces generated by the elements of $\mathcal{X}^\theta$ contains c, it follows that Axiom (H2) is satisfied.

Now let $z \in X \setminus \mathcal{H}$ and reverse the roles of x and z. In other words, define $y^{\theta'}$ with respect to z and z^θ similar to the definition of y^θ with respect to x and x^θ. By considering a solid S of $(X, \mathcal{X})$ through x, y, z such that S and Y generate $(X, \mathcal{X})$, we see similarly as in the previous paragraph that $y^\theta = y^{\theta'}$.

Now let V be any plane of $(X, \mathcal{X})$, with V not contained in $\mathcal{H}$. By including V in a solid as in the previous paragraphs, and by the fact that θ can be defined starting from any point of $V \setminus \mathcal{H}$, we first see that $\dim\langle V^\theta \rangle = 8$ (and hence $c \notin \langle V^\theta \rangle$, otherwise $\dim\langle V \rangle < 8$). This now implies that θ defines a perspectivity from $\langle V \rangle$ onto $\langle V^\theta \rangle$. Hence the sets V and V^θ are projectively equivalent as subsets of $\mathbf{PG}(N+1, q)$. It follows that Axiom (H3) is satisfied for the elements (points and elliptic spaces) of V^θ. Since V was arbitrary, we conclude that (H3) globally holds in X^θ.

An easy induction on N, together with Theorem 5.3, completes the proof of Theorem 1.3. $\square$

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