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# FULL EMBEDDINGS OF THE FINITE DUAL SPLIT CAYLEY HEXAGONS

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In this paper we show that, for each prime power $q$, given an embedding of the finite dual split Cayley hexagon $\mathsf{H}(q)^D$ in the $d$-dimensional projective space $\mathbf{PG}(d,q)$, $d$ satisfies $d \leq 13$. Moreover, if $d=13$, then the embedding is unique and we give an explicit description.

## 1. Introduction and Statement of the Main Result

The main geometric constructions of (finite) geometries $\Gamma$ – such as generalized polygons, partial geometries, semipartial geometries, extended quadrangles – are inside a projective space $\mathbf{PG}(d,q)$. In many cases, points of $\Gamma$ are just points of $\mathbf{PG}(d,q)$, and lines of $\Gamma$ are (subsets of) lines of $\mathbf{PG}(d,q)$. These representations are very useful to detect geometric properties, or to recognize certain specific examples in different circumstances. In the latter case, one additionally needs characterization results for these geometric representations. These results characterize either the geometry amongst other geometries satisfying a same set of axioms, or only the very geometric representation. We will see some examples below, related to our Main Result.

Let us first get down to precise definitions. Let $\Gamma$ be a point-line geometry. We say that $\Gamma$ is *(fully) embedded* in $\mathbf{PG}(d,q)$, for some prime power $q$, if the point set of $\Gamma$ is a subset of the point set of $\mathbf{PG}(d,q)$ generating $\mathbf{PG}(d,q)$, and if the set of points of every line of $\Gamma$ is precisely the set of points of a line in $\mathbf{PG}(d,q)$ (hence every line of $\Gamma$ bears exactly $q+1$ points).

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Now let $\Gamma$ be a generalized polygon (i.e., a point-line geometry with the property that the girth of the incidence graph is twice the diameter $n$ of that graph; in this case we also say that $\Gamma$ is a *generalized $n$-gon*), (fully) embedded in $\mathbf{PG}(d,q)$. If for any point $x$ of $\Gamma$, all points of $\Gamma$ collinear in $\Gamma$ with $x$ are contained in a plane of $\mathbf{PG}(d,q)$, then we say that the embedding is *flat*. Two elements of $\Gamma$ are called *opposite* if they are at distance $n$ in the incidence graph. If for any point $y$ of $\Gamma$, the set of all points not opposite $y$ in $\Gamma$ does not generate $\mathbf{PG}(d,q)$, then we say that the embedding is *polarized* (we used the notion "weak" in earlier papers, following a definition of Lefèvre-Perscy for embedded polar spaces). Note that then necessarily $n$ must be even.

Generalized polygons for which each element is incident with at least three other elements have the property that each line is incident with a constant number of points, say $s+1$, and each point is incident with a constant number of lines, say $t+1$. We say that the generalized polygon has order $(s,t)$; if $s=t$ we say that the polygon has order $s$.

The present paper deals with embedded generalized hexagons, in particular with embeddings in $\mathbf{PG}(d,q)$ of the dual $\mathsf{H}(q)^D$ of the so-called split Cayely hexagons (see next paragraph for a definition). We refer to [4,6] for characterization theorems of the natural embeddings of $\mathsf{H}(q)$. For results on quadrangles, see Chapter 8 of [13].

We now give an explicit construction of $\mathsf{H}(q)$, due to Tits [11], see also [13]. In fact, we define $\mathsf{H}(q)$, or more generally, $\mathsf{H}(\mathbb{K})$, for any commutative field $\mathbb{K}$, as a flat and polarized embedding in $\mathbf{PG}(6,q)$, or more generally, $\mathbf{PG}(6,\mathbb{K})$. We will refer to this embedding as the *natural one*, or as the *standard representation*.

So let $\mathbb{K}$ be any commutative field. The points of $\mathsf{H}(\mathbb{K})$ are the points of $\mathbf{PG}(6,\mathbb{K})$ on the quadric $Q(6,\mathbb{K})$ with equation $X_0X_4+X_1X_5+X_2X_6=X_3^2$; the lines are the lines of this quadric whose Grassmann coordinates satisfy the equations

$$\begin{array}{lll} p_{12}=p_{34}, & p_{54}=p_{32}, & p_{20}=p_{35}, \\ p_{65}=p_{30}, & p_{01}=p_{36}, & p_{46}=p_{31}. \end{array}$$

Incidence is the natural one. This representation of $\mathsf{H}(\mathbb{K})$ has been used to derive many useful geometric properties, see for instance [13]. This hexagon can be recognized as the only generalized hexagon of order $q$ admitting a flat and polarized embedding in $\mathbf{PG}(6,q)$, see [5]. Taking the dual of $\mathsf{H}(\mathbb{K})$ , we obtain $\mathsf{H}(\mathbb{K})^D$. Note that the points of $\mathsf{H}(\mathbb{K})^D$ can be identified, via the Grassmannian of all lines of $\mathbf{PG}(6,\mathbb{K})$, with certain points of $\mathbf{PG}(20,\mathbb{K})$. Since in $\mathsf{H}(\mathbb{K})$, the set of lines through a point $x$ is the set of lines through $x$ in $\mathbf{PG}(6,\mathbb{K})$ in a certain plane of $Q(6,\mathbb{K})$ (see [11], 5.2.3), the lines of

$\mathsf{H}(\mathbb{K})^D$ can be identified with certain lines of $\mathbf{PG}(20,\mathbb{K})$. Hence we have a full embedding of $\mathsf{H}(\mathbb{K})^D$ in some subspace of $\mathbf{PG}(20,\mathbb{K})$. As can easily be seen from the above equations, the set of points of $\mathsf{H}(\mathbb{K})^D$ does not generate $\mathbf{PG}(20,\mathbb{K})$. We will show below that they generate a space of dimension at least 13. So we know that there is at least one embedding of $\mathsf{H}(\mathbb{K})^D$ in $\mathbf{PG}(d,\mathbb{K})$, for $d \geq 13$. Our main result is that in the finite case there is, up to a projective transformation, a unique such embedding and that $d = 13$. Furthermore, as a byproduct, we show that this embedding is polarized (without making explicit computations).

**Main Result.** *Suppose that $\mathsf{H}(q)^D$ is embedded in $\mathbf{PG}(d,q)$. Then $d \leq 13$. If $d = 13$, then there is a unique embedding.*

If $q = 2$ then there is the well-known result that the dimension of the universal (projective) embeddings of $\mathsf{H}(2)$ and $\mathsf{H}(2)^D$ is equal to 13, see for instance [14]. Hence, in order to prove our Main Result, we may assume $q > 2$, and that is what we will do below.

**Remark.** Since there is only one conjugacy class of subgroups of $\mathbf{PGL}_{14}(q)$ isomorphic to $G_2(q)$ and acting irreducibly on $\mathbf{PG}(13,q)$, it follows rather easily that there is a unique embedding of $\mathsf{H}(q)^D$ in $\mathbf{PG}(13,q)$ with the property that every automorphism of $\mathsf{H}(q)^D$ is induced by $\mathbf{PGL}_{14}(q)$. Our Main Result now says that this is the only embedding of $\mathsf{H}(q)^D$ at all in $\mathbf{PG}(13,q)$; hence in particular that every representation of $\mathsf{H}(q)$ in $\mathbf{PG}(13,q)$ enjoys a lot of nice geometric and group-theoretic properties. This is in contrast with the situation in $\mathbf{PG}(d,q)$, for $d < 13$. Indeed, projecting the embedding in $\mathbf{PG}(13,q)$ from a suitable subspace, one can produce (full) embeddings of $\mathsf{H}(q)^D$ in, for instance, $\mathbf{PG}(12,q)$ which are not polarized, and which are certainly not obtained from a subgroup of $\mathbf{PGL}_{13}(q)$ isomorphic to $G_2(q)$.

## 2. The examples

Since the construction of the examples is also valid in the infinite case, we treat the general case here.

Let $\mathbb{K}$ be any commutative field, with $|\mathbb{K}| > 2$, let $\Gamma$ be isomorphic to $\mathsf{H}(\mathbb{K})^D$, and let $\Gamma$ be constructed on the Grassmannian of all lines of $\mathbf{PG}(6,\mathbb{K})$ as described in the introduction.

As already explained in the introduction, the points of $\Gamma$ are the points of $\mathbf{PG}(20,\mathbb{K})$ which lie on the Grassmannian of all lines of $\mathbf{PG}(6,\mathbb{K})$, correspond to lines of the quadric $Q(6,q)$ and satisfy the six linear equations stated in the introduction. In order to show that the embedding we obtain is inside a $\mathbf{PG}(d,\mathbb{K})$ with $d \geq 13$, we have to find 14 points of $\Gamma$ that are linearly independent in $\mathbf{PG}(20,\mathbb{K})$. We simply list them by writing down two points of each corresponding line of $\mathsf{H}(\mathbb{K})$ in $\mathbf{PG}(6,\mathbb{K})$. Let $a \in \mathbb{K}$ with $0 \neq a \neq 1$, and consider the points

| | |
|---|---|
| $(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)$ | $(1,0,0,0,0,0,0),(0,0,0,0,0,1,0)$ |
| $(0,1,0,0,0,0,0),(0,0,0,0,0,0,1)$ | $(0,0,1,0,0,0,0),(0,0,0,0,0,1,0)$ |
| $(0,1,0,0,0,0,0),(0,0,0,0,1,0,0)$ | $(0,0,1,0,0,0,0),(0,0,0,0,1,0,0)$ |
| $(1,0,1,0,0,0,0),(0,0,0,0,1,0,-1)$ | $(0,1,1,0,0,0,0),(0,0,0,0,0,1,-1)$ |
| $(1,0,0,0,0,-1,1),(1,1,1,-1,0,0,1)$ | $(0,1,0,0,0,0,-1),(1,0,0,-1,1,0,0)$ |
| $(0,1,0,-1,0,1,0),(0,0,-1,0,1,0,0)$ | $(0,1,0,-a,0,a^2,0),(0,0,-a,0,1,0,0)$ |
| $(1,0,0,0,0,0,1),(0,1,0,-1,0,1,0)$ | $(a,0,0,0,0,0,1),(0,1,0,-a,0,a^2,0)$. |

An easy elementary calculation now shows that the images on the Grassmannian of the 14 lines defined by these pairs of points generate a projective space of dimension 13.

Note also that, since the full group of automorphisms of $\mathsf{H}(\mathbb{K})$ in $\mathbf{PG}(6,\mathbb{K})$ is induced by the stabilizer of $Q(6,\mathbb{K})$ in $\mathbf{P}\Gamma L_7(\mathbb{K})$, the full automorphism group of $\Gamma$ is induced by $\mathbf{P}\Gamma L_{21}(\mathbb{K})$.

## 3. Some known results

We start with some properties of $\mathsf{H}(q)^D$, and we introduce some notation. All properties we mention are proved in [13], possibly dualized for $\mathsf{H}(q)$. So we put $\Gamma = \mathsf{H}(q)^D$. For any element $x$ (a point or a line) of $\Gamma$, and any positive integer $j \leq n$, we denote by $\Gamma_j(x)$ the set of elements of $\Gamma$ at distance $j$ from $x$ (measured in the incidence graph of $\Gamma$; this is standard notation for any generalized polygon $\Gamma$).

The hexagon $\mathsf{H}(q)^D$ is *line-distance-i-regular*, for $i = 2,3$. That means that, given any line $L$ of $\mathsf{H}(q)^D$, the geometry $(\Gamma_i(L), \{\Gamma_i(L) \cap \Gamma_{6-i}(M) \,|\, M \text{ is opposite } L\}, \in)$ is a partial linear space (two distinct points are contained in at most one common line). The hexagon $\mathsf{H}(q)^D$ is also point-distance-3-regular, with a similar definition.

By the line-distance-2-regularity of $\mathsf{H}(q)^D$, any two lines $L_1, L_2$ at distance 4 from each other determine a unique set of $q+1$ lines (called a *trace*) in $\Gamma_2(L)$, where $L$ meets both $L_1$ and $L_2$. We denote that trace by $[L_1, L_2]$. It is precisely the set of lines of $\Gamma_2(L)$ at distance $\leq 4$ from every line at distance 4 from both $L_1$ and $L_2$.

By the line-distance-3-regularity of $\mathsf{H}(q)^D$, any two opposite lines $L_1, L_2$ determine a unique set of $q+1$ lines in $\Gamma_3(x)$, where $x$ is a point at distance 3 from both $L_1$ and $L_2$. That set is precisely the set of lines at distance 3 from all points that are at distance 3 from both $L_1$ and $L_2$. We call that set the *(line) regulus* through $L_1$ and $L_2$ and denote it by $\mathcal{R}(L_1,L_2)$. Similarly for opposite points $x_1$ and $x_2$, we have a *(point) regulus* $\mathcal{R}(x_1,x_2)$. Every regulus $\mathcal{R}$ has a *complementary regulus* $\mathcal{R}^\perp$ obtained from $\mathcal{R}$ by taking all elements of $\Gamma$ at distance 3 from all elements of $\mathcal{R}$.

For a point $x$ of any generalized hexagon $\Gamma$, we write $x^\perp = \Gamma_2(x) \cup \{x\}$ and $x^{\perp\!\!\perp} = \Gamma_4(x) \cup x^\perp$. If $\Gamma = \mathsf{H}(q)$, then, considering its standard representation, the set $x^\perp$ is the point set of a projective plane of $\mathbf{PG}(6,q)$, for any point $x$. Also, the set $x^{\perp\!\!\perp}$ is the intersection of the quadric $Q(6,q)$ with the tangent hyperplane of $Q(6,q)$ at $x$.

Let $Q^+(5,q)$ be a subquadric of $Q(6,q)$ of hyperbolic type obtained by intersecting $Q(6,q)$ with a hyperplane. Then the points of $Q^+(5,q)$ which are incident with at least two distinct lines of $\mathsf{H}(q)$ that are entirely contained in $Q^+(5,q)$ form the point set of a *thin subhexagon* of order $(1,q)$ of $\mathsf{H}(q)$. This thin subhexagon is determined by any pair of opposite points it contains. Dually, let $\Gamma = \mathsf{H}(q)^D$. Then every pair $\{L,M\}$ of opposite lines determines a unique thin subhexagon $\Gamma(L,M)$ of order $(q,1)$ which contains $L$ and $M$. The subhexagon $\Gamma(L,M)$ is the flag complex of a Desarguesian projective plane of order $q$. All polarized full embeddings of hexagons of order $(q,1)$ have been determined, see [6–10]. For the convenience of the reader, we summarize what we will need from those papers.

Throughout this section, let $\Gamma$ be a hexagon of order $(q,1)$ and suppose that $\Gamma$ is the dual of the double of the Desarguesian projective plane $\mathbf{PG}(2,q)$.

We introduce some notation for the first proposition. If $x_1,x_2,x_3,x_4$ are four collinear points of $\mathbf{PG}(k,q)$, with $|\{x_1,x_2,x_3,x_4\}| \geq 3$, then we denote by $(x_1,x_2;x_3,x_4)$ the usual cross-ratio given by

$$\frac{r_3 - r_1}{r_3 - r_2} : \frac{r_4 - r_1}{r_4 - r_2},$$

where the $r_i$ are non-homogeneous coordinates of the $x_i$ on the line through $x_1,x_2,x_3,x_4$.

**Proposition 3.1.** *If $\Gamma$ is embedded in $\mathbf{PG}(k,q)$, then $k \leq 8$. Also, if $k=8$, then the embedding is polarized and has the following characteristic property.*

($*$) *There exists a non-trivial automorphism $\sigma$ of $\mathbf{GF}(q)$ such that for any pair of opposite lines $L,M$ of $\Gamma$, where $L$ corresponds to a point of $\mathbf{PG}(2,q)$, and hence $M$ corresponds to a line of $\mathbf{PG}(2,q)$, and every* *quadruple* $(x_1,x_2,x_3,x_4)$ *of distinct points of* $\Gamma$ *on* $L$*, one has the following relation between cross-ratios:*

$$(x_1,x_2;x_3,x_4)^\sigma = (\mathrm{proj}_M x_1, \mathrm{proj}_M x_2; \mathrm{proj}_M x_3, \mathrm{proj}_M x_4).$$

*The automorphism* $\sigma$ *is, up to taking inverse, completely determined by a point regulus.*

*If* $k=7$*, then the embedding is polarized whenever the set of points at distance* $3$ *in* $\Gamma$ *from every two opposite lines is contained in a plane; in such a case the plane is uniquely determined by the two lines. Further, this set of points is a conic in that plane. Also, still if* $k=7$*, for any pair of opposite lines, the set of points at distance* $3$ *from both lines is never contained in a line of* $\mathbf{PG}(k,q)$*.*

We remark that the last part of Proposition 3.1 is mainly contained in the proof of Theorem 1.2 of [10].

## 4. Proof of the Main Result for $q \neq 2$

We show our Main Result by a sequence of lemmas. Throughout we assume that $\Gamma \cong \mathsf{H}(q)^D$ is embedded in $\mathbf{PG}(d,q)$, with $d \geq 13$. We want to show that $d=13$ and that the embedding is uniquely determined, up to a projective transformation in $\mathbf{PG}(13,q)$.

**Lemma 4.1.** *Let* $L_1$ *and* $L_1'$ *be two arbitrary opposite lines of* $\Gamma$*. Let* $L_2$ *and* $L_2'$ *be two distinct lines of the regulus* $\mathcal{R}(L_1,L_1')$ *both distinct from* $L_1$ *and from* $L_1'$*. Then the union of the point sets of the three thin subhexagons* $\Gamma(L_1,L_1')$*,* $\Gamma(L_2,L_2')$ *and* $\Gamma(L_1,L_2)$ *generates* $\mathbf{PG}(d,q)$*. Also, the union of the point sets of the two thin subhexagons* $\Gamma(L_1,L_1')$ *and* $\Gamma(L_2,L_2')$ *generates a space of dimension at least* $d-1$*.*

We show that the union of the point sets of $\Gamma(L_1,L_1')$ and $\Gamma(L_2,L_2')$ linearly generates a geometric hyperplane of $\Gamma$ (i.e., a set of points intersecting any line in either one point or all points). Let $\mathcal{S}$ be the subspace of the hexagon generated by that union.

We consider the generalized hexagon $\mathsf{H}(q)$ in its standard representation in 6-dimensional projective space $\mathbf{PG}(6,q)$. The points of $\mathsf{H}(q)$ are all points of a non-singular quadric $Q(6,q)$. Let $\ell_i$ and $\ell_i'$ be the points of $\mathsf{H}(q)$ corresponding to the lines $L_i$ and $L_i'$, respectively, $i=1,2$. With $\Gamma(L_i,L_i')$ there corresponds a hyperplane $W_i$ of $\mathbf{PG}(6,q)$, $i=1,2$ (namely, the hyperplane spanned by the elements of any apartment through the points $\ell_i$ and $\ell_i'$); the set of points of $\Gamma(L_i,L_i')$ corresponds to the set of lines of $\mathsf{H}(q)$ in $W_i$. Then

$W_i$ meets $Q(6,q)$ in a hyperbolic quadric $Q^+_{(i)}(5,q)$, $i=1,2$. We claim that the intersection of these two hyperbolic quadrics is a non-singular quadric $Q(4,q)$ in some 4-dimensional subspace $V$. Indeed, it is easy to see that $\Gamma(L_1,L_1')$ and $\Gamma(L_2,L_2')$ have exactly $q+1$ points in common, hence the spaces $W_1$ and $W_2$ have $q+1$ lines of $\mathsf{H}(q)$ in common; these $q+1$ lines are contained in a unique 3-dimensional subspace $U$ (and they form a system of generators of a non-singular ruled quadric $Q^+(3,q)$ in $U$). Hence the intersection of $Q^+_{(1)}(5,q)$ and $Q^+_{(2)}(5,q)$ is either non-singular, or a cone with basis $Q^+(3,q)$ and with some vertex $v$. Now $v$ is a point of $\mathsf{H}(q)$ collinear on $Q(6,q)$ with all point of $Q^+(3,q)$. Hence $v$ is a point of $\mathsf{H}(q)$ at distance $\leq 4$ in $\mathsf{H}(q)$ of every point of $Q^+(3,q)$. So $v$ must be at distance 3 from every line of $\mathsf{H}(q)$ in $U$. This implies that only the points of the point regulus containing $\ell_1$ and $\ell_1'$ are valid candidates for $v$. But, obviously, none of these points belongs to both $Q^+_{(1)}(5,q)$ and $Q^+_{(2)}(5,q)$. The claim follows.

Now, let $x$ first be a point of $Q(4,q)$ not in $U$. Then $x$ is incident with some line $M_i$ of $\mathsf{H}(q)$ in $W_i$, $i=1,2$ (the line $M_i$ is the intersection of $W_i$ with the plane of $Q(6,q)$ consisting of all points of $\mathsf{H}(q)$ collinear with $x$). Since $x \notin U$, we have $M_1 \neq M_2$. Let $X$, $m_1,m_2$ be the line and points in $\Gamma$ corresponding to $x$, $M_1,M_2$, respectively. Then $X$ contains the points $m_1$ and $m_2$ and hence all points incident with $X$ belong to $\mathcal{S}$.

Secondly, let $x$ be a point of $Q(4,q)$ in $U$ and let $L_x$ be the unique line of $\mathsf{H}(q)$ in $U$ incident with $x$. Consider any line $Z$ of $Q(4,q)$ through $x$ and not contained in $U$ (there are $q-1$ such lines). Let $x_1$ and $x_2$ be two points of $Z$ distinct from $x$. Let $y$ be the unique point of $\mathsf{H}(q)$ collinear in $\mathsf{H}(q)$ with $x,x_1,x_2$. Let the lines $X,X_1,X_2,Y$ of $\Gamma$ correspond with $x,x_1,x_2,y$, respectively, and let the point $\ell_x$ of $\Gamma$ correspond with $L_x$. We have already shown that all points of $X_i$ belong to $\mathcal{S}$, $i=1,2$. Since $Y$ meets the disjoint lines $X_1$ and $X_2$, we see that all points of $Y$ belong to $\mathcal{S}$. The line $X$ meets $Y$ in a point different from $\ell_x$, and hence is contained in $\mathcal{S}$ (and so is every point incident with $X$).

Consequently $\mathcal{S}$ contains all points which correspond to lines of $\mathsf{H}(q)$ having non-empty intersection with $V$. We show that this set of points is indeed a geometric hyperplane. Therefore, let $L$ be any line of $\Gamma$, and let $\ell$ be the corresponding point of $\mathsf{H}(q)$. We have to show that $\ell$ is on either 1 or $q+1$ lines of $\mathsf{H}(q)$ meeting $V$. But this follows immediately by considering the plane $\ell^\perp$.

Hence $\mathcal{S}$ is a geometric hyperplane. The complement of $\mathcal{S}$ is connected by [1]. We claim that there exists a point $p$ in $\Gamma(L_1,L_2)$ not contained in $\mathcal{S}$. Indeed, we can certainly select a point $p$ in $\Gamma(L_1,L_2)$ which is not collinear with any member of the point regulus complementary to the line regulus

$\mathcal{R}(L_1,L_1')$. If $P$ is the line in $\mathsf{H}(q)$ corresponding to $p$, and if $p$ were contained in $\mathcal{S}$, then $P$ would meet $V$ in a point off $Q^+(3,q)$, hence the 5-dimensional subspace corresponding to $\Gamma(L_1,L_2)$ would contain $Q(4,q)$, consequently it would meet $Q^+_{(1)}(5,q)$ in the non-singular quadric $Q(4,q)$, contradicting the fact that this intersection contains a plane, namely the plane determined by $\ell_1^\perp$. The claim is proved. Now we consider such a point $p$ and we see that $\mathcal{S}\cup\{p\}$ generates the whole point set of $\Gamma$ as a subspace. This proves the lemma. ∎

**Proposition 4.2.** *We have $d=13$ and for every pair of opposite lines $L,M$, the thin subhexagon $\Gamma(L,M)$ is fully embedded in a 7-dimensional subspace $\mathbf{PG}(7,q)$ and this embedding is polarized.*

**Proof.** Let $L_0$ and $L_1$ be two arbitrary opposite lines of $\Gamma$. Let $\mathcal{R}(L_0,L_1)=\{L_0,L_1,\ldots,L_q\}$. Let $\{x_0,\ldots,x_q\}$ be the complementary regulus. Suppose the thin subhexagon $\Gamma(L_0,L_1)$ generates the subspace $\mathbf{PG}(k,q)$. We first show that $k\leq 7$. Suppose by way of contradiction that $k\geq 8$; then $k=8$ by Proposition 3.1. Let $\sigma$ be the non-trivial automorphism of $\mathbf{GF}(q)$ corresponding (by Proposition 3.1) to the embedding of $\Gamma(L_0,L_1)$ in $\mathbf{PG}(8,q)$ (where we assume that $L_0$ corresponds with a point of the underlying projective plane). Let $G=(\mathcal{R}(L_0,L_1),E)$ be the undirected graph with the set $E$ of edges defined by: $\{L_i,L_j\}\in E$ if $\Gamma(L_i,L_j)$ is contained in a 7-dimensional subspace of $\mathbf{PG}(d,q)$, $i,j\in\{0,1,\ldots,q\}$, $i\neq j$. If $\{L_i,L_j\},\{L_{i'},L_{j'}\}\in E$, with $|\{i,j,i',j'\}|=4$, then the union of the point sets of $\Gamma(L_i,L_j)$ and $\Gamma(L_{i'},L_{j'})$ generate a space of dimension at most $7+7-3=11$. By Lemma 4.1, $11\geq d-1$, a contradiction. Since $\sigma$ is non-trivial, we have $q\geq 4$ and hence we can find a coclique of size 3 in $G$. Suppose, without loss of generality, that the vertices of that coclique are $L_0,L_1,L_2$. Consider an arbitrary ordered quadruple of pairwise distinct points on $L_0$ for which the cross-ratio $r$ generates the multiplicative group of $\mathbf{GF}(q)$. Then $r^\sigma\neq r$ is the cross-ratio of the projection on $L_1$ of that quadruple. Let $s$ be the cross-ratio of the projection of that quadruple onto $L_2$. Considering $\Gamma(L_0,L_2)$, we have either $s=r^\sigma$ or $r=s^\sigma$ (by Proposition 3.1). Likewise, considering $\Gamma(L_1,L_2)$, we have either $r^\sigma=s^\sigma$ or $r^{\sigma\sigma}=s$. Clearly the combinations (1) $s=r^\sigma=s^\sigma$, (2) $r^\sigma=s=r^{\sigma\sigma}$ and (3) $r=s^\sigma=r^\sigma$ lead to either $s=s^\sigma$ or $r=r^\sigma$, a contradiction since both $r$ and $s$ generate the multiplicative group of $\mathbf{GF}(q)$. Hence $r=s^\sigma=r^{\sigma^3}$, implying $\sigma^3=1$. If we define the directed graph $G'=(\mathcal{R}(L_0,L_1),E')$ by $(L_i,L_j)\in E'$ if $\Gamma(L_i,L_j)$ generates an 8-dimensional subspace of $\mathbf{PG}(d,q)$ and the cross-ratio $s$ of an ordered quadruple of four different points on $L_j$ is equal to the image under $\sigma$ of the cross-ratio of the projection of that quadruple onto $L_i$, $i,j\in\{0,1,\ldots,q\}$, $i\neq j$, then we have just shown that, if $G'$ contains a triangle containing the mutually adjacent vertices $L_i,L_j,L_m$, and $(L_i,L_j)\in E'$, then

$(L_j, L_m), (L_m, L_i) \in E'$. But now $\sigma^3 = 1$ implies that $q \geq 8$. Hence we can find a coclique of at least four points in $G$, and this implies a clique of four points in $G'$, which we may assume to contain $L_0, L_1$. Let $L_i, L_j$ be the other two vertices of the clique. Since $(L_0, L_1) \in E'$, we have $(L_i, L_0), (L_j, L_0) \in E'$. But $(L_i, L_0) \in E'$ implies $(L_0, L_j) \in E'$, a contradiction. Hence we have shown that $k \leq 7$. Similarly, every $\Gamma(L_i, L_j)$, $L_i, L_j \in \mathcal{R}(L_0, L_1)$, $i \neq j$, is contained in a 7-dimensional subspace of $\mathbf{PG}(d, q)$.

Now we show that $k = 7$. Suppose by way of contradiction that $k \leq 6$. If also $\Gamma(L_2, L_3)$ is contained in a 6-dimensional space, then the union of the point sets of $\Gamma(L_0, L_1)$ and $\Gamma(L_2, L_3)$ is contained in a space of dimension $6 + 6 - 1 = 11$, contradicting Lemma 4.1. Hence $\Gamma(L_2, L_3)$ generates a 7-dimensional space and by Proposition 3.1 the set $\{x_0, \ldots, x_q\}$ generates a subspace of dimension at least 2. Hence the union of the point sets of $\Gamma(L_0, L_1)$ and $\Gamma(L_2, L_3)$ is contained in a space of dimension $6+7-2=11$, again contradicting Lemma 4.1. Hence $k=7$. Similarly, $\Gamma(L_2, L_3)$ generates a 7-dimensional subspace of $\mathbf{PG}(d, q)$. If $\{x_0, \ldots, x_q\}$ were not contained in a plane, then $\Gamma(L_0, L_1) \cup \Gamma(L_2, L_3)$ would be contained in a subspace of dimension at most $7+7-3=11$, once again contradicting Lemma 4.1. Hence, by Proposition 3.1, the embedding of $\Gamma(L_0, L_1)$ is polarized and Lemma 4.1 implies that $d \leq 7+7-2+1=13$, and so $d=13$. ∎

Note that, with the notation of the previous proof, this proof implies that the space generated by $\Gamma(L_0, L_1)$ and the one generated by $\Gamma(L_2, L_3)$ intersect in the plane generated by $\{x_0, \ldots, x_q\}$. Also, the space generated by $\Gamma(L_0, L_1) \cup \Gamma(L_2, L_3)$ and the one generated by $\Gamma(L_0, L_2)$ intersect in the space generated by $L_0, L_2$ and $\{x_0, \ldots, x_q\}$. Another immediate consequence is that the space generated by the lines incident with $x_0$ and concurrent with $L_0, L_1, L_2, L_3$, respectively, has dimension 4, hence $\Gamma_2(x_0)$ spans a space of dimension at least 4. Also, the plane containing $\{x_0, x_1, \ldots, x_q\}$ has no point in common with $\langle L_0, L_1, L_2, L_3 \rangle$, and the space $\langle L_0, L_1, L_2, L_3 \rangle$ has dimension 7.

We mention two other corollaries of this proposition and the proof of Lemma 4.1.

**Corollary 4.3.** *Let $x$ be any point of $\Gamma$, let $\mathcal{R}$ be a point regulus containing $x$ and let $L_1, \ldots, L_4$ be four different lines of the complementary regulus $\mathcal{R}^\perp$. Then the subspace $H$ of $\mathbf{PG}(13, q)$ generated by the points of $\Gamma(L_1, L_2)$ and of $\Gamma(L_3, L_4)$ is a hyperplane containing all lines of $\Gamma$ through $x$, and also all elements of both $\mathcal{R}$ and $\mathcal{R}^\perp$.*

**Proof.** By the proof of Lemma 4.1, $H$ is either a hyperplane or the whole space $\mathbf{PG}(13, q)$. But the proof of the previous proposition implies that $H$

is at most 12-dimensional. So $H$ is a hyperplane. It meets the point set of $\Gamma$ in a geometric hyperplane $\mathcal{S}$, and we have shown in the proof of Lemma 4.1 that $\mathcal{S}$ contains all lines of $\Gamma$ through $x$, all points of $\mathcal{R}$, and all lines of $\mathcal{R}^\perp$. ∎

**Corollary 4.4.** *Let $L$ be any line of $\Gamma$. Then the set of points $\Gamma_1(L) \cup \Gamma_3(L)$ generates a subspace of $\mathbf{PG}(13,q)$ of dimension 7.*

**Proof.** We first show that $W := \langle \Gamma_1(L) \cup \Gamma_3(L) \rangle$ has dimension at most 7. To that end, let $x$ and $y$ be two different points on $L$, let $X_1, X_2$ be two different lines through $x$ (distinct from $L$) and let $Y$ be any line through $y$, with $Y \neq L$. Furthermore, let $M$ be a line opposite $L$ such that $S := \Gamma_4(M) \cap \Gamma_2(L)$ contains neither $X_1$ nor $X_2$, nor $Y$ (this is possible since $q > 2$). We know that $S$ generates a 4-dimensional space. Hence the space $W'$ generated by $S$ and $X_1, X_2, Y$ has dimension at most 7. We claim that $W' = W$. Obviously $W' \subseteq W$. To show that $W \subseteq W'$, it suffices to show that every element of $\Gamma_2(L)$ is contained in $W'$. By the line-distance-2-regularity of $\Gamma$, the set $[X_i, Y]$ has a unique element in common with $S$, $i = 1, 2$. Hence it meets $W'$ in a 4-dimensional subspace and consequently, $[X_i, Y]$ is contained in $W'$, for $i = 1, 2$. Now it is easy to see that every line $Z$, $Z \neq L$, concurrent with $L$ is contained in a set $[Z, Z']$, with $Z' \in \Gamma_2(L)$, having at least three elements in common with $S \cup [X_1, Y] \cup [X_2, Y]$. Similarly as before, we conclude that $[Z, Z']$ is contained in $W'$. The claim follows.

Now we show that the dimension of $W$ is at least 7. Suppose that $W$ is contained in a 6-dimensional space. Let $M_0$ be a line at distance 4 from $L$, let $N_0$ be the line concurrent with both $L$ and $M_0$, and let $u$ be a point on $M_0$ at distance 5 from $L$. Let $M_1, M_2, M_3$ be three different lines through $u$ opposite $L$. Also, let $\mathcal{R}$ be a point regulus containing $u$ and some point $v$ of $L$. Finally, for $i = 1, 2, 3$, let $N_i$ be the unique line at distance 3 from $v$ and concurrent with $M_i$. Then by Lemma 4.1 the space $\mathbf{PG}(13,q)$ is generated by $W$, $M_0, M_1, M_2, M_3$, $\mathcal{R}$ and $\Gamma(N_1, N_2)$ (indeed, this space contains $\Gamma(N_3, N_0)$, $\Gamma(N_0, N_1)$ and $\Gamma(N_1, N_2)$), which has dimension at most $6 + 6 = 12$, a contradiction.

The corollary is proved. ∎

Our next aim is to show that the set of lines of $\Gamma$ through a fixed point $x$ of $\Gamma$ is contained in a 4-dimensional subspace of $\mathbf{PG}(13,q)$. Corollary 4.3 already says that $\Gamma_1(x)$ is contained in many hyperplanes. Corollary 4.4 says that it is contained in many 7-dimensional spaces. The idea is now to intersect these subspaces until we get a space of dimension 4.

**Lemma 4.5.** *Let $x$ be any point of $\Gamma$, let $\mathcal{R}$ be a point regulus containing $x$ and let $\mathcal{R}^\perp$ be the complementary regulus. Then the subspace $U_{\mathcal{R}}$ of $\mathbf{PG}(13,q)$ generated by $\Gamma_1(x) \cup \mathcal{R} \cup \mathcal{R}^\perp$ has dimension 10.*

**Proof.** Put $\mathcal{R}^\perp =: \{L_0, L_1, \ldots, L_q\}$. Let $H_1$ and $H_2$ be the hyperplanes of $\mathbf{PG}(13,q)$ spanned by the union of the point sets of $\Gamma(L_0,L_1)$ and $\Gamma(L_2,L_3)$, and of $\Gamma(L_0,L_3)$ and $\Gamma(L_1,L_2)$, respectively. By Corollary 4.3, both $H_1$ and $H_2$ contain all elements of $\Gamma_1(x)\cup\mathcal{R}\cup\mathcal{R}^\perp$. Moreover, by the proof of Lemma 4.1, the points of $\Gamma$ in the union $H_1\cup H_2$ linearly generate the point set of $\Gamma$, so $H_1\neq H_2$. Hence $H_1\cap H_2$ is a space of dimension 11 containing all elements of $\Gamma_1(x)$. Now let $i\in\{3,4\}$ and consider the hyperplane $H_i$ spanned by the union of the point sets of $\Gamma(L_0,L_2)$ and $\Gamma(L_1,L_i)$. Suppose that $H_1\cap H_2\subseteq H_i$, $i\in\{3,4\}$. Let $Q^{(j)}(4,q)$ be the non-singular quadric contained in $Q(6,q)$ such that the geometric hyperplane $\mathcal{S}_j$ of $\Gamma$ induced by $H_j$ consists of the set of points of $\Gamma$ which corresponds with the set of lines of $\mathsf{H}(q)$ having non-empty intersection with $Q^{(j)}(4,q)$ (see the proof of Lemma 4.1), $j\in\{1,2,i\}$, and suppose that $Q^{(j)}(4,q)$ is contained in the 4-space $V_j$ of $\mathbf{PG}(6,q)$. Each $V_j$ contains the 3-space $V$ generated by the lines of $\mathsf{H}(q)$ which correspond with the elements of $\mathcal{R}$ of $\Gamma$. Let $z$ be a point of $\mathcal{S}_1\cap\mathcal{S}_2$ which corresponds with a line $Z$ of $\mathsf{H}(q)$ not meeting $V$. Then clearly, $Q^{(1)}(4,q)\cup Q^{(2)}(4,q)$ is contained in the 5-space $V_{12}$ generated by $V$ and $Z$. Our assumption $H_1\cap H_2\subseteq H_i$ implies that $Z$ meets $Q^{(i)}(4,q)$ and hence $Q^{(i)}(4,q)$ is also contained in $V_{12}$. Now let the point $\ell_k$ of $\mathsf{H}(q)$ correspond with the line $L_k$ of $\Gamma$, $k\in\{0,\ldots,i\}$. Let $\ell'_k$ be the projection of $\ell_k$ from $V$ onto any plane $\Pi$ of $\mathbf{PG}(6,q)$ skew to $V$. By construction, $V_1$ contains the intersection of the lines $\ell'_0\ell'_1$ and $\ell'_2\ell'_3$; $V_2$ contains the intersection of the lines $\ell'_0\ell'_3$ and $\ell'_1\ell'_2$; $V_i$ contains the intersection of the lines $\ell'_0\ell'_2$ and $\ell'_1\ell'_i$. The space $V_{12}$ meets $\Pi$ in a line, hence, if $i=3$, we see that $q$ must be even (since $\Pi$ contains a quadrangle whose diagonal points are collinear). But then $q\geq 4$. If also $H_1\cap H_2\subseteq H_4$, then $\ell'_1,\ell'_3$ and $\ell'_4$ are collinear, a contradiction as the points $\ell_0,\ell_1,\ell_2,\ell_3,\ell_4$ are contained in a non-singular conic and hence so are the points $\ell'_0,\ell'_1,\ell'_2,\ell'_3,\ell'_4$. Hence $H_i$, for the appropriate choice of $i$, does not contain $H_1\cap H_2$, and so $H_1\cap H_2\cap H_i$ is a 10-dimensional space containing $\Gamma_1(x)\cup\mathcal{R}\cup\mathcal{R}^\perp$.

Now suppose the space $W$ generated by $\Gamma_1(x)\cup\mathcal{R}\cup\mathcal{R}^\perp$ has dimension $m$, $m\leq 10$. Then $W' := \langle W, \Gamma(L_0,L_1), \Gamma(L_2,L_3), \Gamma(L_0,L_2)\rangle$ has dimension at most $m+3$. But $W'$ must coincide with $\mathbf{PG}(13,q)$ by Lemma 4.1. Hence $m+3\geq 13$, implying $m=10$.

The proof of the lemma is complete. ∎

**Lemma 4.6.** *Let $x$ be any point of $\Gamma$, and let $\Sigma$ be any apartment containing $x$. Suppose $\Gamma_1(x)$ is not contained in a 4-dimensional subspace of $\mathbf{PG}(13,q)$. Then $\Sigma\cup\Gamma_1(x)$ is contained in a subspace of dimension 7.*

**Proof.** Let $L_1, L_2$ be the two lines of $\Sigma$ incident with $x$. Since $\langle\Gamma_3(L_1)\rangle$ and $\langle\Gamma_3(L_2)\rangle$ are 7-dimensional, and $\langle\Gamma_1(x)\rangle$ is at least 5-dimensional, we deduce that $W:=\langle\Gamma_3(L_1)\cup\Gamma_3(L_2)\rangle$ is at most 9-dimensional. Hence $W':=\langle W,\Sigma\rangle$ is at most 10-dimensional. Denote by $y$ the point of $\Sigma$ opposite $x$, and let $\mathcal{R}$ be the point regulus containing $x$ and $y$. Let $V$ be the 10-dimensional subspace of $\mathbf{PG}(13,q)$ generated by $\Sigma\cup\Gamma_1(x)\cup\Gamma_1(y)\cup\mathcal{R}\cup(\Gamma_3(x)\cap\Gamma_3(y))$ (this is indeed 10-dimensional as follows from Lemma 4.5). It is easy to see that $\langle V\cup W'\rangle$ contains all thin subhexagons $\Gamma(X,Y)$, where $X\in\Gamma_3(x)\cap\Gamma_3(y)\cap\Sigma$, and $Y\in\Gamma_3(x)\cap\Gamma_3(y)$, $X\neq Y$. By Lemma 4.1, $\langle V\cup W'\rangle$ is equal to $\mathbf{PG}(13,q)$. Hence $V\cap W'$ is at most 7-dimensional. But this contains $\Sigma$ and $\Gamma_1(x)$. The lemma is proved. ∎

**Proposition 4.7.** *For every point $x$ of $\Gamma$, the subspace generated by $\Gamma_1(x)$ is 4-dimensional.*

**Proof.** We already know that $\langle\Gamma_1(x)\rangle$ is at least 4-dimensional. Suppose now that $\langle\Gamma_1(x)\rangle$ is at least 5-dimensional. With the notation of the proof of Lemma 4.6, we then know that $U:=\langle\Gamma_1(x)\cup\Sigma\rangle$ is at most 7-dimensional. Let $z$, $x\neq z\neq y$, be any element of the regulus $\mathcal{R}$ determined by $x$ and $y$, and let $L_1,L_2,L_3,L_4$ be four different lines belonging to the complementary regulus, with $L_1,L_2\in\Sigma$. As $\langle L_1\cup L_2\cup L_3\cup L_4\rangle$ is a 7-dimensional subspace which has no point in common with the plane $\langle\mathcal{R}\rangle$, we see that the subspace generated by $U,z,L_3,L_4$ is at least 10-dimensional. Hence $U$ is 7-dimensional and $\langle U,z\rangle$ is 8-dimensional. Interchanging the roles of $y$ and $z$, we see that $U':=\langle\Gamma_1(x)\cup L_1\cup L_2\cup\{z\}\rangle$ is also 7-dimensional. But this means that $\langle U\cup U'\rangle$ is 8-dimensional, and so $U\cap U'$ is 6-dimensional. Hence $\langle\Gamma_1(x)\cup L_1\cup L_2\rangle$ is at most 6-dimensional. Now, since the subspace $\langle\Gamma_1(x),L_1\cup L_2\cup L_3\cup L_4\rangle$ has dimension at least 8, the subspace $\langle\Gamma_1(x),L_1\cup L_2\cup L_3\rangle$ has dimension 7 and the subspace $\langle\Gamma_1(x),L_1\cup L_2\rangle$ has dimension 6. Similarly $\langle\Gamma_1(x),L_1\cup L_3\rangle$ is 6-dimensional. It follows that $\langle\Gamma_1(x),L_1\rangle$ is 5-dimensional (since $\langle\Gamma_1(x)\rangle$ is assumed to be at least 5-dimensional). Similarly, $\langle\Gamma_1(x),L_2\rangle$ is 5-dimensional, and so $\langle\Gamma_1(x),L_1\cup L_2\rangle=\langle\Gamma_1(x),L_1\rangle=\langle\Gamma_1(x),L_2\rangle$ is 5-dimensional, a contradiction.

The proposition is proved. ∎

Next we show that the embedding is polarized. To that end, we need another lemma.

**Lemma 4.8.** *Any line regulus of $\Gamma$ is contained in a unique 7-dimensional subspace of $\mathbf{PG}(13,q)$, which is generated by any four elements of the regulus.*

**Proof.** After Proposition 4.2 we remarked that every 4 lines of a line regulus generate a 7-dimensional space.

Denote the lines of a fixed line regulus by $L_0, L_1, \ldots, L_q$, and let $x, y, z$ be three different points of the complementary point regulus. The spaces $W_z$ and $W$ generated by $L_0, L_1, \ldots, L_q, x, y$ and $L_0, L_1, \ldots, L_q, x, y, z$, respectively, are at most 9-dimensional and 10-dimensional, respectively (indeed, the first one is generated by $\Gamma_2(x) \cup \Gamma_2(y)$, the second one by the first one and the point $z$). Lemma 4.1 implies that we can generate $\mathbf{PG}(13, q)$ with $W$ and three additional points, hence $W$ is 10-dimensional and, consequently, $W_z$ is 9-dimensional. Interchanging the roles of $y$ and $z$, we see that also $W_y$ (defined by swapping $y$ and $z$) is 9-dimensional. Since $W = \langle W_y, W_z \rangle$, it follows that $W'_x = W_y \cap W_z$ is 8-dimensional. Similarly, we obtain that $W'_y$ (obtained by interchanging roles of $x$ and $y$) is 8-dimensional. As $\langle W'_x, W'_y \rangle$ is 9-dimensional, the subspace $W'_x \cap W'_y$ is 7-dimensional. But this space contains our fixed line regulus. The lemma is proved. ∎

**Proposition 4.9.** *Let $x$ be any point of $\Gamma$. Then the subspace generated by $\Gamma_2(x) \cup \Gamma_4(x)$ is 12-dimensional.*

**Proof.** First we remark the following. Let $L$ and $M$ be two opposite lines of $\Gamma$ and let $L$ be incident with $x$. Let $\mathcal{R}$ be a point regulus in $\Gamma(L, M)$ containing $x$ and a point of $M$. Let $N$ be the line at distance 3 from $x$ and concurrent with $M$. Note that $\mathcal{R}$ is a conic in the plane $\pi$ it generates. Let $T$ be the tangent line of that conic at $x$ (in $\pi$). Clearly the space $U$ generated by all lines of $\Gamma(L, M)$ concurrent with $L$ and by $N$ has dimension 5. If it would meet $\pi$ in a line (necessarily $T$), then by adding one point of $\mathcal{R} \setminus \{x\}$ to $U$, we would generate a space of dimension 6 containing all lines of $\Gamma(L, M)$ concurrent with $L$, containing $M$ and $\mathcal{R}$, hence containing $\Gamma(L, M)$ itself, a contradiction. So the space generated by all lines of $\Gamma(L, M)$ which meet $L$, by $N$ and by $T$ is 6-dimensional and hence is the space generated by all points of $\Gamma(L, M)$ at distance $\leq 4$ from $x$.

Now put $\Gamma_1(x) = \{L_0, L_1, \ldots, L_q\}$. Let $U_i$ be the 7-dimensional space generated by all lines of $\Gamma$ concurrent with $L_i$, $i \in \{0, 1, \ldots, q\}$. Let $y$ be any point opposite $x$, and let $\mathcal{R}$ be the line regulus $\Gamma_3(x) \cap \Gamma_3(y)$. Using Lemma 4.8, Proposition 4.7 and Corollary 4.4, we deduce that the space $W$ generated by $U_0, U_1$ and $\mathcal{R}$ has dimension (at most) 12. But $W$ contains the 6-dimensional space generated by the points of $\Gamma(M_0, M_1)$ not opposite $x$, with $M_i$ the unique element of $\mathcal{R}$ concurrent with $L_i$, $i = 0, 1, \ldots, q$. Hence $W$ contains the line $T$ tangent at $x$ to the point regulus $\mathcal{R}'$ which is complementary to the line regulus $\mathcal{R}$. But that implies, by the previous paragraph, that, for all $i \in \{2, \ldots, q\}$, the space $W$ contains the 6-dimensional space generated by the points not opposite $x$ of $\Gamma(M_j, M_i)$, $j = 0, 1$. This means that, for all $i \in \{2, \ldots, q\}$, $W$ contains $[M_i, L_0]$, $[M_i, L_1]$ and $\Gamma_1(x)$, hence $W$ contains all lines concurrent with $L_i$, by an argument already used in the proof of

Corollary 4.4. Consequently $W$ contains all points not opposite $x$. It is now also clear that $W$ is a hyperplane. ∎

Before we can actually start to prove uniqueness of the embedding, we show some more lemmas.

**Lemma 4.10.** *Let $x$ be any point of $\Gamma$. Then $\langle x^{\perp}\rangle\cap\Gamma=x^{\perp}$ and $\langle x^{\perp\!\!\perp}\rangle\cap\Gamma=x^{\perp\!\!\perp}$.*

**Proof.** Assume that $y\in\langle x^{\perp}\rangle\cap\Gamma$ with $y\notin x^{\perp}$. First, let $y\in\Gamma_4(x)$. Consider a point regulus $\mathcal{R}$ containing $x$ for which $y$ belongs to one of the lines of $\mathcal{R}^{\perp}$. If $z\in\mathcal{R}\setminus\{x\}$, then $\langle x^{\perp},z^{\perp}\rangle$ is 9-dimensional, but $y\in\langle x^{\perp}\rangle$ implies that $\langle x^{\perp},z^{\perp}\rangle$ is at most 8-dimensional, a contradiction. If $y\in\Gamma_6(x)$, then a similar argument applies (considering $\langle x^{\perp},y^{\perp}\rangle$).

Next, assume that $y\in\langle x^{\perp\!\!\perp}\rangle\cap\Gamma$ with $y\notin x^{\perp\!\!\perp}$. By the connectivity of the complement of $x^{\perp\!\!\perp}$ in $\Gamma$ (see [1]), it easily follows that $\Gamma$ is contained in $\langle x^{\perp\!\!\perp}\rangle$, clearly a contradiction. ∎

**Lemma 4.11.** *Let $\mathcal{R}$ be a point regulus of $\Gamma$, and let $L_0,L_1,L_2,L_3$ be four distinct lines of the complementary regulus. Let $L$ be any line of $\Gamma$ containing a unique point $x_{01}$ of $\Gamma(L_0,L_1)$ and a unique point $x_{23}$ of $\Gamma(L_2,L_3)$, with $x_{01}\neq x_{23}$. Then for every point $x$ on $L$, $x_{01}\neq x\neq x_{23}$, we can find a point $u$ in $\Gamma$ opposite both $x_{01}$ and $x_{23}$, not opposite $x$, and opposite at least one element of $\mathcal{R}$.*

**Proof.** We dualize the situation and consider a natural embedding of $\Gamma^D$ in $\mathbf{PG}(6,q)$. So we are given a line regulus $\mathcal{R}'$, four points $\ell_0,\ell_1,\ell_2,\ell_3$ of the complementary regulus, two different concurrent lines $X_{01}$ and $X_{23}$ of $\Gamma^D(\ell_0,\ell_1)$ and $\Gamma^D(\ell_2,\ell_3)$, respectively, the intersection point $\ell$ of these two lines, and a line $X$ through $\ell$ different from $X_{01}$ and from $X_{23}$. We look for a line $U$ opposite both $X_{01}$ and $X_{23}$, not opposite $X$, and opposite at least one element of $\mathcal{R}'$. Then with $\Gamma^D(\ell_0,\ell_1)$ and with $\Gamma^D(\ell_2,\ell_3)$, there correspond hyperbolic quadrics which intersect in a non-singular parabolic quadric Q in $\mathbf{PG}(6,q)$, as before. The point $\ell$ lies on $Q$, and all elements of $\mathcal{R}'$ are contained in $Q$. Let $R$ be any element of $\mathcal{R}'$, and consider any point $z\neq\ell$ on $X$. The shortest path from $z$ to $R$ does not pass through $\ell$, as otherwise we obtain a line of $\Gamma$ in $Q$ not belonging to $\mathcal{R}'$, a contradiction. If $X$ and $R$ are not opposite, then this implies that we may take $U=R$. Suppose now that $X$ and $R$ are opposite. Then we may take $U$ concurrent with $W:=\mathrm{proj}_z R$, but not incident with $z$, nor with $\mathrm{proj}_W R$. Clearly, $U$ is opposite $R$ and the lemma is proved. ∎

**Lemma 4.12.** *There exists a natural embedding of the dual $\Gamma^D$ of $\Gamma$ in $\mathbf{PG}(6,q)$ such that, if $x_1,x_2,x_3,x_4$ are four different points on a line of $\Gamma$,* *and $X_1,X_2,X_3,X_4$ are the four corresponding concurrent lines of $\Gamma^D$ in $\mathbf{PG}(6,q)$ (which lie in a plane $\mathbf{PG}(2,q)$ of $\mathbf{PG}(6,q)$), then the cross-ratio $(x_1,x_2;x_3,x_4)$ in $\mathbf{PG}(13,q)$ is equal to the cross-ratio $(X_1,X_2;X_3,X_4)$ in the plane $\mathbf{PG}(2,q)$.*

**Proof.** Consider two opposite lines $L,M$ in $\Gamma$. The generalized hexagon $\Gamma(L,M)$ is fully embedded in some 7-dimensional space $\mathbf{PG}(7,q)$ and the embedding is polarized. It follows easily that a natural embedding of $\Gamma^D$ in $\mathbf{PG}(6,q)$ can be chosen such that the cross-ratio of four given lines of $\Gamma^D$ in $\mathbf{PG}(6,q)$ through the point $l$ corresponding in $\Gamma^D$ with $L$, is equal to the cross-ratio of the four corresponding points of $\Gamma(L,M)$ on $L$ in $\mathbf{PG}(7,q)$. Since the cross-ratio is preserved by projecting $\Gamma_1(L)$ onto $\Gamma_1(M)$, and the dual statement holds in $\Gamma^D$, we see that the lemma also holds for points on $M$. Moreover, since every line of $\Gamma$ is opposite some line opposite $L$, we conclude by applying the foregoing twice that the lemma holds for every four points on all lines of $\Gamma$. ∎

We can now prove our Main Result for $q\geq 3$.

**Theorem 4.13.** *For $q>2$, the embedding of $\Gamma$ in $\mathbf{PG}(13,q)$ is projectively unique.*

**Proof.** Throughout, we consider a fixed natural embedding of $\Gamma^D$ in $\mathbf{PG}(6,q)$ satisfying the conditions of Lemma 4.12.

Let $x_0$ and $x_1$ be two opposite points of $\Gamma$ and put $\mathcal{R}=\mathcal{R}(x_0,x_1)=\{x_0,x_1,\ldots,x_q\}$. These points form a conic $\mathcal{C}$ in some plane $\mathbf{PG}(2,q)$ of $\mathbf{PG}(13,q)$. Now let $L_0,L_1,L_2,L_3$ be four different, arbitrary, but fixed, lines of the complementary regulus of $\mathcal{R}(x_0,x_1)$. Using Lemma 4.12, and since $\langle\Gamma(L_0,L_1)\rangle\cap\langle\Gamma(L_2,L_3)\rangle=\mathbf{PG}(2,q)$, the correspondence between elements of $\Gamma(L_0,L_1)\cup\Gamma(L_2,L_3)$ and the corresponding elements of $\Gamma^D$ is unique up to a linear projectivity in $\mathbf{PG}(13,q)$ (and $\Gamma(L_0,L_1)\cup\Gamma(L_2,L_3)$ generates a hyperplane $\mathbf{PG}(12,q)$ of $\mathbf{PG}(13,q)$). Let us denote by $\theta$ the considered anti-isomorphism from $\Gamma^D$ onto $\Gamma$. As the union of the point sets of $\Gamma(L_0,L_1)$ and $\Gamma(L_2,L_3)$ linearly generate a geometric hyperplane $\mathcal{S}$ of $\Gamma$, where $\mathcal{S}\subseteq\mathbf{PG}(12,q)$, and as the complement of $\mathcal{S}$ in $\Gamma$ is connected, it follows that $\mathcal{S}$ is the set of all points of $\Gamma$ in $\mathbf{PG}(12,q)$. We now show that we can derive in a unique way the image under $\theta$ of any point of $\mathcal{S}^{\theta^{-1}}$.

Let $Q$ be the (parabolic) quadric in some space $\mathbf{PG}(4,q)$ of $\mathbf{PG}(6,q)$ which is the intersection of $\Gamma^D$ with the two hyperplanes of $\mathbf{PG}(6,q)$ determined by the subhexagons $\Gamma^D(L_0,L_1)$ and $\Gamma^D(L_2,L_3)$ of $\Gamma^D$ corresponding to $\Gamma(L_0,L_1)$ and $\Gamma(L_2,L_3)$, respectively. Let the line $X_i$ in $Q$ correspond with the point $x_i$ of $\Gamma$, $i\in\{0,1,\ldots,q\}$. Then, by the proof of Lemma 4.1, the points of $\Gamma$ in $\mathbf{PG}(12,q)$ correspond to the lines of $\Gamma^D$ which meet $Q$. Suppose first that $x$ is a point of $\Gamma$ in $\mathbf{PG}(12,q)$, but not in $\Gamma(L_0,L_1)\cup\Gamma(L_2,L_3)$, which corresponds to a line $X$ meeting $Q$, but not meeting one of the lines $X_0,X_1,\ldots,X_q$. Let $a$ be the point of $X$ in $Q$, and let $A$ be the corresponding line of $\Gamma$. As $a$ is incident with one line of $\Gamma^D(L_0,L_1)$, and one line of $\Gamma^D(L_2,L_3)$, and as these lines are distinct (see the proof of Lemma 4.1 again), the line $A$ contains two distinct points $x_{01}$ and $x_{23}$ of $\Gamma(L_0,L_1)$ and $\Gamma(L_2,L_3)$, respectively, and hence is uniquely determined by these points. Now, by Lemma 4.11, there exists a point $y$ of $\Gamma$ opposite both $x_{01}$ and $x_{23}$, not opposite $x$, and opposite at least one element of $\mathcal{R}$. It is easy to see that, for every $i\in\{0,1,2,3\}$, the set of points of $y^{\perp\!\!\perp}$ in $\langle L_i,\mathcal{R}\rangle$ either spans a 3-dimensional space, or it is a plane containing $L_i$ and a line of $\Gamma$ meeting $L_i$ and incident with an element $x_j$ of $\mathcal{R}$. In the latter case, it is clear that $\langle y^{\perp\!\!\perp}\rangle$ intersects $\langle L_i,\mathcal{R}\rangle$ in the 3-dimensional space generated by $L_i$ and the tangent at $x_j$ to the conic $\mathcal{R}$ in the plane $\mathbf{PG}(2,q)$. Now, these four 3-dimensional spaces thus obtained clearly intersect mutually in a common line of $\langle\mathcal{R}\rangle$) (because otherwise $\langle y^{\perp\!\!\perp}\rangle$ would contain $\mathcal{R}$, hence $y^{\perp\!\!\perp}$ would contain $\mathcal{R}$, contradicting Lemma 4.10 and the fact that $y$ is opposite at least one element of $\mathcal{R}$). Hence $\langle y^{\perp\!\!\perp}\rangle$ intersects $\langle L_0,L_1,\mathcal{R}\rangle$ in a 5-dimensional space. Likewise, $\langle y^{\perp\!\!\perp}\rangle$ intersects $\langle L_2,L_3,\mathcal{R}\rangle$ in a 5-dimensional space. By our condition on $y$, we may assume that $y$ is opposite $x_0$, and hence there exists a line $K$ of $\Gamma(L_0,L_1)$ at distance 3 from $x_0$, which has a unique point not opposite $y$, and not contained in $\langle L_0,L_1,\mathcal{R}\rangle$. Similarly there exists a line $K'$ of $\Gamma(L_2,L_3)$ at distance 3 from $x_0$, which has a unique point not opposite $y$, and not contained in $\langle L_2,L_3,\mathcal{R}\rangle$. We conclude that $\langle y^{\perp\!\!\perp}\rangle\cap\mathbf{PG}(12,q)$ contains two 6-dimensional spaces, meeting in $\mathbf{PG}(2,q)$ in at most a line (otherwise, by Lemma 4.10, $x_0$ would not be opposite $y$, a contradiction). Hence $\langle y^{\perp\!\!\perp}\rangle\cap\mathbf{PG}(12,q)$ is an 11-dimensional space, which is completely determined by previously constructed elements of $\Gamma$, and which meets $A$ in $x$. Hence $x$ is uniquely determined by its image $X$ in $\Gamma^D$.

We now claim that also all lines through the points $x_0,x_1,\ldots,x_q$ are uniquely determined by their images in $\Gamma^D$. Indeed, let $X$ be a line through, say, $x_0$, and let $x$ be the corresponding point of $\Gamma^D$. We may assume that $X$ is not a line in $\Gamma(L_0,L_1)\cup\Gamma(L_2,L_3)$. Let, as in the proof of Lemma 4.1, $Z$ be any line of $Q$ through $x$ not in the space generated by $X_0$ and $X_1$. Let the set of points incident in $\mathbf{PG}(4,q)$ with $Z$ be equal to $\{x,z_1,\ldots,z_q\}$. Let $y$ be the unique point of $\Gamma^D$ collinear in $\Gamma^D$ with $x$ and $z_j$, for all $j=1,\ldots,q$. Let $Y$ and $Z_j$, $j=1,\ldots,q$, correspond with $x,y,z_j$, respectively, in $\Gamma$. From the previous paragraph it follows that all points on $Z_j$, $j=1,\ldots,q$, are determined by their image on $\Gamma^D$; hence $Y$ is uniquely determined by $y$, and so are all points of $Y$ that are incident with one of the $Z_j$. Now $X$ is the unique line containing $x_0$ and meeting $Y$ in a point not on any $Z_j$, $j=1,\ldots,q$. Our claim is proved.

Now, with the notation of the previous paragraph, we show that every point of $X$ is determined by its image in $\Gamma^D$. So let $w\in\Gamma_1(X)\setminus\{x_0\}$ and let $W$ be the corresponding line of $\Gamma^D$. Let $W_1,W_2,W_3,W_4$ be the lines of $\Gamma$ in $\Gamma(L_0,L_1)\cup\Gamma(L_2,L_3)$ through $x_0$. Suppose that $w_j$ is the point of $\Gamma^D$ which corresponds to $W_j$, $j=1,2,3,4$. Now we choose an arbitrary (dual) trace $X_0^V$ in $\Gamma^D$ containing $W$ (and $V$ is opposite $X_0$). Let $V_j$ be the unique line of that trace incident with $w_j$, $j=1,2,3,4$. If $v_j$ is the point of $\Gamma$ which corresponds to $V_j$, $j=1,2,3,4$, then, by Propositions 4.7 and 4.9, the points $w,v_1,v_2,v_3,v_4$ are contained in a common 3-dimensional space $\mathbf{PG}(3,q)$, which is already determined by $v_1,v_2,v_3,v_4$. Hence $\{w\}=\mathbf{PG}(3,q)\cap X$, and so $w$ is uniquely determined by its image $W$ on $\Gamma^D$.

Since all points on every line of $\Gamma$ through any $x_i$, $i=0,1,\ldots,q$, meeting either $L_1$ or $L_2$ are uniquely determined by their images (under $\theta^{-1}$) in $\Gamma^D$, there is a unique way, up to a linear projectivity of $\mathbf{PG}(13,q)$, to construct the images of the elements of $\Gamma(L_1,L_2)^{\theta^{-1}}$ in $\mathbf{PG}(13,q)$ (remark that $\Gamma(L_1,L_2)$ and $\mathbf{PG}(12,q)$ generate $\mathbf{PG}(13,q)$); this follows from Section 4 of [7]. So we have at least one line $K$ of $\Gamma$ outside $\mathbf{PG}(12,q)$ all points of which are uniquely determined by their corresponding elements in $\Gamma^D$. We show that also all points at distance 3 of $K$ (but not collinear in $\Gamma$ with the intersection point of $K$ with $\mathbf{PG}(12,q)$) in $\Gamma$ are uniquely determined by the corresponding elements in $\Gamma^D$. By connectivity of the complement in $\Gamma$ of the set of points of $\Gamma$ in $\mathbf{PG}(12,q)$, we then conclude that the embedding is unique.

Each point of $\Gamma$ in $\mathbf{PG}(12,q)$ is incident with a line of $\Gamma$ contained in $\mathbf{PG}(12,q)$. So the unique point $w$ on $K$ in $\mathbf{PG}(12,q)$ is incident with a line $K'$ of $\Gamma$ contained in $\mathbf{PG}(12,q)$. Since all points of $K$ are uniquely determined by their images in $\Gamma^D$, and since every line through such a point meets $\mathbf{PG}(12,q)$ in a unique point, all lines of $\Gamma$ meeting $K$ in a point distinct from $w$ are determined by their corresponding elements in $\Gamma^D$. So we know all cubic scrolls with base line $K$ containing $K'$. Since all points on $K'$ are uniquely determined, since on every line of every such scroll, there is at least one point not on $K$ uniquely determined (the intersection of that line with $\mathbf{PG}(12,q)$), and since every conic on every scroll is determined by two points, we see that all points on all these scrolls are uniquely determined by their images in $\Gamma^D$.

The proof is complete. ∎

## References


[1] A. E. Brouwer: The complement of a geometric hyperplane in a generalized polygon is usually connected, *Finite Geometry and Combinatorics*, ed. F. De Clerck *et al.*, Cambridge University Press, *London Math. Soc. Lect. Notes Ser.* **191** (1993), 53–57.

[2] E. Govaert and H. Van Maldeghem: Some combinatorial and geometric characterizations of the finite dual classical generalized hexagons, *J. Geom.* **68** (2000), 87–95.

[3] S. E. Payne and J. A. Thas: *Finite Generalized Quadrangles*, Pitman, Boston, London, Melbourne, 1984.

[4] J. A. Thas and H. Van Maldeghem: Embedded thick finite generalized hexagons in projective space, *J. London Math. Soc. (2)* **54** (1996), 566–580.

[5] J. A. Thas and H. Van Maldeghem: Flat lax and weak lax embeddings of finite generalized hexagons, *European J. Combin.* **19** (1998), 733–751.

[6] J. A. Thas and H. Van Maldeghem: On embeddings of the flag geometry of projective planes in finite projective spaces, *Des. Codes Cryptogr.* **17** (1999), 97–104.

[7] J. A. Thas and H. Van Maldeghem: Classification of embeddings of the flag geometry of projective planes in finite projective spaces, Part 1; *J. Combin. Theory Ser. A* **90** (2000), 159–172.

[8] J. A. Thas and H. Van Maldeghem: Classification of embeddings of the flag geometry of projective planes in finite projective spaces, Part 2; *J. Combin. Theory Ser. A* **90** (2000), 241–256.

[9] J. A. Thas and H. Van Maldeghem: Classification of embeddings of the flag geometry of projective planes in finite projective spaces, Part 3; *J. Combin. Theory Ser. A* **90** (2000), 173–196.

[10] J. A. Thas and H. Van Maldeghem: Some remarks on embeddings of the flag geometry of projective planes in finite projective spaces, *J. Geom.* **67** (2000), 217–222.

[11] J. Tits: Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60.

[12] J. Tits: *Buildings of Spherical Type and Finite BN-Pairs*, Springer Verlag, Berlin, Heidelberg, New York, *Lecture notes in Math.* **386**, 1974.

[13] H. Van Maldeghem: *Generalized Polygons*, Birkhäuser Verlag, Basel, Boston, Berlin, *Monographs in Mathematics* **93**, 1998.

[14] H. Völklein: On the geometry of the adjoint representation of a Chevalley group, *J. Algebra* **127** (1989), 139–154.



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