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Sharp Homogeneity in Affine Planes, and in some Affine Generalized Polygons

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Abstract. Let G be a collineation group of a generalized $(2n+1)$-gon Γ and let L be a line such that every symmetry σ of any ordinary $(2n+1)$-gon in Γ containing L with $\sigma(L) = L$ extends uniquely to a collineation in G. We show that Γ is then a Desarguesian projective plane. We also describe the groups G that arise. As a corollary, we treat the analogous problem without the restriction $\sigma(L) = L$.

1 Introduction

The classification of geometries satisfying homogeneity conditions has a rich history. Often transitivity conditions alone are not restrictive enough to allow a classification (see [9, 14, 12]); one needs also some kind of rigidity. In this paper, we continue earlier investigations [4, 5, 18] by considering affine planes Γ with collineation groups G which act sharply transitively on the triangles in Γ; we also consider affine generalized polygons with collineation groups satisfying an analogous condition (see Section 2). In contrast to the situation in [5] (and also to a certain extent in [4]), all Desarguesian planes arise here.

We recall some definitions. Let $m \geq 2$ be a positive integer. A *(thick) generalized m-gon* is a point-line incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with point set $\mathcal{P}$, line set $\mathcal{L}$ and symmetric incidence relation $\mathtt{I}$, whose incidence graph (i.e., the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and adjacency relation $\mathtt{I}$) has diameter m (the maximal distance between two vertices) and girth $2m$ (the length of the smallest cycle), and which contains an ordinary $(m+1)$-gon as a subgeometry (see the monograph [16]). In this paper, we consider only the case where m is odd, because it is difficult to control the involutive collineations if m is even (cp. the results in [18] for $m = 4$ and $m = 6$), in particular, there are too many possibilities for the fixed point structure of an involution. If $m = 3$, then Γ is a projective plane in the usual sense. Distances in Γ refer to the distances measured in the incidence graph of Γ and the distance function is denoted with δ.

An ordinary m-gon (viewed as a subgeometry) of Γ is called an *apartment* of Γ. An *ordered ordinary k-gon* of Γ (with $k \geq m$) is a cycle $(x_0, x_1, \ldots, x_{2k-1}, x_0)$ in

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the incidence graph of Γ, with $x_0 \in \mathcal{P}$, with $x_i \mathtt{I} x_{i+1}$ (indices to be considered modulo $2k$), and such that $x_i \neq x_j$ for $i \not\equiv j \bmod 2k$. We emphasize that we view a cycle (and hence an ordered ordinary k-gon) as a closed path with a distinguished origin (x_0) and a distinguished direction (from x_0 to x_1 and *not* to x_{2k-1}). A *simple path* in Γ *of length* k is a sequence $(x_0, x_1, \ldots, x_k)$ of points and lines, with x_0 a point, such that $x_{i-1} \mathtt{I} x_i$ for all $i \in \{1, 2, \ldots, k\}$, and such that $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$.

A *collineation* of Γ is a pair of permutations, one of $\mathcal{P}$ and one of $\mathcal{L}$, such that two elements are incident if and only if their images are incident. The set of all collineations forms a group, the *full collineation group of* Γ. Every subgroup of that full collineation group will be called a *collineation group*. An *axial collineation* of a generalized m-gon Γ, with $m = 2n + 1$ odd, is a collineation of Γ fixing all elements at distance $\leq n$ from some point p and fixing all elements at distance $\leq n$ from some line L. If $p \mathtt{I} L$, then we call this collineation an *axial elation*. If it is not the identity, then p and L are unique with the above defining property and we call them *center* and *axis*, respectively. Finally, a *Baer involution* is defined to be a collineation of order 2 the fixed points of which form the point set of a thick subpolygon of Γ.

For each alternative division ring K there is a unique projective plane $\mathrm{PG}(2, K)$ constructed using coordinates (see e.g. [7]). If K is a skew field, then $\mathrm{PG}(2, K)$ arises from a 3-dimensional vector space V over K by taking as points all vector lines of V and as lines all vector planes of V, with natural incidence relation. This plane is often referred to as the *Desarguesian projective plane over* K.

An *affine plane* Γ is an incidence structure obtained from a projective plane by deleting a fixed line (called the *line at infinity*) and all points incident with it. The projective plane can be uniquely and canonically recovered from Γ and is called the *projective completion* of Γ, denoted Γ^c. The points of Γ^c not belonging to Γ are called the *points at infinity* of Γ. The affine plane $\mathrm{AG}(2, K)$ over a skew field K has the point set $K^2 = K \times K$. The projective completion of $\mathrm{AG}(2, K)$ is isomorphic to $\mathrm{PG}(2, K)$.

A *planar* collineation π of $\mathrm{PG}(2, K)$, or of $\mathrm{AG}(2, K)$, is a collineation which fixes all points of a subplane; this means that π is conjugate to a collineation induced by a semilinear mapping with identity matrix and non-identity companion skew field automorphism. Note that it suffices to conjugate with *linear* collineations, i.e., collineations with trivial companion skew field automorphism.

2 Statement of the results

The following theorem is the main result of this paper. The stabilizer $G_{\gamma,L}$ in Condition $(*)$ below is the subgroup of G leaving invariant the point set $\{x_0, x_2, \ldots, x_{2m}\}$ and the line set $\{x_1, x_3, \ldots, x_{2m+1}\}$ of

$$\gamma = (\{x_0, x_2, \ldots, x_{2m}\}, \{x_1, x_3, \ldots, x_{2m+1}\}, \mathtt{I})$$

and the set consisting of the two points of γ on L.

Theorem 2.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized m-gon, where $m \geq 3$ is odd, let $L \in \mathcal{L}$ be a line and let G be a collineation group of Γ fixing L with the following property:*

($*$) *for every ordinary $(m+1)$-gon γ of Γ containing L, the stabilizer $G_\gamma = G_{\gamma,L}$ has order 2 and acts faithfully on γ.*

Then $m = 3$ and the projective plane Γ is Desarguesian; hence the affine plane arising from Γ by taking L as the line at infinity is of the form $\mathrm{AG}(2, K)$ for some skew field K. The group G contains the affine group

$$\mathrm{ASL}^{\pm}(2, K) := \mathrm{ASL}(2, K)\langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle .$$

Conversely, let K be any skew field, let $\Gamma := \mathrm{PG}(2, K)$ and let $\mathrm{AG}(2, K)$ be the affine plane arising from $\mathrm{PG}(2, K)$ by declaring any line L to be the line at infinity. Then the group $\mathrm{ASL}^{\pm}(2, K)$ and each larger group of collineations of $\mathrm{AG}(2, K)$ which contains no planar element except the identity satisfy the assumption on G in ($*$).

The group $\mathrm{ASL}^{\pm}(2, K)$ is the group of all affine maps

$$x \mapsto Mx + b \in \mathrm{AGL}(2, K)$$

where the linear map M has determinant 1 or -1; for non-commutative K one has to read this as the Dieudonné determinant, which takes values in the commutator factor group $K^\times/[K^\times, K^\times]$, compare ARTIN [1] Chapter 4 or DIEUDONNÉ [3] II.1. It is convenient to consider the determinant as a group homomorphism defined on $\mathrm{AGL}(2, K)$ which is trivial on translations $x \mapsto x + b$.

In the following we also need the semilinear affine group $\mathrm{A\Gamma L}(2, K) = \{(A, \phi) \mid A \in \mathrm{AGL}(2, K), \phi \in \mathrm{Aut}\, K\}$; here a pair (A, ϕ) denotes the composition of A with the map $K^2 \to K^2 : (x, y) \mapsto (x^\phi, y^\phi)$.

Let K be a skew field. A mapping $\alpha : K^\times \to \mathrm{Aut}\, K$ is called a *Dickson map* if $\alpha(x)\alpha(y) = \alpha(yx^{\alpha(y)})$ for $x, y \in K^\times$. Then $\{x \mapsto ax^{\alpha(a)} \mid a \in K^\times\}$ is a subgroup of $\Gamma\mathrm{L}(1, K)$ which is sharply transitive on $K^\times$. Thus Dickson maps describe the (not necessarily normal) subgroups of $\Gamma\mathrm{L}(1, K) = \mathrm{GL}(1, K). \mathrm{Aut}\, K$ which are complementary to $\mathrm{Aut}\, K$; in other terminology, these groups are the multiplicative groups of the Dickson nearfields constructed from K, compare [8, 19].

Corollary 2.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized m-gon, where $m \geq 3$ is odd, and let G be a collineation group of Γ which fixes a distinguished line L of Γ and acts sharply transitively on the set of all ordered ordinary $(m + 1)$-gons of Γ of the form $(x, L, \dots)$ with $x \in \mathcal{P}$. Then $m = 3$ and the projective plane Γ is Desarguesian; hence the affine plane arising from Γ by taking L as the line at infinity is of the form $\mathrm{AG}(2, K)$ for some skew field K. The group G is one of the groups*

$$G_\alpha := \{(A, \alpha(\det A)) \mid A \in \mathrm{AGL}(2, K)\} \leq \mathrm{A\Gamma L}(2, K) ,$$

where $\alpha : K^\times \to \mathrm{Aut}\, K$ is a Dickson map which is trivial on the commutator subgroup $[K^\times, K^\times]$.

Conversely, let K be a skew field and let $\alpha : K^\times \to \mathrm{Aut}\, K$ be a Dickson map which is trivial on $[K^\times, K^\times]$. Then the group G_α as above is a collineation group of *the Desarguesian projective plane* $\Gamma = \mathrm{PG}(2, K)$ *fixing a line* L *and acting sharply transitively on the set of all ordered ordinary* 4*-gons of* Γ *of the form* $(x, L, \dots)$.

Note that the trivial Dickson map gives the group $\mathrm{AGL}(2, K)$. The group G_α is the product of the two groups $\mathrm{ASL}(2, K)$ and

$$N_\alpha := \{(x, y) \mapsto (ax^{\alpha(a)}, y^{\alpha(a)}) \mid a \in K^\times\};$$

if K is commutative, then this is a semidirect product $G_\alpha = \mathrm{ASL}(2, K).N_\alpha$.

Corollary 2.3. *Let* Γ *be an affine plane and let* G *be a collineation group of* Γ *acting sharply transitively on the set of ordered triangles of* Γ. *Then* Γ *is a Desarguesian affine plane and* G *is one of the groups* G_α *as in Corollary* 2.2.

Conversely, in every Desarguesian affine plane Γ *each collineation group* G_α *as in Corollary* 2.2 *acts sharply transitively on the set of ordered triangles of* Γ.

In the special case $m = 3$, Theorem 2.1 will be deduced from the following proposition on affine planes.

Proposition 2.4. *Let* Γ *be an affine plane and let* G *be a collineation group of* Γ *acting in such a way that, for each triangle* Δ*, the stabilizer* G_Δ *acts faithfully on* Δ *as the symmetric group* S_3 *of degree* 3. *Then* Γ *is a Desarguesian affine plane, i.e.* $\Gamma = \mathrm{AG}(2, K)$ *for some skew field* K*, and* G *contains the affine group* $\mathrm{ASL}^{\pm}(2, K)$.

Conversely, for every skew field K*, the group* $\mathrm{ASL}^{\pm}(2, K)$ *and each larger group of collineations of* $\mathrm{AG}(2, K)$ *which contains no planar element except the identity induces faithfully the symmetric group* S_3 *on each triangle of the Desarguesian plane* $\mathrm{AG}(2, K)$.

Concerning the groups arising in Theorem 2.1 and in Proposition 2.4, we remark that each group G with $\mathrm{ASL}^{\pm}(2, K) \le G \le G_\alpha$, with G_α as in Corollary 2.2, satisfies the (equivalent) assumptions on G in Theorem 2.1 and Proposition 2.4. But there are also examples of groups G as in Theorem 2.1 (and Proposition 2.4) which are not contained in any of the groups G_α from Corollary 2.2. For example, let p be a prime with $p \equiv 1 \pmod{16}$; let $q = p^4$, choose a generator σ of $\mathrm{Aut}\,\mathrm{GF}(q)$, let a be a primitive 4th root of unity in $\mathrm{GF}(p) \subseteq \mathrm{GF}(q)$, and let b be a primitive 16th root of unity such that $b^4 = a$. We consider the group

$$G := \{(A, \sigma^{2i}) \mid A \in \mathrm{AGL}(2, q), 0 \le i < 4, \det A = a^i\} \le \mathrm{A\Gamma L}(2, q)\,.$$

Since σ fixes a, it is easy to see that G is actually a group, of order $4.|\mathrm{ASL}(2, q)|$, and G contains $\mathrm{ASL}^{\pm}(2, q)$, since $a^2 = -1$. Furthermore we claim that the identity is the only planar element of G; for this it suffices to show that no collineation (A, σ^2) with $\det A = a$ is planar. If not, then $B^{-1}AB^{[\sigma^2]} = \mathrm{id}$, for some $B \in \mathrm{AGL}(2, q)$, where $B^{[\sigma^2]}$ is obtained from B by applying σ^2 to each entry of a coordinate description of B. Now taking the determinant gives a contradiction since $\det A = a$ does not belong to the group $\{b^{-1}b^{\sigma^2} \mid 0 \ne b \in \mathrm{GF}(q)\}$ of order $p^2 + 1 \equiv 2 \bmod 4$. The claim follows. Suppose G is contained in a collineation group G_α which acts sharply transitively on the ordered triangles of $\mathrm{AG}(2, q)$. Then the Dickson map α is an extension of the group homomorphism $a^i \mapsto \sigma^{2i}$. Now $b^\sigma = b$ implies $\alpha(b^4) = \alpha(b)^4 = 1$, as $\sigma^4 = 1$. Hence $\sigma^2 = \alpha(a) = \alpha(b^4) = 1$, a contradiction.

We remark that Corollary 2.2 is the counterpart of the main result of [4] for affine planes. However, the counterpart of Theorem 2.1 for projective planes, and more generally generalized $(2n+1)$-gons, has not yet been considered in the literature. For completeness' sake, we include it here. In fact, it follows as a further consequence of Theorem 2.1. We split it up in two corollaries.

A quadrangle in a projective plane is a set of four points no three of which are contained in a line. This is not quite the same thing as an ordinary 4-gon, which has also four lines (and the dihedral group of order 8 as its automorphism group).

Corollary 2.5. *Let Γ be a projective plane and let G be a collineation group of Γ acting in such a way that, for each quadrangle Q of Γ, the set-wise stabilizer G_Q acts faithfully on Q as the symmetric group S_4 of degree* 4. *Then Γ is a Pappian projective plane, i.e. $\Gamma = \mathrm{PG}(2, F)$ for some field F, and G contains the little projective group* $\mathrm{PSL}(3, F)$.

Conversely, for every field F, the group $\mathrm{PSL}(3, F)$ *and each larger group of collineations of* $\mathrm{PG}(3, F)$ *which contains no planar element except the identity satisfy the assumptions on G.*

Corollary 2.6. *Let Γ be a generalized m-gon, where $m \geq 3$ is odd. Suppose that Γ admits a collineation group G which has the following property.*

(∗∗) *For each ordinary $(m+1)$-gon γ of Γ, the stabilizer G_γ acts faithfully on the point set of γ as the dihedral group of order* $2(m+1)$ *in its natural action of degree $m+1$.*

Then $m = 3$ and Γ and G satisfy the assumptions and conclusions of Corollary 2.5.

Concerning the groups arising in Corollary 2.5, we observe the following: each group G which contains $\mathrm{PSL}(3, F)$ and is contained in a collineation group of $\mathrm{PG}(2, F)$ acting sharply transitively on the set of all ordered quadrangles (as described in [4]) satisfies the stated assumptions. If F is finite, then all groups G as in Corollary 2.5 are of this type, because the index of $\mathrm{PSL}(3, F)$ in $\mathrm{PGL}(3, F)$ is then 1 or 3. But, just as in the affine plane case above, there are also (infinite) examples of groups G as in Corollary 2.5 which are not contained in any of the groups from [4]. Here we describe just one example. Let $F = \mathbb{Q}(\sqrt[3]{2}, i)$ and let $\sigma \in \mathrm{Aut}\, F$ be the involution which fixes $\sqrt[3]{2}$ and maps i to $-i$. The semilinear map $F^3 \to F^3 : (x, y, z) \mapsto ((1+i)x^\sigma, y^\sigma, z^\sigma)$ has the square $(x, y, z) \mapsto (2x, y, z)$, and 2 is a cube in F, hence the collineation g induced by that semilinear map satisfies $g^2 \in \mathrm{PSL}(3, F)$. Thus $G := \mathrm{PSL}(3, F) \cup \mathrm{PSL}(3, F)g$ is a (non-split) extension of $\mathrm{PSL}(3, F)$ by a cyclic group of order 2. This group G satisfies the assumptions in Corollary 2.5, because the coset $\mathrm{PSL}(3, F)g$ contains no planar collineation (otherwise some matrix $M^{-1}SM^{[\sigma]}$ with $\det S = 1+i$ would be a multiple of the identity, where $M^{[\sigma]}$ is obtained from M by applying σ to each entry of M, and then taking the norms with respect to $\langle\sigma\rangle$ of the determinants would give a contradiction to the fact that 2 is not a cube of a norm). Furthermore, G is not contained in any of the groups from [4] because the quotients of these groups modulo $\mathrm{PSL}(3, F)$ are finite 3-groups or infinite.

One could wonder why we do not consider as an analogue of Corollary 2.3 a regular action on ordered ordinary m-gons in the "affine" part of a generalized m-gon, m odd (the "affine" part would be the set of points and lines in general position with respect to a fixed line L). The reason is that this set need not be connected, and there might even be no ordinary m-gon inside it. Hence the generalization we consider in Theorem 2.1 and Corollary 2.2 is the only natural and reasonable one.

In Section 3 we prove that necessarily $m = 3$ in Theorem 2.1 and in Corollary 2.2. Using this fact, we start in Section 4 by reducing Theorem 2.1 and Corollary 2.2 to Proposition 2.4 and Corollary 2.3, respectively (note that the hypotheses of these results are not trivially equivalent: in Theorem 2.1 the group induced on the three lines distinct from L of an ordinary 4-gon containing L has order 2, while in Proposition 2.4, the group under consideration is assumed to have order 6). Then we prove Proposition 2.4 and Corollary 2.3 in Section 4. In Section 5 we prove Corollaries 2.5 and 2.6.

3 The case $m > 3$ is impossible

In this section we show that the assumptions of Theorem 2.1 and of Corollary 2.2 lead to a contradiction for $m > 3$. For this it suffices to prove Proposition 3.1 below, because the assumptions of Corollary 2.2 imply Condition $(*)$ in Theorem 2.1. Indeed, the points and lines of an ordinary $(m+1)$-gon γ containing L yield precisely two ordered ordinary $(m+1)$-gons of the form $(x, L, \ldots)$, hence $|G_\gamma| = 2$ for each group G as in Corollary 2.2.

Proposition 3.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized m-gon, where $m \geq 3$ is odd, let $L \in \mathcal{L}$ be a line and let G be a collineation group of Γ fixing L with the following property:*

$(*)$ *for every ordinary $(m+1)$-gon γ of Γ containing L, the stabilizer $G_\gamma = G_{\gamma,L}$ has order 2 and acts faithfully on γ.*

Then $m = 3$, thus Γ is a projective plane.

We proceed indirectly, so we assume that $m > 3$, and we write $m = 2n + 1$ with $n > 1$.

Claim 1: Let x, y be two elements of Γ at distance i from L, with $1 \leq i \leq n + 1$. Then there exists an element $\theta \in G$ taking x to y.

To prove this, let x_L, y_L be the points of L closest to x, y, respectively. Since each line is incident with at least three points, there is some point $z' \mathtt{I} L$ distinct from both x_L and y_L. Let Σ_x be an arbitrary apartment containing x and z'. Note that L is contained in Σ_x. Let z be the unique element of Σ_x with $\delta(L, z) = i$ and $\delta(z, z') = i - 1$. The unique simple path connecting z with y via L has length $2i \leq m + 1$, hence it is contained in an apartment Σ_y of Γ. Now we replace in Σ_x the simple path of length $2m - 2i$ between x and z by a simple path of length $2m-2i+2$ having only x and z in common with the simple path of length $2m-2i$ in Σ_x connecting x with z. This way we obtain an ordinary $(m+1)$-gon γ_x containing

x, z, L. Likewise, we construct an ordinary $(m+1)$-gon γ_y containing y, z, L. Multiplying the unique involutions in G which stabilize these $(m+1)$-gons in Γ, we see that we can map x to y, and Claim 1 is established.

The replacement procedure of the last arguments in the previous paragraph actually proves the following assertion.

Claim 2: Each apartment containing L is stabilized by an involution in G.

Now let Σ be an arbitrary ordinary $(m+1)$-gon containing L. Let σ be the unique involution in G stabilizing Σ. According to Theorem 3.2 of [17], there are three possibilities for σ. One of these is that σ is a Baer involution. This is impossible here since this would mean that σ fixes an ordinary $(m+1)$-gon containing L, contradicting our main assumption. A second possibility is that σ fixes a non-thick *solid* subpolygon Γ' pointwise, where "solid" means that, whenever an element of Γ' is incident in Γ' with at least three elements of Γ', then it is incident with all elements of Γ incident with it in Γ, and Γ does not consist of just one apartment. If we call an element of Γ' *thick* if it is incident with at least three elements of Γ', then it follows from the fact that m is odd, that thick elements of Γ have odd minimal distance from each other (see Theorem 1.6.2 of [16]). Hence, if Σ' is an apartment of Γ' containing L, then no involution σ' of G stabilizing Σ' (and such involutions exist by Claim 2) maps a thick element of Γ' belonging to Σ' onto a thick element of Γ' (indeed, considering a shortest path to L from these elements, this would mean that the ones nearest to L lie symmetrically with respect to L, and hence that the minimal distance between thick elements is even). Hence conjugating σ with σ' we obtain an involution σ'' whose fixed point structure inside the thick generalized polygon naturally associated with Γ' contains an apartment without thick elements. This contradicts the fact that σ'' is an involution in this polygon and the classification of types of involutions stated in Theorem 3.2 of [17].

Consequently only the third possibility remains, and so σ is a central collineation. Let M be the line opposite to L in the ordinary $(m+1)$-gon Σ. Then both L and M are fixed by σ, but there are points on these lines that are not fixed. Hence the unique element x_n at distance n from both L and M in Γ is the center or axis of σ (according whether n is odd or even), and the axis or center is an element x'_{n+1} of Γ incident with x_n, but not belonging to Σ. If n is even, then let $x_{n/2}$ be the element of Σ at distance $n/2$ from both L and x_n. Let x'_n be a line of Γ at distance $n/2$ from $x_{n/2}$ and at distance n from both x_n and L (it is easy to construct such an element using the thickness of Γ). By Claim 1 there is an element $\theta \in G$ taking x_n to x'_n. Since $\delta(x_n, x'_n) = n$, and clearly $\delta(x'_{n+1}, {x'_{n+1}}^{\theta}) = n+2$, the coexistence of the central collineations σ and σ^{θ} leads to a contradiction, as is shown in Section 3, Case 1 of [13].

If n is odd, then we consider an ordinary $(m+1)$-gon Σ' in Γ sharing with Σ two points on L and exactly one point on M, and also M itself. The corresponding involution σ' has the same center x_n as σ, but a different axis. Indeed, σ and σ' have the same action on the points of Σ on L, hence if their axes were equal, they would be the same. So σ and σ' commute and the product $\sigma\sigma'$ is again an involution. Clearly it must again be a central involution with axis x_n. But since this product fixes at least three points incident with L, it fixes all points on L, and so the axis is the line x_{n-1} belonging to Σ at distance $n-1$ from L and incident with x_n. We now consider the element $y = x_{(n-1)/2}$ of Σ at distance $(n-1)/2$ from L and at distance $(n+1)/2$ from x_n. Exploiting the thickness of Γ, we see that there exist elements x'_n, x'_{n-1} at distance $(n-1)/2$ and $(n+1)/2$, respectively, from y, and at distance $n-1$ and n, respectively, from L, and at distance n and $n+1$, respectively, from x_n. Since $\delta(x_n, x'_{n-1}) = n$ and $\delta(x'_{n+1}, x'_n) = n+2$, the coexistence of the central collineations σ and $(\sigma\sigma')^\theta$, where $\theta \in G$ maps x_n to x'_n (θ exists by Claim 1), and the assumption $n > 1$ contradict Section 3, Case 1 of [13].

This completes the proof of Proposition 3.1.

4 Affine planes

In this section we prove Theorem 2.1, Corollaries 2.2, 2.3 and Proposition 2.4. By Section 3 we have $m = 3$ in Theorem 2.1 and in Corollary 2.2. Therefore Theorem 2.1 and Corollary 2.2 can be reduced to Proposition 2.4 and Corollary 2.3, respectively, by the following consideration.

Let Δ be a triangle in an affine plane Γ, with points p_1, p_2, p_3. It suffices to show that, under the hypothesis that every ordinary 4-gon of Γ^c containing the line L at infinity is stabilized by just two elements of $G = G_L$, there exists a unique element in G_Δ which exchanges p_1 and p_2. But this follows immediately by considering the quadrangle $p_1, p_2, L \cap p_2p_3, L \cap p_1p_3$.

Now we prove Proposition 2.4. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be an affine plane and G a collineation group acting on Γ in such a way that, for every triangle Δ of Γ, the stabilizer G_Δ has order 6 and acts faithfully on the set of vertices of Δ.

We claim that G acts 2-transitively on the point set $\mathcal{P}$. Indeed, the stabilizer G_x of a point x can map a point $y \neq x$ onto any point z not on the line xy (this can be done using an involution in the triangle with vertices x, y, z). Applying this twice, we can map y onto any point of the line xy distinct from x. The claim follows. In particular, G acts transitively on the line set $\mathcal{L}$.

We will freely talk about points at infinity and the line L_∞ at infinity of Γ, referring in fact to the points and line of the projective completion Γ^c of Γ that are not contained in Γ.

Let p be any point at infinity of Γ, and let L be any line of Γ incident with p. Let $\Delta = \{a, b, c\}$ be a triangle of Γ chosen in such a way that both a and b are incident with L. There is a unique involution $\sigma(a, b, c) \in G$ fixing c and interchanging a with b. This involution cannot be a Baer involution (otherwise it would fix a triangle pointwise), hence it has an axis and a center. Considering it as an involution in Γ^c, we see that, since the lines L, pc and L_∞ are fixed, the point p is the center. Hence the axis is some line M incident with the point c.

We claim that Γ is (p, L)-transitive. To show this, let R be any line of Γ through a distinct from L, and not parallel to bc. We show that there exists a (p, L)-perspectivity mapping the line ac to R. Indeed, let c' be the intersection of R with bc and consider the involution $\sigma(a, b, c')$, which has also center p. The collineation $\theta := \sigma(a, b, c)\sigma(a, b, c')$ has center p, fixes both a and b (and so the axis of θ

is L) and maps ac to R. The claim follows if we can show that there is a (p, L)-perspectivity mapping ac to the line R' through a parallel to bc. But this follows from a similar argument interchanging the roles of b with some point b' on L, where $a \neq b' \neq b$. For this, we need at least three points on every line of Γ. But if Γ has only two points per line, then it is automatically Desarguesian and the theorem holds. We have shown the claim.

Since L was chosen arbitrarily, Γ is (p, L)-transitive for every line L through p, $L \neq L_\infty$. Hence p is a translation point (see for instance Theorem 4.19 of [7]). So all points at infinity of Γ are translation points of Γ^c and hence Γ^c is a Moufang plane (see e.g. [7], page 151, Theorem 6.18). This means that Γ is coordinatized by an alternative division ring K (compare [7], page 139, or [15], Chapter 19, or [10] 3.5 and 6.1). The shears

$$\sigma_a : (x, y) \mapsto (x, y + ax) \text{ and } \sigma'_a : (x, y) \mapsto (x + ay, y)$$

with $a \in K$ are in G, as we have shown. The collineation

$$\tau_a := \sigma'_1 \sigma_{a-1} \sigma'_{-a^{-1}} \sigma_{a-a^2} \in G,$$

with $0 \neq a \in K$, is the map

$$\tau_a(x, y) = (a^{-1}x, ay),$$

see [11] 12.14, page 37, for the computation.

Lemma 4.1. *Let K be an alternative division ring, define $\tau_a : K^2 \to K^2$ by $\tau_a(x, y) = (a^{-1}x, ay)$ for $0 \neq a \in K$, and let $H := \langle \tau_a \mid 0 \neq a \in K \rangle$. If the stabilizer $H_{(1,1)}$ is trivial, then K is associative (hence a skew field).*

Proof of Lemma 4.1. Proceeding indirectly, we assume that K is not associative. Then $K = B + eB$, with B a quaternion skew field such that $\overline{e} = -e$, $be = e\overline{b}$ and $(ea)(eb) = e^2(b\overline{a})$, for all $a, b \in B$, where $\bar{\cdot}$ denotes an involutory antiautomorphism of K stabilizing B, compare [15], page 196, or [10] Chapter 6, or [16] Appendix B; we will also use the fact that $e^2 = -\overline{e}e$ is in the center of K.

We consider nonzero elements $a, b \in B$ with $\overline{a} = -a$ and $\overline{b} = -b$ and the two collineations $\alpha := \tau_b \tau_a \in H$ and $\beta := \tau_{(ba)e} \tau_{e^{-1}} \in H$. We compute

$$\begin{aligned}\beta(1,1) &= (((ba)e)^{-1}e, ((ba)e)e^{-1}) = ((e\overline{ba})^{-1}e, ba) = (((\overline{ba})^{-1}e^{-1})e, ba) = \\ &= ((\overline{a}\overline{b})^{-1}, ba) = (((-a)(-b))^{-1}, ba) = ((ab)^{-1}, ba) = (b^{-1}a^{-1}, ba) = \\ &= \alpha(1,1).\end{aligned}$$

From our assumption on H we conclude that $\alpha = \beta$, in particular $\alpha(0, a) = \beta(0, a)$, and this means that

$$ba^2 = ((ba)e)(e^{-1}a) = (e\overline{ba})(ea)e^{-2} = e^2 a\overline{\overline{ba}}e^{-2} = aba,$$

hence $ba = ab$.

Thus any two elements of the set $\{a \in B \mid \overline{a} = -a\}$ commute. This set is a vector space of dimension 3 over the center of B (the latter coincides with the center of K) and therefore a generating set of the algebra B. Hence the quaternion skew field B is commutative, a contradiction, which completes the proof of Lemma 4.1. $\square$

The collineations $\tau_a \in G$ fix the point $(0,0)$ and the two lines $K \times \{0\}$ and $\{0\} \times K$ of Γ, hence also the points at infinity of these two lines. Therefore $H_{(1,1)} \leq G$ fixes the triangle $(0,0)$, $(1,0)$, $(0,1)$ of Γ pointwise. Now Lemma 4.1 together with our assumption on G implies that K is a skew field, hence $\Gamma = \mathrm{AG}(2, K)$ is Desarguesian.

We have seen that G contains all translations $(x, y) \mapsto (a + a, y + b)$ and the shears σ_a, σ'_a as above. These collineations generate the special affine group $\mathrm{ASL}(2, K)$, see ARTIN [1] Theorem 4.3, p. 156 or DIEUDONNÉ [3] II.1. Furthermore G contains axial involutions $\sigma(a, b, c)$ as above; each involution $\sigma(a, b, c)$ belongs to $\mathrm{AGL}(2, K)$ and has (Dieudonné) determinant -1 (modulo $[K^\times, K^\times]$), even if K has characteristic 2. Hence $\mathrm{ASL}^{\pm}(2, K) \leq G$. Since $\mathrm{ASL}^{\pm}(2, K)$ induces S_3 on each triangle of $\mathrm{AG}(2, K)$ (note that $\begin{pmatrix} 0 & a^{-1} \\ a & 0 \end{pmatrix}$ exchanges $(1, 0)$ and $(0, a)$), this completes the proof of Proposition 2.4.

For Corollary 2.3, we can use Proposition 2.4 to conclude that $\Gamma = \mathrm{AG}(2, K)$ and $\mathrm{ASL}(2, K) \leq G$. Now $\mathrm{ASL}(2, K)$ acts doubly transitively on K^2 and on the line at infinity, hence $G = \mathrm{ASL}(2, K) G_{(0,0),(0,1),K \times \{0\}}$. By our assumption on G, the stabilizer $G_{(0,0),(0,1),K \times \{0\}} \leq \Gamma\mathrm{L}(2, K)$ acts faithfully and sharply transitively on $K^\times \times \{0\}$ as a subgroup of $\Gamma\mathrm{L}(2, K) = \{x \mapsto ax^\alpha \mid a \in K^\times, \alpha \in \mathrm{Aut}\, K\}$, hence $G_{(0,0),(0,1),K \times \{0\}} = \{(x, y) \mapsto (ax^{\alpha(a)}, y^{\alpha(a)}) \mid a \in K^\times\} =: N_\alpha$ for a uniquely determined map $\alpha : K^\times \to \mathrm{Aut}\, K$. In fact, α is a Dickson map since $G_{(0,0),(0,1),K \times \{0\}}$ is closed under multiplication. We obtain $G = \mathrm{ASL}(2, K) N_\alpha = \{(A, \alpha(\det A)) \mid A \in \mathrm{AGL}(2, K)\} = G_\alpha$. The intersection $\mathrm{ASL}(2, K) \cap G_{(0,0),(0,1),K \times \{0\}} = \mathrm{SL}(2, K)_{(0,1),K \times \{0\}}$ consists of all matrices $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \in [K^\times, K^\times]$, hence α is trivial on $[K^\times, K^\times]$.

Conversely, each group G_α acts sharply transitively on the set of ordered triangles in $\mathrm{AG}(2, K)$, which completes the proof of Corollary 2.3.

Thus we have proved Theorem 2.1, Corollaries 2.2, 2.3 and Proposition 2.4.

5 Proofs of Corollaries 2.5 and 2.6

Concerning Corollary 2.6, we fix a line L of Γ and consider any ordinary $(m + 1)$-gon γ containing L. The stabilizer $G_{\gamma, L}$ clearly has order 2. It follows from Theorem 2.1 (or more directly from Proposition 3.1) that $m = 3$. Let Q be a quadrangle of Γ. There are three ways to make Q into an ordinary 4-gon, hence Condition $(**)$ implies that G_Q contains three distinct dihedral subgroups of order 8, which generate the symmetric group S_4 on Q. Thus we have reduced Corollary 2.6 to Corollary 2.5.

Now we prove Corollary 2.5. We apply Theorem 2.1 (or more directly, Proposition 2.4) to the stabilizer G_L of any line L, and we conclude that Γ is a Desarguesian projective plane, i.e. $\Gamma \cong \mathrm{PG}(2, K)$ for some skew field K, and that G contains all elations of Γ. Hence G contains the little projective group $\mathrm{PSL}(3, K)$, see e.g. ARTIN [1] Theorem 4.3, p. 156 or DIEUDONNÉ [3] II.1. In particular, G contains all collineations which are induced by diagonal matrices $\mathrm{diag}(c, c, c)$ with $c \in [K^*, K^*]$ (compare [1] Theorem 4.2, p. 155 or [3] II.1). These collineations fix a quadrangle, hence they are trivial by our assumption on G. This means that each commutator $c \in [K^*, K^*]$ is contained in the center of K. According to a theorem of Hua [6], compare also [2] 3.9.1, p. 144, this implies that K is commutative, i.e. a field, and then Γ is Pappian.

For each field F, the group $G = \mathrm{PSL}(3, F)$ satisfies the assumptions of Corollaries 2.5 and 2.6, because the matrix $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SL}(3, F)$ and its conjugates under $\mathrm{GL}(3, F)$ induce any given involution on any given quadrangle of $\mathrm{PG}(2, F)$.

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