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Maps between buildings that preserve a given Weyl distance

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ABSTRACT

Let Δ and Δ' be two buildings of the same type (W, S), viewed as sets of chambers endowed with "distance" functions δ and δ', respectively, admitting values in the common Weyl group W, which is a Coxeter group with standard generating set S. For a given element $w \in W$, we study surjective maps $\varphi : \Delta \to \Delta'$ with the property that $\delta(C, D) = w$ if and only if $\delta'(\varphi(C), \varphi(D)) = w$. The result is that the restrictions of φ to all residues of certain spherical types—determined by w—are isomorphisms. We show with counterexamples that this result is optimal. We also demonstrate that, in many cases, this is enough to conclude that φ is an isomorphism. In particular, φ is an isomorphism if Δ and Δ' are 2-spherical and every reduced expression of w involves all elements of S.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The theorem of Beckman and Quarles (see, for instance, [3]) states that one can identify an isometry of real Euclidean space by checking whether a given surjective mapping preserves a certain fixed distance. Generalizations and analogous statements for other (types of) spaces have been considered in the literature. For discrete geometries, one is led to consider the distance in one of the graphs associated with the geometry. For instance, for the class of generalized n-gons (which are the spherical buildings of rank 2), it is shown in [5] that, up to some well understood exceptions, surjective maps preserving a certain arbitrary, but fixed, distance i, with

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$1 \leqslant i < n$ (measured in the incidence graph), on the set of points, the set of lines (i even), or on both (i odd), can be extended to isomorphisms. The same conclusion holds if one considers maps on the flag set of such geometries preserving a given distance d in the flag graph of the generalized n-gon. For $n = d$ (the maximal distance possible), this result is a special case of a more general result that has been proved for all spherical buildings by the authors (see [1]). Since buildings are metric spaces with a natural group valued metric (in its associated Weyl group, which is a Coxeter group), it is natural to ask whether a map preserving a given Weyl distance is necessarily an isometry, i.e., an isomorphism of buildings. In this paper, we answer this question. The result is that in "many" cases (for the details, see below) the preservation of a single given Weyl distance w indeed already leads to isomorphisms between the corresponding buildings. This is rather surprising since chambers at distance w from each other are not in "general position" as in the case of opposite chambers in a spherical building, which was treated in [1]. In the course of the proof we establish two lemmas which are of independent interest, and we state these as separate propositions. They essentially state a relation between w-distance for some w in the Weyl group and, respectively, apartments and adjacency of chambers. Note that our results hold for arbitrary (thick) buildings (and not just for spherical ones). While the proof of Proposition 1.6 is a more or less straightforward generalization of the proof of the analogous result for w being the longest word in a spherical Coxeter group, the proof of Proposition 1.5 requires new ideas in comparison with the analogous statement for spherical buildings and w the longest element.

We use standard notation of building theory. In particular, (W, S) will always denote a pair consisting of a Coxeter group W and a standard set S of (involutive) generators. A building Δ is of type (W, S) if its apartments are Coxeter complexes isomorphic to the thin building naturally associated to (W, S). For the definition of a building using chamber systems, we will mainly refer to [6], Chapter 3. However, we shall repeat that definition in Section 2 below, but without first introducing the notion of a chamber system (see also [8] and [9]). We introduce all the notions we need in the sequel, to make the paper self-contained, and to fix notation. Alternative definitions of a building can be found in [4] and [7]. For a building Δ, we denote by $\mathcal{C}(\Delta)$ its set of chambers. Let us also remark that we do not require a building to be thick; we will always explicitly mention thickness where we need it.

We will also use standard terminology with regard to Coxeter groups. In particular, we will talk about words, reduced expressions and the length $\ell(w)$ of an element w of W (with respect to S). If $w \in W$, and if $s \in S$, then we say that *s is involved in w* if some (and hence every) reduced expression of w in elements of S contains s. We put

$$S(w) = \{s \in S \mid s \text{ is involved in } w\}.$$

For two arbitrary chambers $C, D \in \mathcal{C}(\Delta)$ we write $d(C, D) = \ell(\delta(C, D))$ (and call this the *gallery distance* between C and D). If $d(C, D) = 1$, then C and D are *adjacent*, and we more specifically say that C and D are *$\delta(C, D)$-adjacent*. If

$\delta(C, D) \in S' \subseteq S$, then we also say that C and D are *S'-adjacent*. The Coxeter system (W, S) will be called *2-spherical* if for every two generators $r, s \in S$, the order of the product rs is finite.

If Δ and Δ' are both buildings of type (W, S), and if $\varphi : \mathcal{C}(\Delta) \to \mathcal{C}(\Delta')$ is a map, we say that φ *preserves S'-adjacency*, for some $S' \subseteq S$, if $\varphi(C), \varphi(D)$ are S'-adjacent whenever C and D are S'-adjacent, for $C, D \in \mathcal{C}(\Delta)$.

More notation is introduced and explained in Section 2. With the above notions, we may now state some results that we will prove in this paper (for more detailed, but also more technical statements, see Section 4 below).

Theorem 1.1. *Let Δ and Δ' be two thick buildings of type (W, S), and let $w \in W$ be arbitrary. Suppose that $\varphi : \mathcal{C}(\Delta) \to \mathcal{C}(\Delta')$ is a surjective mapping with the additional property that $\delta(C, D) = w$ if and only if $\delta'(\varphi(C), \varphi(D)) = w$, for all $C, D \in \mathcal{C}(\Delta)$. Then φ is a bijection and both φ and its inverse preserve $S(w)$-adjacency of chambers.*

There are two immediate consequences that can be mentioned.

Corollary 1.2. *Let Δ and Δ' be two thick buildings of type (W, S), and let $w \in W$ be such that $S(w) = S$. Suppose that $\varphi : \mathcal{C}(\Delta) \to \mathcal{C}(\Delta')$ is a surjective mapping with the additional property that $\delta(C, D) = w$ if and only if $\delta'(\varphi(C), \varphi(D)) = w$, for all $C, D \in \mathcal{C}(\Delta)$. Then φ is a bijection and both φ and its inverse preserve adjacency of chambers.*

We remark that the condition that every $s \in S$ is involved in w is necessary, here and in Corollary 1.3 below (for counterexamples see Example 4.2 in Section 4).

Corollary 1.3. *Let Δ, Δ', w and φ be as in Corollary 1.2. Suppose moreover that (W, S) is 2-spherical. Then there is a permutation $\theta : S \to S$ such that two chambers $C, D \in \mathcal{C}(\Delta)$ are s-adjacent if and only if $\varphi(C), \varphi(D)$ are $\theta(s)$-adjacent (in other words, φ induces a not necessarily type preserving isomorphism between Δ and Δ', in the sense of Section 2).*

We also want to mention that Corollary 1.3 is not true if we drop the assumption "2-spherical". This will be demonstrated by means of another counterexample, see Example 4.4 in Section 4.

The following is a corollary of our more detailed result Theorem 4.1 below.

Corollary 1.4. *Let again Δ, Δ', w and φ be as in Corollary 1.2. Suppose moreover that w has a unique reduced expression with respect to S. Then φ induces an isometry (a type preserving isomorphism) between Δ and Δ'.*

For other applications, see Section 4.

Our results will follow from the following proposition.

Proposition 1.5. *Let Δ be a thick building of type (W, S), and let $w \in W$ be arbitrary. Define $S_1(w) = \{s \in S \mid \ell(sw) < \ell(w)\}$. Let C, D, E be three different chambers of Δ and suppose that they satisfy the following condition.*

($*$) *For any chamber $X \in \mathcal{C}(\Delta)$, the number of chambers $Y \in \{C, D, E\}$ such that $\delta(Y, X) = w$ is not equal to* 1.

Then C, D, E are pairwise s-adjacent, for some $s \in S_1(w)$. Conversely, if the three chambers C, D, E of Δ are pairwise s-adjacent, for some $s \in S_1(w)$, then they satisfy Condition ($*$).

This proposition is really the heart of the proof, and of our paper. It generalizes in a rather tricky way Proposition 4.1 of [1], which not only restricts to spherical buildings, but also assumes that w is the unique longest element of W (implying in particular $S_1(w) = S$). Hence new ideas are needed in the proof to replace the special properties of opposition that were used in [1]. However, one tool that was used in [1] will also be needed here, although in a more general form, and with a proof that more or less generalizes in a standard way the analogous result for spherical buildings and w the longest element ([1], Proposition 3.2, implication (a) $\Rightarrow$ (b)). Nevertheless we present the full proof in Section 3 since some arguments require new reasonings and references. We mention this result here since it might also be interesting in its own.

For a building Δ of type (W, S), a chamber $C \in \mathcal{C}(\Delta)$, an apartment Σ of Δ, and an element w of W, we write $n_{\Sigma,w}(C)$ for the number of chambers $X \in \Sigma$ such that $\delta(C, X) = w$.

Proposition 1.6. *Let Δ be a building of type (W, S), let Σ be an apartment of Δ, and let C be any chamber of Δ. Let $w \in W$ and put $n(w) = n_{\Sigma,w}(C)$. Then $n(w) = 1$ if and only if $C \in \Sigma$. Also, $n(w)$ is even if and only if $C \notin \Sigma$. In particular, $n(w)$ is never an odd integer bigger than* 2.

Propositions 1.5 and 1.6 will be proved in Section 3. In Section 4 we state and prove our main results in the fullest detail and generality, mentioning some more consequences and counterexamples that show that our hypotheses are best possible. In the next section we gather some standard notation from building theory and prove two easy basic lemmas.

Finally we remark that similar results for a given gallery distance are not available. This could be investigated in the future. But the present paper shows that a complete answer is available in the case of W-valued distance.

2. PRELIMINARIES

In the following, a Coxeter group W with distinguished set of generators S is given. The *length* function $\ell : W \to \mathbb{N} \cup \{0\}$ is defined as usual. Recall that a *word* $(s_1, \ldots, s_n)$, with $s_i \in S$, $1 \leqslant i \leqslant n$, $n \in \mathbb{N} \cup \{0\}$, is *reduced* if $\ell(w) = n$ for $w = s_1 \ldots s_n$.

We now define the notion of a building. We consider a pair $\Delta = (\mathcal{C}(\Delta), \delta)$ consisting of a set $\mathcal{C}(\Delta)$, the elements of which are called *chambers*, and a map $\delta : \mathcal{C}(\Delta) \times \mathcal{C}(\Delta) \to W$, subject to the following conditions (where $C, D, E \in \mathcal{C}(\Delta)$):

(1) $\delta(C, D) = 1$ if and only if $C = D$,
(2) if $\delta(C, D) \in S$, then $\delta(D, C) = \delta(C, D)$,
(3) if $\delta(C, D) = s = \delta(D, E)$ and $s \in S$, then $\delta(C, E) \in \{1, s\}$.

Two chambers $C, D \in \mathcal{C}(\Delta)$ will be called *s-adjacent* if $\delta(C, D) = s \in S$. *Adjacent* chambers are chambers which are s-adjacent for some $s \in S$. A *gallery* γ is a sequence of chambers $\gamma = (C_0, C_1, \ldots, C_n)$ such that C_{i-1} and C_i are adjacent, for all $i > 0$. The word $(s_1, \ldots, s_n)$, where $s_i = \delta(C_{i-1}, C_i)$, for all $i > 0$, is called the *type* of γ. We also say that the gallery γ *connects* C_0 with C_n, and that its *length* is equal to n. If every gallery connecting C_0 with C_n is of length at least n, then we say that γ is *minimal*.

We call $\Delta = (\mathcal{C}(\Delta), \delta)$ a *building* (*of type* (W, S)) if it satisfies additionally the following two properties.

(4) For every chamber C, and every $s \in S$, there exists a chamber D which is s-adjacent to C.
(5) For any two chambers $C, D \in \mathcal{C}(\Delta)$ and any reduced word $(s_1, \ldots, s_n)$, $n \in \mathbb{N} \cup \{0\}$, we have $\delta(C, D) = s_1 \ldots s_n$ if and only if there exists a gallery of type $(s_1, \ldots, s_n)$ connecting C with D.

We remark that a gallery in Δ is minimal if and only if its type is reduced (see [6], Theorem 3.1). This means in particular that $d(C, D) := \ell(\delta(C, D))$ is in fact the length of a minimal gallery connecting C with D.

The following well known consequence of the properties (1) to (5) of the building Δ will be often tacitly used in our paper:

(6) If $\delta(C, D) = w$ and $\delta(D, E) = s \in S$, then $\delta(C, E) \in \{w, ws\}$ and $\delta(C, E) = ws$ whenever $\ell(ws) > \ell(w)$.

There is a canonical building $\Sigma(W, S)$ associated with the Coxeter system (W, S). It is defined as $\Sigma(W, S) = (W, \delta)$, with $\delta : W \times W \to W : (u, w) \mapsto u^{-1}w$. For every $w \in W$, left multiplication with w defines a permutation λ_w of W, which is s-adjacency preserving for all $s \in S$. If w is a conjugate of an element of S, then we call λ_w a *reflection of* $\Sigma(W, S)$. For any pair (w, ws) of s-adjacent chambers, $s \in S$, in $\Sigma(W, S)$, there is a (unique) reflection interchanging w and ws, namely $\lambda_{wsw^{-1}}$.

In the sequel, $\Delta = (\mathcal{C}(\Delta), \delta)$ will always denote a building of type (W, S).

A *panel* (*of type* $s \in S$) of Δ is a maximal set of pairwise (s-)adjacent chambers and is determined by any two members of it. The building Δ is called *thin* (respectively, *thick*) if each panel contains precisely two (respectively, at least three) chambers. For $J \subseteq S$, a *J-residue* is a maximal set of chambers $R \subseteq \mathcal{C}(\Delta)$ with the property $\delta(C, D) \in W_J$ (where $W_J := \langle J \rangle \leqslant W$) for all $C, D \in R$. The pair $(R, \delta|_{R\times R})$ is a building of type (W_J, J). A *residue* is a J-residue for some $J \subseteq S$. Let R be a residue of the building Δ, and let $C \in \mathcal{C}(\Delta)$. Then there is a unique chamber D in R closest to C (with respect to the gallery distance) and this is usually denoted by $D = \mathrm{proj}_R\, C$ and called the *projection of* C *onto* R. Now let Δ and Δ' be two buildings of type (W, S). Let θ be a permutation of S. Then a bijection $\varphi : \mathcal{C}(\Delta) \to \mathcal{C}(\Delta')$ is a θ-*isomorphism* if for any pair of chambers $C, D \in \mathcal{C}(\Delta)$, $\delta(C, D) = s \in S$ is equivalent to $\delta(\varphi(C), \varphi(D)) = \theta(s)$. If θ is the identity, then we talk about a *special isomorphism*. An *isometry* ψ from $\mathcal{C}$ to $\mathcal{C}'$, where $\mathcal{C} \subseteq \mathcal{C}(\Delta)$ and $\mathcal{C}' \subseteq \mathcal{C}(\Delta')$, is a map from $\mathcal{C}$ to $\mathcal{C}'$ such that for any two elements $C, D \in \mathcal{C}$, we have $\delta(C, D) = \delta(\psi(C), \psi(D))$. In particular, a special isomorphism from Δ to Δ' is also an isometry from Δ to Δ'. An *apartment* in Δ is an isometric image of $\Sigma(W, S)$ in Δ. Let Σ be such an apartment. Let C, D be two adjacent chambers in Σ. We define

$$\alpha_{C,D} = \{X \in \Sigma \mid d(X, C) < d(X, D)\},$$

and call this a *root*. Note that Σ is the disjoint union of the roots $\alpha_{C,D}$ and $\alpha_{D,C}$. The latter are called *opposite roots in* Σ. We remark that there is always a special automorphism σ of Σ exchanging C and D, and hence also $\alpha_{C,D}$ and $\alpha_{D,C}$ (this is clear by our definition of reflections in $\Sigma(W, S)$ above). We will also call σ a *reflection*. We shall need the following well known fact (see, e.g., [7], Chapter 2 and [6], Proposition 2.6).

Fact 2.1. *If* $X, Y \in \Sigma$ *are adjacent chambers with* $X \in \alpha_{C,D}$ *and* $Y \in \alpha_{D,C}$, *then* $\alpha_{X,Y} = \alpha_{C,D}$, $\alpha_{Y,X} = \alpha_{D,C}$, $\sigma(X) = Y$ *and* $\sigma(Y) = X$.

Notice that we view roots and apartments as sets of chambers (for convenience of notation), but it is clear that they have an additional structure induced by Δ. A basic property of apartments is that every pair of chambers is contained in at least one apartment (see 3.7 of [6]), and every minimal gallery between two such chambers is completely contained in it. Also, if C, D, E are three chambers in a common apartment, then $\delta(C, E) = \delta(C, D)\delta(D, E)$.

Let W be a Coxeter group with distinguished set S of involutive generators. For $w \in W$ and $S_1(w)$ as defined in Proposition 1.5 above we set $W_1(w) := W_{S_1(w)}$. This is a finite Coxeter group (see Theorem 2.16 of [6]) and thus has a unique element w_1^0 of maximal length, i.e., $\ell(w_1^0) = \max\{\ell(w) \mid w \in W_1(w)\}$. It is also shown in [6], Theorem 2.16, that there exists a reduced expression of w of the form $w = w_1^0 w_1$, i.e., $w_1 \in W$ and $\ell(w) = \ell(w_1^0) + \ell(w_1)$.

We now prove two elementary lemmas that we shall need later on. The first one is a more general version of Lemma 2.6 of [1], the proof of which differs from the one in [1] in that we cannot take advantage of the properties of the opposition relation. Hence some other arguments have to be used, and this justifies a detailed proof in the present paper.

Lemma 2.2. *Let C_1, C_2, C_3 be three different pairwise s-adjacent chambers of a building Δ of type (W, S), $s \in S$. Let Σ be an apartment of Δ which contains C_1 and C_2. Let α_i, $i = 1, 2$, be the unique root in Σ containing C_i but not C_{3-i}. Then there exists a root α_3 of Δ containing C_3 such that $\alpha_i \cap \alpha_3 = \emptyset$, $i = 1, 2$, and such that $\alpha_i \cup \alpha_3$ is an apartment Σ_{i3}, $i = 1, 2$.*

Proof. It is clear that $\alpha_1 \cup \{C_3\}$ is isometric to $\alpha_1 \cup \{C_2\}$. Hence, by Theorem 3.6 of [6], $\alpha_1 \cup \{C_3\}$ is contained in an apartment Σ_{13} of Δ. Define α_3 as the root in Σ_{13} containing C_3 but not C_1. Then, clearly, $\alpha_1 \cap \alpha_3 = \emptyset$. We now claim that also $\alpha_2 \cap \alpha_3 = \emptyset$. Indeed, if $X_i \in \alpha_i$, $i = 2, 3$, then the projection of X_i onto the s-panel defined by C_1, C_2 and C_3 is obviously C_i, hence $X_2 \neq X_3$ and the claim follows.

Now we show that $\alpha_2 \cup \alpha_3$ is isometric to $\alpha_1 \cup \alpha_2$, implying the lemma. Let σ be the reflection in Σ_{13} interchanging C_1 and C_3 (and hence also interchanging α_1 and α_3). We define the following map $\rho : \alpha_3 \cup \alpha_2 \to \alpha_1 \cup \alpha_2$. If $X \in \alpha_3$, then $\rho(X) = \sigma(X)$; if $X \in \alpha_2$, then $\rho(X) = X$. We show that ρ is an isometry. So consider two chambers X, Y in $\alpha_3 \cup \alpha_2$. If $\{X, Y\} \subseteq \alpha_3$ or $\{X, Y\} \subseteq \alpha_2$, then clearly $\delta(X, Y) = \delta(\rho(X), \rho(Y))$. So suppose $X \in \alpha_2$ and $Y \in \alpha_3$. Choose a minimal gallery $\gamma = (X = X_0, X_1, \ldots, X_{j-1}, X_j, \ldots, X_m = \rho(Y))$, with j the unique positive integer $\leqslant m$ such that $X_{j-1} \in \alpha_2$ and $X_j \in \alpha_1$ (this is well defined since $\alpha_1 \cup \alpha_2 = \Sigma$ is an apartment).

We claim that X_{j-1} and $\sigma(X_j)$ are t-adjacent, with $t := \delta(X_j, X_{j-1})$. Let Z be the unique chamber in Σ_{13} such that $\delta(X_j, Z) = t$. If Z were not in α_3, then it would be in α_1, and hence in Σ, implying $Z = X_{j-1}$. But this is impossible since $\Sigma_{13} \cap \alpha_2 = \emptyset$. Therefore $Z \in \alpha_3$, and since $X_j \in \alpha_1$, Fact 2.1 above implies $\sigma(X_j) = Z$, proving our claim.

Thus we can consider the gallery $\gamma' = (X_0, X_1, \ldots, X_{j-1}, \sigma(X_j), \ldots, \sigma(X_m) = \sigma(\rho(Y)) = \sigma(\sigma(Y)) = Y))$. Then γ' is contained in $\alpha_2 \cup \alpha_3$. Since the type of γ is reduced and the same as the type of γ', we obtain $\delta(X, \rho(Y)) = \delta(X, Y)$. The lemma is proved. $\square$

We remark that the map ρ of the previous proof is nothing else than the restriction to $\alpha_2 \cup \alpha_3$ of the retraction with center C_2 onto the apartment Σ (see p. 32 of [6]).

Lemma 2.3. *Let Δ be a building of type (W, S), let $A, B \in \mathcal{C}(\Delta)$ and let $u, v, w \in W$ with $w = uv$. Suppose, for all $X \in \mathcal{C}(\Delta)$, that $\delta(B, X) = v$ implies $\delta(A, X) = w$. Then $\delta(A, B) = u$.*

Proof. Choose an apartment Σ of Δ containing both A, B. Let X_0 be the unique chamber of Σ with $\delta(B, X_0) = v$. Our assumption implies $\delta(A, X_0) = w$. Consequently $\delta(A, B) = \delta(A, X_0)\delta(X_0, B) = wv^{-1} = u$. $\square$

3.1. Proof of Proposition 1.6

We now give the proof of Proposition 1.6. First note that, if $C \in \Sigma$, then $n(w) = 1$. We show that, if $C \notin \Sigma$, then $n(w) \equiv 0 \bmod 2$.

We will use an inductive argument based on the gallery distance, say m, from C to Σ. So let $(C = C_0, C_1, \ldots, C_m)$ be a minimal gallery with the property that $C_m \in \Sigma$. Then there exists a unique chamber C' in Σ such that C_m, C_{m-1} and C' are pairwise s-adjacent, for some $s \in S$. By Lemma 2.2, there are pairwise disjoint roots α_1, α_2 and α_3 such that $\alpha_1 \cup \alpha_2 = \Sigma$, $\Sigma' := \alpha_1 \cup \alpha_3$ and $\Sigma'' := \alpha_2 \cup \alpha_3$ are apartments, and such that $C_{m-1} \in \alpha_3$, $C' \in \alpha_1$ and $C_m \in \alpha_2$. There are two possibilities.

(1) Suppose $C \in \alpha_3$. This happens in particular when $m = 1$, providing the first step of the induction process. Then there are unique chambers $D' \in \Sigma'$ and $D'' \in \Sigma''$ with $\delta(C, D') = \delta(C, D'') = w$. If $D' \in \alpha_3$, then $D' = D''$, and consequently $n(w) = 0$; if $D' \in \alpha_1$, then $D'' \in \alpha_2$ and $D' \neq D''$ (because $\alpha_1 \cap \alpha_2 = \emptyset$). Hence in the latter case $n(w) = 2$.
(2) Suppose now $C \notin \alpha_3$. Then $C \notin \Sigma'$ and $C \notin \Sigma''$. Since the gallery distance from C to both, Σ' and Σ'' is less than m, we may apply the induction hypothesis, which gives us $n_{\Sigma',C}(w) \equiv n_{\Sigma'',C}(w) \equiv 0 \bmod 2$. Let x be the number of chambers D of $\Sigma' \cap \Sigma'' = \alpha_3$ with $\delta(C, D) = w$. We then have $n(w) = n_{\Sigma,C}(w) = n_{\Sigma',C}(w) + n_{\Sigma'',C}(w) - 2x \equiv 0 \bmod 2$.

The proof of Proposition 1.6 is complete.

So the proof of the implication (a) $\Rightarrow$ (b) of Proposition 3.2 in [1] directly carries over to arbitrary buildings and arbitrary $w \in W$ once Lemma 2.2 is established. One might ask about the converse implication (b) $\Rightarrow$ (a), i.e., about an analogous combinatorial characterization of apartments in buildings using chambers at distance w instead of opposite chambers. This question will be studied in detail in a forthcoming paper (see [2]). Let us just mention here that one only gets new characterizations of apartments if one considers “sufficiently many” Weyl distances w at the same time, not just a single one.

3.2. Proof of Proposition 1.5

Throughout this section, let Δ be a thick building of type (W, S), and let $w \in W$ be arbitrary. Put $n := \ell(w)$. Let C, D, E be three chambers of Δ. If C, D, E are pairwise s-adjacent, for some $s \in S_1(w)$, then they satisfy Condition $(*)$. Indeed, let P be the panel containing C, D, E. If $\delta(Y, X) = w$, for some $Y \in \{C, D, E\}$, then (since $s \in S_1(w)$), $\delta(Z, X) = sw$ for $Z = \mathrm{proj}_P X$. So $\delta(Y, X) = w$ for all $Y \in \{C, D, E\} \setminus \{Z\}$.

Therefore, we suppose from now on that, for any chamber $X \in \mathcal{C}(\Delta)$, the number of chambers $Y \in \{C, D, E\}$ such that $\delta(Y, X) = w$, is not equal to 1 (we refer to this as Condition $(*)$, as in the statement of the proposition). Our aim is to show that C, D, E are pairwise s-adjacent, for some $s \in S_1(w)$.

We break up the proof in a series of steps, which we number for future reference.

Step 1. We start by choosing an arbitrary but fixed apartment Σ of Δ containing the chambers C and D. Let C_w be the unique chamber of Σ such that $\delta(C, C_w) = w$. Then $\delta(D, C_w) \neq w$ and hence Condition $(*)$ implies $\delta(E, C_w) = w$. We now define the positive integer i as the smallest integer with the property that there is a minimal gallery $\gamma = (C = C_0, C_1, \ldots, C_n = C_w)$ from C to C_w (thus completely contained in Σ), with $\delta(C, C_i) = \delta(E, C_i)$. Note that i is well defined and $i \leqslant n$, since we have $\delta(E, C_n) = w = \delta(C, C_n)$.

We fix some more notation. Let $(s_1, \ldots, s_n)$ be the (reduced) type of the minimal gallery γ introduced above. Put $w_0 = s_1 s_2 \ldots s_{i-1}$, $w_1 = s_{i+1} \ldots s_{n-1} s_n$ and $s = s_i$. Let p be the panel of type s determined by C_{i-1} and C_i. We denote by E_p the projection of E onto p. Our assumption on i implies that $\delta(E, E_p) = w_0$ and $E_p \notin \Sigma$. We set $C_{i-1} = C_p$ and $C_i = D_p$. We have $\delta(C, C_p) = w_0$ and C_p is the projection of C onto p. We shall justify the notation for D_p in Step 2 below with a similar property of D_p with respect to D.

Step 2. We show that $\delta(D, D_p) = w_0$.

We first prove that $\delta(D, E_p) = w_0 s$. Let $X \in \mathcal{C}(\Delta)$ be arbitrary, but such that $\delta(E_p, X) = w_1$. We have $\delta(C, X) = w_0 s w_1 = w$ and $d(E, X) \leqslant \ell(w_0) + \ell(w_1)$, hence $\delta(E, X) \neq w$. Therefore, Condition $(*)$ implies $\delta(D, X) = w$. Since X was arbitrary, Lemma 2.3 implies $\delta(D, E_p) = w_0 s$.

So $\delta(D, D_p) \in \{w_0 s, w_0\}$. But, as $C, D, C_p, D_p \in \Sigma$ and $\delta(C, D_p) = w_0 s$, we have $\delta(D, D_p) \neq w_0 s$. We conclude that $\delta(D, D_p) = w_0$.

Note that $\delta(D, D_p) = w_0$ implies that $\delta(C, D) = w_0 s w_0^{-1}$, since $C, D, D_p \in \Sigma$.

Step 3. Our aim is to show that $w_0 = 1$. This will be eventually achieved in Step 5. To this end we assume throughout that $w_0 \neq 1$ and we choose arbitrarily a decomposition $w_0 = w_0' s'$ with $s' \in S$ and $\ell(w_0') = \ell(w_0) - 1$. Note also that $s \neq s'$.

Here we show that $\ell(w_0 s s') = \ell(w_0) + 2$.

Indeed, assume by way of contradiction that $\ell(w_0 s s') < \ell(w_0) + 2$. Since $\ell(w_0 s) = \ell(w_0) + 1$, we have $\ell(w_0 s s') = \ell(w_0)$. Set $\widetilde{w}_0 := w_0 s s'$, i.e., $\widetilde{w}_0 s' = w_0 s$. Let D' be the projection of D onto the panel of type s' containing D_p. Since $D, D_p \in \Sigma$, we also have $D' \in \Sigma$. Hence $\delta(C, D') = \delta(C, D_p) s' = w_0 s s' = \widetilde{w}_0$.

For any chamber X satisfying $\delta(D', X) = s' w_1$, we have $\delta(C, X) = \widetilde{w}_0 s' w_1 = (w_0 s s') s' w_1 = w_0 s w_1 = w$ (since $\ell(w) = \ell(\widetilde{w}_0) + \ell(s' w_1)$), and $d(D, X) \leqslant \ell(w_0') + \ell(s' w_1) = \ell(w_0) + \ell(w_1) < \ell(w)$. So $\delta(D, X) \neq w$. Condition $(*)$ implies $\delta(E, X) = w$ and Lemma 2.3 implies $\delta(E, D') = \widetilde{w}_0$. So, in Σ, we have a minimal gallery $\gamma' = (C, \ldots, D', D_p, \ldots, C_w)$, with $\delta(C, D') = \widetilde{w}_0 = \delta(E, D')$. But $\ell(\widetilde{w}_0) = \ell(w_0) = i - 1$, contradicting the minimality assumption on i in Step 1.

Step 4. Now we prove that $\ell(s' w_1) < \ell(w_1)$.

We keep the notation for C_w, C_p, D_p, E_p and D' of the previous steps. Let D'' be a chamber s'-adjacent to both D' and D_p (D'' exists in view of the thickness of Δ), and let X be an arbitrary chamber such that $\delta(D'', X) = s w_1$. Then, since $\delta(D, D'') = w_0' s'$, we have $\delta(D, X) = w_0' s' s w_1 = w_0 s w_1 = w$. Condition $(*)$ implies that $\delta(C, X) = w$ or $\delta(E, X) = w$.

First we assume that $\delta(C, X) = w$. Consider a gallery $(C, \ldots, C_p, D_p, D'', \ldots, X)$ with $\delta(C, C_p) = w_0$, $\delta(C_p, D_p) = s$, $\delta(D_p, D'') = s'$ and $\delta(D'', X) = sw_1$. By Step 3, the product w_0ss' is reduced, hence $\delta(C, D'') = w_0ss'$. Choose a reduced decomposition $w_1 = t_1t_2 \ldots t_k$ of w_1, with $t_j \in S$, for $1 \leqslant j \leqslant k$, and set $t_0 := s$. Consider the gallery $(D'' = D_0, D_1, \ldots, D_k = X)$ of type $(t_0, t_1, \ldots, t_k)$. On one hand, it follows from this that $\delta(C, D_k)$ is equal to $w_0ss't$, where $t = t_{i_0}t_{i_1} \ldots t_{i_m}$, with $0 \leqslant i_0 < i_1 < \cdots < i_m \leqslant k$, for some $m \leqslant k$. On the other hand, $\delta(C, D_k) = w = w_0sw_1$, and this is a reduced expression. Since the product $w_0ss't_0 \ldots t_k$ is of length $\ell(w_0) + 2 + \ell(sw_1)$, only the following possibilities have to be considered.

(i) $w_0ss't_0 \ldots t_k = w$. So, $w_0ss'(sw_1) = w_0sw_1$, hence $ss' = 1$, which implies $s = s'$, a contradiction.
(ii) $w_0ss't_0 \ldots \widehat{t_l} \ldots t_k = w$, for some l, $0 \leqslant l \leqslant k$ (here, the notation $\hat{x}$ means, as usual, that x is deleted). But then the lengths of the products on the two sides of the equation have different parity, again a contradiction.
(iii) $w_0ss't_0 \ldots \widehat{t_l} \ldots \widehat{t_{l'}} \ldots t_k = w$, with $0 \leqslant l < l' \leqslant k$. Then comparing lengths, we see that $w_0ss't_0 \ldots \widehat{t_l} \ldots \widehat{t_{l'}} \ldots t_k$ is a reduced decomposition of w, and hence $s't_0 \ldots \widehat{t_l} \ldots \widehat{t_{l'}} \ldots t_k$ gives a reduced decomposition of w_1, showing $\ell(s'w_1) = \ell(w_1) - 1$, and this we wanted to prove.

A completely similar argument applies if $\delta(E, X) = w$. We only have to start with a gallery $(E, \ldots, E_p, D_p, D'', \ldots, X)$ in this case.

Step 5. We now derive a final contradiction (still assuming $w_0 \neq 1$). We consider again the apartment Σ containing C and D. From Step 2 we infer $\delta(C, D) = w_0sw_0^{-1}$. We also have

$$w_0 = w_0's' \text{ with } \ell(w_0') = \ell(w_0) - 1;$$
$$w_1 = s'w_1' \text{ with } \ell(w_1') = \ell(w_1) - 1 \text{ (see Step 4);}$$
$$w = w_0sw_1, \text{ with } w_0sw_1 \text{ a reduced decomposition.}$$

Denote by $\widetilde{C}$ the projection of C onto the panel of type s' of C_p and consider the following (not necessarily minimal) gallery $(C, \ldots, \widetilde{C}, C_p, D_p, \ldots, D)$ in Σ, where $(C, \ldots, \widetilde{C})$ and $(\widetilde{C}, C_p, D_p, \ldots, D)$ are minimal (recall that w_0ss' is reduced by Step 3). Denote by α and $\widetilde{\alpha}$ the roots of Σ containing C_p but not $\widetilde{C}$, and containing $\widetilde{C}$ but not C_p, respectively. If D_w is the unique chamber of Σ such that $\delta(D, D_w) = w$, then $C_w \in \alpha$ and $D_w \in \widetilde{\alpha}$ (the latter because $w = w_0ss'w_1'$ is a reduced decomposition of w by Steps 3 and 4, and because $\delta(D, C_p) = w_0s$, $\delta(D, \widetilde{C}) = w_0ss'$). Let F be a chamber s'-adjacent to $\widetilde{C}$ and C_p (F exists by the thickness assumption), and let α' be a third root of Δ containing F, disjoint with α and $\widetilde{\alpha}$, and such that $\alpha \cup \alpha'$ and $\widetilde{\alpha} \cup \alpha'$ are apartments of Δ (the existence of α' is guaranteed by Lemma 2.2). Let C_w' and D_w' be the chambers in α' satisfying $\delta(C, C_w') = w = \delta(D, D_w')$. Then clearly $\delta(F, C_w') = sw_1$ and $\delta(F, D_w') = w_1'$; in particular $C_w' \neq D_w'$. So

$$\delta(C, C_w) = w \neq \delta(D, C_w), \text{ implying } \delta(E, C_w) = w \text{ by Condition } (*);$$

$$\delta(D, D_w) = w \neq \delta(C, D_w), \text{ implying } \delta(E, D_w) = w \text{ by Condition } (*);$$
$$\delta(C, C'_w) = w \neq \delta(D, C'_w), \text{ implying } \delta(E, C'_w) = w \text{ by Condition } (*);$$
$$\delta(D, D'_w) = w \neq \delta(C, D'_w), \text{ implying } \delta(E, D'_w) = w \text{ by Condition } (*).$$

Suppose now $\delta(E, X) = w$, with X a chamber in $\alpha \cup \widetilde{\alpha} \cup \alpha'$. Then, again by Condition $(*)$, either $\delta(C, X) = w$ or $\delta(D, X) = w$. But, considering the apartments Σ, $\widetilde{\alpha} \cup \alpha'$ and $\alpha \cup \alpha'$ we obtain $X \in \{C_w, C'_w\}$ and $X \in \{D_w, D'_w\}$. Hence the apartments $\alpha \cup \alpha'$ and $\widetilde{\alpha} \cup \alpha'$ contain exactly 3 chambers at distance w from E, contradicting Proposition 1.6.

So we conclude $w_0 = 1$, which immediately implies that C and E are s-adjacent. But then Step 2 (which does not assume $w_0 \neq 1$), says that D and C are s-adjacent. Also, $w = w_0 s w_1 = s w_1$, and the latter is a reduced expression; therefore $s \in S_1(w)$. Hence Proposition 1.5 is proved.

4. MAIN RESULTS, SOME FURTHER CONSEQUENCES AND EXAMPLES

We shall now state a precise version of our main result. For that, we need some additional terminology and notation.

Let (W, S) be a Coxeter system, and let $w \in W$ be arbitrary. Recall from Section 2 that $S_1(w)$ generates a spherical Coxeter group W_1 and, denoting the longest element in that group by w_1^0, that w can be written as $w = w_1^0 w_1$, with $\ell(w) = \ell(w_1^0) + \ell(w_1)$. But now $S_1(w_1) =: S_2(w)$ again generates a spherical Coxeter group W_2 with some unique longest element w_2^0, and hence we may write $w = w_1^0 w_2^0 w_2$, with $\ell(w) = \ell(w_1^0) + \ell(w_2^0) + \ell(w_2)$. Going on like that, we obtain a unique reduced decomposition of $w \in W_1 W_2 \ldots W_k$ as $w = w_1^0 w_2^0 \ldots w_k^0$, for some natural number k, where w_j^0 is the longest word of the spherical Coxeter subgroup $W_{S_j(w)} =: W_j$, $1 \leqslant j \leqslant k$. We now have $S_j(w) = S(w_j^0)$ and $S(w)$ is the union of all $S_j(w)$. A similar reduced decomposition $v = v_1^0 \ldots v_m^0$ can be defined for $v = w^{-1}$, but note that $m \neq k$ is possible (see Example 4.3 below).

We can now state and prove a sharp version of our main result.

Theorem 4.1. *Let Δ and Δ' be two thick buildings of type (W, S) and let $w \in W$. Let $\varphi : \mathcal{C}(\Delta) \to \mathcal{C}(\Delta')$ be a surjective map such that $\delta(C, D) = w$ if and only if $\delta'(\varphi(C), \varphi(D)) = w$, for all $C, D \in \mathcal{C}(\Delta)$. Then φ is a bijection and both φ and its inverse preserve $S_i(w)$-adjacency, for all $i \in \{1, 2, \ldots, k\}$. Similarly for w^{-1} and $S_j(w^{-1})$-adjacency, for all $j \in \{1, \ldots, m\}$. Finally, $\delta(C, D) = u$ if and only if $\delta'(\varphi(C), \varphi(D)) = u$, for all $u \in \{w_1^0, \ldots, w_k^0, v_1^0, \ldots, v_m^0\}$, with the decompositions $w = w_1^0 \ldots w_k^0$ and $w^{-1} = v_1^0 \ldots v_m^0$ introduced above.*

Proof. We show the theorem for w, the result for w^{-1} then follows, because $\delta(C, D) = w$ if and only if $\delta(D, C) = w^{-1}$, which is an immediate consequence of property (5) of the definition of buildings. First we claim that φ is injective. Indeed, if $\varphi(C) = \varphi(D)$, then our condition implies that every chamber in Δ at distance w from C is also at distance w from D, clearly false if $C \neq D$ (consider an apartment containing C and D).

We now show that φ preserves $S_1(w)$-adjacency. Indeed, if C, D are $S_1(w)$-adjacent chambers, then by the thickness assumption there is a third chamber E such that C, D, E are three different chambers satisfying Condition $(*)$. Since φ and its inverse preserve the distance w, Condition $(*)$ is satisfied for $\varphi(C), \varphi(D), \varphi(E)$, hence by Proposition 1.5, the chambers $\varphi(C)$ and $\varphi(D)$ are $S_1(w)$-adjacent. Similarly for φ^{-1}.

Hence $S_1(w)$-residues are mapped bijectively onto $S_1(w)$-residues, with adjacency being preserved. It now follows from Theorem 3.21 of [7] that these bijections are θ-isomorphisms for some permutation $\theta : S_1(w) \to S_1(w)$, which might however depend on the chosen residues (see Example 4.4 below). Now let w_1^0 be the longest word in $W_1 = \langle S_1(w) \rangle$, and put $w = w_1^0 w_1$, as above. Let C, D be two arbitrary chambers of Δ with $\delta(C, D) = w_1$. We claim that $\delta'(\varphi(C), \varphi(D)) = w_1$. Indeed, let $X \in \mathcal{C}(\Delta')$ be arbitrary but such that $\delta'(X, \varphi(C)) = w_1^0$. This means that X and $\varphi(C)$ belong to a common (spherical) $S_1(w)$-residue R of Δ' and are opposite in R. Since φ^{-1} induces an isomorphism from R to $\varphi^{-1}(R)$, $\varphi^{-1}(X)$ and C are opposite in $\varphi^{-1}(R)$, and therefore $\delta(\varphi^{-1}(X), C) = w_1^0$. Since $w_1^0 w_1$ is reduced, this implies that $\delta(\varphi^{-1}(X), D) = w$, and hence $\delta'(X, \varphi(D)) = w$. Now Lemma 2.3 (applied to w^{-1}) yields $\delta'(\varphi(C), \varphi(D)) = w_1$, whence the claim. Conversely, one shows in the same manner that $\delta'(A, B) = w_1$ implies that $\delta(\varphi^{-1}(A), \varphi^{-1}(B)) = w_1$. Hence the assumptions of the theorem are also satisfied for w_1 instead of w. Applying what we just proved, we obtain that also $S_1(w_1)$-adjacency, i.e., $S_2(w)$-adjacency, is preserved by φ and φ^{-1}. Going on like this, the assertion follows. □

Theorem 4.1 is the best one can assert in the general case. In more specific cases, it is possible that this implies that φ is a (special) isomorphism. For instance, in the spherical case, if $S(w) = S$, then φ is automatically a (not necessarily special) isomorphism. Clearly, if $S(w) = S$, and if W has a unique decomposition with respect to S, then every $S_j(w)$ is a singleton, and so φ is a special isomorphism. This is Corollary 1.4.

Obviously, Theorem 1.1 follows from Theorem 4.1. Also, Corollary 1.2 is clearly true. Corollary 1.3 follows from Corollary 1.2 and Theorem 3.21 of [7].

We now present a counterexample showing that, in Corollaries 1.2 and 1.3, the condition that every $s \in S$ is involved in w is necessary.

Example 4.2. Let Δ be a thick 2-spherical building of type (W, S) having two different but isomorphic residues R and R' of type $S \setminus \{s\}$, for some $s \in S$ with the property that $st \neq ts$ for some $t \in S$ (it is obviously easy to find examples of this situation!). Choose an element $w \in W$ which does not involve s. If we define φ on the set of chambers of Δ as the identity on $\mathcal{C}(\Delta) \setminus (\mathcal{C}(R) \cup \mathcal{C}(R'))$, and on the set of chambers of R and R' as corresponding with a pair of isomorphisms from R to R' and from R' to R, then we see that the conditions of the corollaries are satisfied, but φ does not preserve adjacency of chambers, since s-adjacency is clearly not preserved.

Concerning Theorem 4.1 again, we remark that the information obtained by considering w is not always the same as the information obtained by considering w^{-1}, and so for a specific w, it is worthwhile to calculate the decompositions in both cases and apply the theorem.

Example 4.3. Let Δ be a thick building of type (W, S), with $S = \{s_1, s_2, s_3, s_4\}$ and $W = \langle s_1, s_2, s_3, s_4 : s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1s_3)^2 = (s_1s_4)^2 = (s_2s_4)^2 = (s_1s_2)^3 = (s_3s_4)^3 = 1\rangle$. Consider first $w = s_1s_2s_1s_4s_3s_4$. Then Theorem 4.1 applied to w says that, if φ preserves w-distance, then it preserves $\{s_1, s_2\}$-adjacency, s_3-adjacency and s_4-adjacency, since $S_1(w) = \{s_1, s_2, s_4\}$, $S_2(w) = \{s_3\}$ and $S_3(w) = \{s_4\}$. From this we can not directly derive that φ is an isomorphism. (Note that Δ is not 2-spherical since the order of s_2s_3 is infinite.) However, also using that $S_1(w^{-1}) = \{s_1, s_3, s_4\}$, $S_2(w^{-1}) = \{s_2\}$ and $S_3(w^{-1}) = \{s_1\}$, we can conclude that φ is a special isomorphism.

As a second (asymmetrical) case let us now consider the same building Δ but with φ and φ^{-1} preserving the w-distance for $w = s_1s_2s_1s_4s_3$. We observe that $S_1(w) = \{s_1, s_2, s_4\}$, $S_2(w) = \{s_3\}$ and $S_1(w^{-1}) = \{s_1, s_3\}$, $S_2(w^{-1}) = \{s_2, s_4\}$, $S_3(w^{-1}) = \{s_1\}$, demonstrating first of all that the decompositions for w and w^{-1} can indeed have different lengths as remarked directly before Theorem 4.1. Secondly, we conclude with this theorem that φ and φ^{-1} preserve s_1-adjacency, s_3-adjacency and $\{s_2, s_4\}$-adjacency (which follows already from considering w^{-1} alone). However, we can even do better. Recall from Theorem 4.1 that also the Weyl distance $(v_2^0v_3^0)^{-1} = v_3^0v_2^0 = s_1s_2s_4$ is preserved by φ and φ^{-1}. Using the preservation of $s_1s_2s_4$, this theorem implies that φ also preserves $\{s_1, s_4\}$-adjacency and s_2-adjacency, yielding finally that also in this case φ must be a special isomorphism.

We remark in passing that additional information might also be obtained from the fact that certain spherical buildings only admit special isomorphisms. For instance, if the $S_1(w)$ residues are of type C_n with $n \geqslant 3$, then we do not only know that $S_1(w)$-adjacency is preserved but even s-adjacency for all $s \in S_1(w)$. So one might get the impression that, if $S(w) = S$, then we always have an isomorphism, but that we cannot show it in general. This is false, and we give a counterexample now.

Example 4.4. Let Δ be a thick building of type (W, S), where $S = \{s_1, s_2, s_3\}$ and $W = \langle s_1, s_2, s_3\colon\ s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^n = 1\rangle$, with $n \geqslant 2$ any positive integer. We define a second building Δ' of type (W, S) as follows. We take a second copy of the set $\mathcal{C}(\Delta)$ and for $C \in \mathcal{C}(\Delta)$, we denote by C' the corresponding chamber of Δ' (and we gather all these chambers in the set $\mathcal{C}(\Delta')$). We choose freely a $\{1, 2\}$-residue R of Δ and define adjacency of chambers in Δ' as follows. If C and D are s_i-adjacent in Δ, and not both C and D belong to R, then C' and D' are s_i-adjacent in Δ', for all $i \in \{1, 2, 3\}$. Also, if C, D are s_1-adjacent (s_2-adjacent) in Δ and $C, D \in R$, then C and D are s_2-adjacent (s_1-adjacent) in Δ'. We show that Δ' is indeed a thick building of type (W, S). First we have to define the W-valued distance δ'. In order to do so, we consider two chambers $C', D' \in \mathcal{C}(\Delta')$. Put $W_{12} = \langle s_1, s_2\rangle \leqslant W$, and let $W_{12}^{\times} = W_{12} \setminus \{1\}$. The element $w := \delta(C, D)$ has a reduced decomposition as a product in $W_{12}s_3W_{12}^{\times}s_3W_{12}^{\times}\ldots s_3W_{12}^{\times}s_3W_{12}$, and we claim that the number of factors in $W_{12}^{\times}$ and the number of factors equal to s_3 is uniquely determined by w. Moreover, the length of the factor in each W_{12} (respectively, $W_{12}^{\times}$) is fixed, and the decomposition in s_1, s_2 of that factor is consequently unique except if it is equal to the longest word of W_{12}. These claims follow immediately from the fact that two reduced decompositions may be transformed into each other by a sequence of elementary homotopies, see Theorem 2.11 of [6]. And clearly, in view of the presentation of W, the only elementary homotopy available is the replacement of the longest element of W_{12} by one of its two equivalent expressions. Using these facts about $w = \delta(C, D)$, it now follows that every minimal gallery between C and D must pass through the same $\{s_1, s_2\}$-residues, and, while passing, the first (respectively, the last) chamber of the subgallery obtained by restriction to such a residue, is also always the same. If none of these residues coincides with R, then we put $\delta'(C', D') = \delta(C, D)$. Otherwise, we interchange in any reduced expression $\delta(C, D) \in W_{12}s_3W_{12}^{\times}\ldots s_3W_{12}$ the s_1 and s_2 in the positions that correspond with the subgallery that has its chambers in R, and we define the resulting new product as $\delta'(C', D')$. This is well defined by the foregoing discussion. In order to show that Δ' is a building, we have to check whether for two arbitrary chambers $C', D' \in \mathcal{C}(\Delta')$, and whenever $(s_{i_1}, \ldots, s_{i_t})$, $i_j \in \{1, 2, 3\}$ for $1 \leqslant j \leqslant t$, $t \in \{1, 2, \ldots\}$, is a reduced word, then $\delta'(C', D') = s_{i_1}\ldots s_{i_t}$ if and only if there is gallery of type $(s_{i_1}, s_{i_2}, \ldots, s_{i_t})$ between C' and D'. But this follows immediately from our definition of δ', and the fact that Δ is a building of type (W, S).

Now consider the map $\varphi: \mathcal{C}(\Delta) \to \mathcal{C}(\Delta'): X \mapsto X'$. Let w_{12}^0 be the longest element of W_{12}. Then it is easy to see that for any two chambers C, D of Δ we have $\delta(C, D) = w_{12}^0 s_3$ if and only if $\delta'(C', D') = w_{12}^0 s_3$ (this essentially follows from the fact that, if two chambers X, Y of Δ are opposite in R, then X' and Y' are opposite in the corresponding $\{s_1, s_2\}$-residue of Δ'). But clearly, φ does not define an isomorphism between Δ and Δ' since some s_1-adjacent chambers are mapped onto s_1-adjacent chambers (those not belonging to R), and others are mapped onto s_2-adjacent chambers (those of R). This concludes our counterexample. Note that φ preserves adjacency of chambers, as claimed by Corollary 1.2.

We now mention some other consequences of Theorem 4.1. The *chamber graph* of a building is the graph (V, E), where the set of vertices is the set of chambers of the building, and adjacency is adjacency of chambers.

Corollary 4.5. *Let Δ be a thick building of type (W, S), and let $w \in W$ be arbitrary but such that $S(w) = S$. Then the chamber graph of Δ is completely and uniquely determined by the set $\Omega(w)$ of ordered pairs $(C, D) \in \mathcal{C}(\Delta) \times \mathcal{C}(\Delta)$ with $\delta(C, D) = w$. If Δ is additionally 2-spherical, then Δ is (up to a permutation of the types which induces an automorphism of the Weyl group) uniquely determined by $\Omega(w)$. Finally, if w has a unique reduced decomposition (and Δ is arbitrary), then Δ is completely and uniquely determined by $\Omega(w)$.*

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