Arch. Math. 84 (2005) 375–384
0003–889X/05/040375–10
DOI 10.1007/s00013-004-1161-5

Archiv der Mathematik

# Generalized quadrangles with an ovoid that is translation with respect to opposite flags

By

Alan Offer, Koen Thas and Hendrik Van Maldeghem

**Abstract.** The article [6] contains the result that if a finite generalized quadrangle $\Gamma$ of order $s$ has an ovoid $\mathcal{O}$ that is translation with respect to two opposite flags, but not with respect to any two non-opposite flags, then $\Gamma$ is self-polar and $\mathcal{O}$ is the set of absolute points of a polarity. In particular, if $\Gamma$ is the classical generalized quadrangle $Q(4, q)$ then $\mathcal{O}$ is a Suzuki-Tits ovoid. In this article, we remove the need to assume that $\Gamma$ is $Q(4, q)$ in order to conclude that $\mathcal{O}$ is a Suzuki-Tits ovoid by showing that the initial assumptions in fact imply that $\Gamma$ is $Q(4, q)$. At the same time, we also relax the requirement that $\Gamma$ have order $s$.

**1. Introduction.** Generalized quadrangles are incidence geometries in which there are neither digons nor triangles, but through any two elements there is a quadrangle. These belong to the class of geometries known as generalized polygons, which includes projective planes as generalized triangles. Here we are concerned with finite generalized quadrangles, and just as a finite projective plane has an order, a finite generalized quadrangle has an order which is a pair $(s, t)$ of integers. This is often abbreviated to $s$ when $s = t$. In this paper, we are particularly interested in the classical finite generalized quadrangle $Q(4, q)$, which has order $q$.

An ovoid $\mathcal{O}$ of a generalized quadrangle $\Gamma$ is a set of points such that on each line of $\Gamma$ there is exactly one point of $\mathcal{O}$. One way of obtaining an ovoid is as the set of absolute points of a polarity. In the context of $Q(4, q)$, such an ovoid is called a Suzuki-Tits ovoid.

Translation ovoids are essentially those that possess a certain amount of symmetry. There are two notions: that of being translation with respect to a point and that of being translation with respect to a flag. The former notion is the stronger as it is equivalent to being translation with respect to every flag containing that point. Ovoids of $Q(4, q)$ that are translation with respect to a point have drawn a lot of attention for their correspondence to translation generalized quadrangles, semifield flocks and semifield planes. Here we shall need only the weaker symmetry condition given by the latter notion.

---

*Mathematics Subject Classification* (2000): 51E12, 20B25, 05B25.

The second author is a Postdoctoral Fellow of the Fund for Scientific Research—Flanders (Belgium).

In this work, our objective has been to show that if a finite generalized quadrangle $\Gamma$ has an ovoid $\mathcal{O}$ that is translation with respect to two opposite flags but not with respect to two non-opposite flags, then $\Gamma$ is $Q(4, q)$ and $\mathcal{O}$ is a Suzuki-Tits ovoid. However, we stop a little short of this by requiring that the order $(s, t)$ of $\Gamma$ be such that $t$ is even or $s = t$. This improves on the result [6, Theorem 5.3] which asserts that $\mathcal{O}$ arises from a polarity and which applies only when $s = t$.

## 2. Background.

**2.1. Generalized quadrangles.** A *flag* in a geometry of points and lines is a pair $\{p, L\}$ consisting of an incident point $p$ and line $L$. A *generalized quadrangle* $\Gamma$ is a geometry of points and lines in which (i) every point is on at least three lines and every line passes through at least three points, (ii) there exists a point and a line that are not incident with each other, and (iii) given any point $p$ and line $L$ that are not incident with each other, there is a unique flag $\{q, M\}$ with $p$ on $M$ and $q$ on $L$. The point $q$ is the *projection* of $p$ onto $L$, denoted here by $q = p \rhd L$, and the line $M$ is the *projection* of $L$ onto $p$, denoted by $M = L \rhd p$. In addition, if $p$ and $L$ are incident then we put $p \rhd L = p$ and $L \rhd p = L$. The following details can be found in [8].

A generalized quadrangle $\Gamma$ is *finite* if it has finitely many points. In this case, there are integers $s, t > 1$ such that every line has exactly $s + 1$ points and every point is on exactly $t + 1$ lines. The ordered pair $(s, t)$ is the *order* of $\Gamma$ and the inequalities of Higman [4] assert that $s \leqq t^2$ and $t \leqq s^2$. When $s = t$, it is customary to just say that $\Gamma$ has order $s$.

**Subquadrangles.** Let $\Gamma$ be a finite generalized quadrangle of order $(s, t)$. A *subquadrangle* $\Gamma'$ of $\Gamma$ is a generalized quadrangle whose point and line sets are subsets of those of $\Gamma$ and whose incidence relation is induced by that of $\Gamma$. As $\Gamma$ is finite, so too is $\Gamma'$ which then also has an order $(s', t')$, with $s', t' > 1$. If $\Gamma' \neq \Gamma$ then it is a *proper* subquadrangle of $\Gamma$.

Let $\Gamma'$ be a proper subquadrangle of $\Gamma$ of order $(s', t)$. Then $s' \leqq t < s$. If $s' = t$, then $s = t^2$ and every point of $\Gamma$ that is not a point of $\Gamma'$ lies on a unique line of $\Gamma'$ (see [8, §2.2]).

**Ovoids and spreads.** Let $\Gamma$ be a finite generalized quadrangle of order $(s, t)$. Two points of $\Gamma$ are *opposite* when they are not collinear, and two lines are *opposite* when they are not concurrent. Two flags $\{p, L\}$ and $\{q, M\}$ are *opposite* if their points $p$ and $q$ are opposite and their lines $L$ and $M$ are opposite.

An *ovoid* of $\Gamma$ is a set $\mathcal{O}$ of points such that (i) the points of $\mathcal{O}$ are pairwise opposite, (ii) every line of $\Gamma$ contains a point of $\mathcal{O}$ and (iii) $|\mathcal{O}| = st + 1$. In fact, any two of these conditions imply the third. Dually, a *spread* is a set $\mathcal{S}$ of lines satisfying any two, and therefore all three, of: (i) the lines of $\mathcal{S}$ are pairwise opposite, (ii) every point of $\Gamma$ is on a line of $\mathcal{S}$ and (iii) $|\mathcal{S}| = st + 1$.

One source of ovoids and spreads is polarities. A *polarity* of $\Gamma$ is an involutory incidence preserving map that interchanges the points and lines of $\Gamma$. Given a polarity $\rho$, an element

$u$ of $\Gamma$ is *absolute* if it is incident with its image $u^\rho$. The sets of absolute points and absolute lines form an ovoid and a spread, respectively (see [8, 1.8.2]).

Hence $\Gamma$ has both an ovoid and a spread if it has a polarity. On the other hand, $\Gamma$ does not admit an ovoid if its order is $(s, s^2)$, and dually, it does not admit a spread if its order is $(t^2, t)$ (see [8, 1.8.3]).

**Translation ovoids.** Translation ovoids are ovoids that have a certain amount of symmetry. As mentioned in the introduction, there are two notions and it is the weaker of being translation with respect to a flag that concerns us here.

Let $\Gamma$ be a finite generalized quadrangle of order $(s, t)$ and let $\mathcal{O}$ be an ovoid of $\Gamma$. Let $p \in \mathcal{O}$ and let $L$ be a line incident with $p$. Let $G_{\{p,L\}}$ be the set of all collineations of $\Gamma$ that fix $p$ linewise, fix $L$ pointwise and stabilize the ovoid $\mathcal{O}$. For each point $z \neq p$ on $L$, let $V_z$ be the set of $t$ points of $\mathcal{O}\setminus\{p\}$ collinear with $z$. Notice that the $s$ sets $V_z$ partition $\mathcal{O}\setminus\{p\}$.

An action of a group $G$ on a set $X$ is *semiregular* if $G_x = \{1\}$ for all $x \in X$. In particular, it is regular if and only if it is transitive and semiregular. By [8, 2.4.1], the group of all collineations of $\Gamma$ that fix $p$ linewise and $L$ pointwise is semiregular on the set of points opposite $p$. It follows that the group $G_{\{p,L\}}$ is semiregular on each of the sets $V_z$.

The ovoid $\mathcal{O}$ is *translation with respect to the flag* $\{p, L\}$ if the group $G_{\{p,L\}}$ acts transitively (and hence regularly) on one (and hence all) of the sets $V_z$, or equivalently, if $|G_{\{p,L\}}| = t$. This group $G_{\{p,L\}}$ is then called the *associated group* with respect to the flag $\{p, L\}$.

**The generalized quadrangle $Q(4, q)$.** The generalized quadrangle $Q(4, q)$, which has order $q$, is the geometry of points and lines on a non-degenerate quadric $\mathcal{P}_4$ in $\mathrm{PG}(4, q)$. If $\mathcal{E}_3$ is an elliptic hyperplane section of $\mathcal{P}_4$ then the points of $\mathcal{E}_3$ form an ovoid of $Q(4, q)$. Thus $Q(4, q)$ admits ovoids for all $q$. By [11] however, it admits spreads only if $q = 2^h$, which is when it is self-dual and isomorphic to the symplectic generalized quadrangle $\mathrm{W}(q)$ (see [8, 3.2.1]). From that paper comes the following characterization.

**Theorem 2.1** (Thas [11])**.** *Let $\Gamma$ be a finite generalized quadrangle of order s and let $\mathcal{O}$ be an ovoid of $\Gamma$. Suppose that for any three distinct points $a, b, c \in \mathcal{O}$ there is a point $w$ of $\Gamma$ that is collinear with all of a, b and c. Then $\Gamma$ is isomorphic to $Q(4, 2^h)$, for some h.*

Finally, when $q = 2^{2e+1}$ for some $e \geqq 0$, the generalized quadrangle $Q(4, q)$ admits polarities and the ovoids and spreads that arise as a result are known as *Suzuki-Tits* ovoids and spreads, for the Suzuki group $\mathrm{Sz}(q)$ acts as an automorphism group and the existence of polarities was shown by Tits [12, §5].

**2.2. The Suzuki groups.** Here we present the necessary facts about the Suzuki groups $\mathrm{Sz}(q)$ and the inversive planes belonging to them. These groups were first identified by Suzuki in [9]. The properties used here are taken from [5, Chapter IV] and [10]. For the relationship with inversive planes, we refer the reader to [5, §26]. As the parameter here is always $q$, for brevity we shall mostly just write Sz for Sz$(q)$.

The Suzuki group Sz$(q)$ exists for $q = 2^{2e+1} \geqq 2$, its order is $(q^2+1)q^2(q-1)$ and it has a natural 2-transitive action on a set $\Omega$ of size $q^2+1$ in which only the identity leaves three distinct points fixed. The group Sz(2) is sharply 2-transitive in its natural action. In the following, it is always assumed that $q > 2$.

Let $a$ be a point in $\Omega$. The stabilizer $\mathrm{Sz}_a$ is transitive but not regular on $\Omega \setminus \{a\}$ and only the identity in $\mathrm{Sz}_a$ leaves two distinct points of this set fixed. Such groups are called *Frobenius groups* as Frobenius provided a structure theorem for them (see [13, Theorem 5.1]). From that theorem, we have $\mathrm{Sz}_a = P\mathrm{Sz}_{a,b}$ where $P \trianglelefteqq \mathrm{Sz}_a$ is a regular normal subgroup and $b \neq a$. Consequently, $P$ is the unique Sylow 2-subgroup of $\mathrm{Sz}_a$.

Let $Z = Z(P)$ be the centre of $P$. Then $Z \trianglelefteqq \mathrm{Sz}_a$ and $|Z| = q$. The non-trivial elements of $Z$ are precisely the involutions in $P$ and so they are all of the involutions in $\mathrm{Sz}_a$ since $P$ is the unique Sylow 2-subgroup of $\mathrm{Sz}_a$. Each subgroup $\mathrm{Sz}_{a,b}$, and so also the larger group $\mathrm{Sz}_a$, acts transitively on $Z \setminus \{1\}$ by conjugation.

An *inversive plane* is an incidence structure, whose elements are usually called points and circles, with the property (among others) that for every three distinct points there is a unique circle containing them. Each finite inversive plane has an order $n$, such that there are $n+1$ points on each circle and $n^2+1$ points in total. These are discussed in [1, Chapter 6] and in [5, §25], although in the latter they are called *Möbius planes*.

For each $q = 2^{2e+1} > 2$, there is a unique inversive plane $\mathcal{M}$ of order $q$ (up to isomorphism) that admits Sz$(q)$ as an automorphism group. Furthermore, the action of Sz$(q)$ on the point set $\Omega$ of $\mathcal{M}$ is its usual 2-transitive action.

Let $\mathcal{C}$ be a circle in $\mathcal{M}$. The stabilizer $\mathrm{Sz}_{\mathcal{C}}$ has order $q(q-1)$. Let $Q$ be a Sylow 2-subgroup of $\mathrm{Sz}_{\mathcal{C}}$, let $P$ be a Sylow 2-subgroup of Sz containing $Q$, and let $a$ be the unique point that is fixed by $P$. Since $P$ is semiregular on $\Omega \setminus \{a\}$, so too is $Q$. Together with $|Q| = q$ and $|\mathcal{C}| = q+1$, we see that $a \in \mathcal{C}$ and $Q$ is regular on $\mathcal{C} \setminus \{a\}$.

The group $\mathrm{Sz}_{\mathcal{C}}$ is not transitive on $\mathcal{C}$ since $|\mathrm{Sz}_{\mathcal{C}}|$ is not divisible by $|\mathcal{C}|$, so from the transitivity of $Q$ on $\mathcal{C} \setminus \{a\}$, it follows that every element of $\mathrm{Sz}_{\mathcal{C}}$ leaves the point $a$ fixed. Thus $\mathrm{Sz}_{\mathcal{C}} \leqq \mathrm{Sz}_a$. Now $\mathrm{Sz}_{\mathcal{C},b} \leqq \mathrm{Sz}_{a,b}$ for $b \neq a$ in $\mathcal{C}$, but both of these groups have order $q-1$, so we conclude that $\mathrm{Sz}_{\mathcal{C},b} = \mathrm{Sz}_{a,b}$.

Now $\mathrm{Sz}_{\mathcal{C}}$ is transitive but not regular on $\mathcal{C} \setminus \{a\}$ and only the identity fixes two points of this set. Thus $\mathrm{Sz}_{\mathcal{C}}$ is a Frobenius group and so $Q \trianglelefteqq \mathrm{Sz}_{\mathcal{C}}$. In particular, $Q$ is normalized by $\mathrm{Sz}_{\mathcal{C},b} = \mathrm{Sz}_{a,b}$. Since $Q \leqq \mathrm{Sz}_a$ has even order, it contains some involution that belongs to $\mathrm{Sz}_a$ and so also to $Z = Z(P)$. Since $\mathrm{Sz}_{a,b}$ is transitive on $Z \setminus \{1\}$, we have $Z \leqq Q$, and as these groups both have order $q$, it follows that $Z = Q$.

In summary, let $\mathcal{C}$ be a circle in $\mathcal{M}$. Then:

- the stabilizer $\mathrm{Sz}_{\mathcal{C}}$ has a unique Sylow 2-subgroup $Q$;
- $|Q| = q$;
- there is a point $a \in \mathcal{C}$ such that $Q$ fixes $a$ and is regular on $\mathcal{C} \setminus \{a\}$;
- $Q = Z(P)$, where $P$ is the unique Sylow 2-subgroup of $\mathrm{Sz}_a$;
- the non-trivial elements of $Q$ are precisely the involutions in $\mathrm{Sz}_a$;
- $\mathrm{Sz}_a$ acts transitively on $Q \setminus \{1\}$.

In addition, we will use the following classification due to Hering.

**Theorem 2.2** (Hering [3]). *Let $G$ be a transitive permutation group on a set $\Omega$, where $|\Omega| > 1$, and let $\alpha \in \Omega$. Suppose $G_\alpha$ has a normal subgroup $Q$ of even order that is semiregular on $\Omega \backslash \{\alpha\}$. Then $\langle Q^G \rangle$ either has a transitive normal subgroup of odd order, or acts 2-transitively on $\Omega$ as one of the groups* $\mathrm{SL}(2, 2^h)$, $\mathrm{PSU}(3, 2^h)$, *for some $h$, or* $\mathrm{Sz}(2^{2e+1})$, *for some $e \geqq 1$, in its usual 2-transitive representation.*

**3. The theorem.** Let $\Gamma$ be a finite generalized quadrangle with order $(s, t)$ and let $\mathcal{O}$ be an ovoid in $\Gamma$ that is translation with respect to two opposite flags $\mathcal{F} = \{u, U\}$ and $\mathcal{G} = \{v, V\}$, where $u, v \in \mathcal{O}$. Let $G = \langle G_{\{u,U\}}, G_{\{v,V\}} \rangle$ be the group generated by the associated groups with respect to each of the flags $\mathcal{F}$ and $\mathcal{G}$. Suppose that the flags in the union $\mathcal{F}^G \cup \mathcal{G}^G$ of orbits are pairwise opposite.

Given these definitions and assumptions, we can state [6, Theorem 5.3] as follows.

**Theorem 3.1** (Offer and Van Maldeghem [6]). *If $s = t$ then $G$ is transitive on $\mathcal{O}$ and the flags of $\mathcal{F}^G$ determine an ovoid and a spread that arise from a polarity.*

This is actually a slightly stronger statement than the original theorem, but the proof is the same except for a small redefinition of the set $\mathcal{S}$ in the second half of the proof (let $\mathcal{S}$ be the set of lines in the flags of $\mathcal{F}^G$ rather than the set of all lines that belong to a flag with respect to which $\mathcal{O}$ is translation).

A corollary of this theorem is that if $\Gamma \cong Q(4, q)$ then $\mathcal{O}$ is a Suzuki-Tits ovoid. However, as the theorem stands, so much structure is deduced without making this additional assumption that one is led to wonder if perhaps $\Gamma$ *must* be $Q(4, q)$. This leads us to the main result of this paper.

**Theorem 3.2** (Main Theorem). *Let $\Gamma$ be a finite generalized quadrangle of order $(s, t)$ and let $\mathcal{O}$ be an ovoid of $\Gamma$ such that the conditions described above are satisfied. If $s = t$ or $t$ is even then $\Gamma \cong Q(4, 2^{2e+1})$, for some $e \geqq 0$, and $\mathcal{O}$ is a Suzuki-Tits ovoid.*

In the case that $s = t$, we already know from Theorem 3.1 that $\Gamma$ admits a polarity, which implies that $t$ is even (see [8, 1.8.2]). Thus to prove our theorem, we need only address the case that $t$ is even. We proceed in three steps, represented by Lemmas 3.1, 3.2 and 3.3. First we show that $G$ is transitive on $\mathcal{O}$, then that $G \cong \mathrm{Sz}(q)$ and finally that $\Gamma \cong Q(4, q)$. We remark that the first lemma is proved without the assumption that $t$ is even.

Let $\mathcal{O}^* = u^G \cup v^G$ and $\mathcal{S} = U^G \cup V^G$, so $\mathcal{O}^*$ and $\mathcal{S}$ are the sets of points and lines, respectively, in the flags of $\mathcal{F}^G \cup \mathcal{G}^G$. As the flags in $\mathcal{F}^G \cup \mathcal{G}^G$ are pairwise opposite, for each point $a \in \mathcal{O}^*$ there is a unique line $L \in \mathcal{S}$ incident with it, and conversely, for each line $L \in \mathcal{S}$ there is a unique point $a \in \mathcal{O}^*$ on $L$. Represent this relationship between such a point $a$ and line $L$ by $L = a^\rho$ and $a = L^\rho$.

For each $a \in \mathcal{O}^*$, there is a $g \in G$ such that either $a = u^g$ or $a = v^g$. The ovoid $\mathcal{O}$ is then translation with respect to the flag $\{a, L\}$, where $L = a^\rho$, and the associated group $G_{\{a,L\}}$ is correspondingly either $(G_{\{u,U\}})^g$ or $(G_{\{v,V\}})^g$. As it is uniquely determined by either $a$ or $L$, we shall simply write $H(a)$ or $H(L)$ for the associated group $G_{\{a,L\}}$ with respect to the flag $\{a, L\}$. Thus, for instance, $G = \langle H(u), H(v)\rangle$. Notice that by the uniqueness of $a^\rho$, the stabilizer $G_a$ fixes $a^\rho$, and so $H(a) \leqq G_a$ since $H(a)$ fixes $a$ linewise and $a^\rho$ pointwise.

**Lemma 3.1.** *The group $G$ is transitive on $\mathcal{O}$. In particular, $\mathcal{O}^* = \mathcal{O}$ and the set $\mathcal{S}$ is a spread of $\Gamma$. In addition, $t|s$ and $s < t^2$.*

Proof. Let $\Gamma^*$ be the incidence structure whose point and line sets are

$$\mathcal{P} = \{a \rhd L \mid a \in \mathcal{O}^* \text{ and } L \in \mathcal{S}\}$$
$$\mathcal{L} = \{L \rhd a \mid a \in \mathcal{O}^* \text{ and } L \in \mathcal{S}\}$$

and whose incidence is inherited from that of $\Gamma$.

First we show that $\Gamma^*$ is a subquadrangle of $\Gamma$ and then that $\Gamma^* = \Gamma$. Thence $\mathcal{O}^* = \mathcal{O}$ and this has at most two orbits under $G$, namely $u^G$ and $v^G$. Finally, we demonstrate that each of these orbits contains more than half the points of $\mathcal{O}$ and so prove the proposition. The additional facts that $t|s$ and $s < t^2$ will arise along the way.

Notice that $\mathcal{O}^* \subseteqq \mathcal{P}$ and $\mathcal{S} \subseteqq \mathcal{L}$ as $a = a \rhd a^\rho$ and $L = L \rhd L^\rho$ for $a \in \mathcal{O}^*$ and $L \in \mathcal{S}$. Let $\mathcal{P}^* = \mathcal{P} \backslash \mathcal{O}^*$ and $\mathcal{L}^* = \mathcal{L} \backslash \mathcal{S}$.

Consider a point $p \in \mathcal{P}^*$. Then there is a point $a \in \mathcal{O}^*$ and a line $L \in \mathcal{S}$ such that $p = a \rhd L$. Let $K \neq L$ be a line through $p$ and let $b$ be the unique point of $\mathcal{O}$ on $K$. As the ovoid $\mathcal{O}$ is translation with respect to the flag $\{L^\rho, L\}$, there is a collineation in the associated group $H(L)$ that maps $a$ to $b$. Hence $b \in \mathcal{O}^*$ and so $K = L \rhd b \in \mathcal{L}$. Thus every line of $\Gamma$ through a point $p \in \mathcal{P}^*$ is a line of $\Gamma^*$.

Consider now a line $L \in \mathcal{S}$. Let $s'$ be the number of points of $\mathcal{P}^*$ on $L$, so together with the point $L^\rho$, there are exactly $s' + 1$ points on $L$ in $\Gamma^*$. By the previous paragraph, for each point $p \in \mathcal{P}^*$ on $L$, all of the $t$ lines $K \neq L$ through $p$ belong to $\mathcal{L}$ and so the unique point of $\mathcal{O}$ on each such line $K$ belongs to $\mathcal{O}^*$. Conversely, if $a \in \mathcal{O}^*$ is not on $L$ then $a \rhd L$ is one of the points of $\mathcal{P}^*$ on $L$. Thus there are precisely $s't + 1$ points in $\mathcal{O}^*$. As $|\mathcal{O}^*|$ is independent of the line $L$, it follows that every line of $\mathcal{S}$ is incident with the same number $s' + 1$ of points of $\mathcal{P}$.

Now consider a point $a \in \mathcal{O}^*$. Let the $t$ lines of $\Gamma$ distinct from $a^\rho$ through $a$ be $L_1, L_2, \dots, L_t$. For each $i = 1, \dots, t$, let $s_i$ be the number of points of $\mathcal{P}^*$ on $L_i$. As each point of $\mathcal{P}^*$ lies on a line of $\mathcal{S}$, and for each line $K \in \mathcal{S}$ distinct from $a^\rho$ there is a unique line $L_i$ through $a$ that is concurrent with $K$, it follows that $1 + \sum\limits_1^t s_i = |\mathcal{S}| = |\mathcal{O}^*| = s't + 1$. Consider one of the lines $L_i$. As there are $s_i$ points of $\mathcal{P}^*$ on $L_i$ and every line of $\Gamma$ through each of these points belongs to $\mathcal{L}$, and each of these lines is incident with a unique and distinct point of $\mathcal{O}^*$, there are $1 + s_i t$ points of $\mathcal{O}^*$ that are collinear with some point of $\mathcal{P}^*$ on $L_i$. Hence $1 + s_i t \leqq |\mathcal{O}^*| = 1 + s't$. It follows that $0 \leqq s_i \leqq s'$ and hence $s_1 = s_2 = \ldots = s_t = s'$. Consequently, every line of $\Gamma$ through a point $a \in \mathcal{O}^*$ is also a line of $\Gamma^*$ and every point of $\mathcal{O}^*$ is collinear with a point of $\mathcal{P}$ on $L_i$. Also, given a line $M \in \mathcal{L}^*$, we may choose the point $a$ above to be the unique point of $\mathcal{O}^*$ on $M$, so then $M = L_i$, for some $i$, and $M$ has exactly $1 + s_i = 1 + s'$ points of $\mathcal{P}$ on it.

Thus $\Gamma^*$ is an incidence structure in which there are exactly $1+t$ lines through each point and $1+s'$ points on each line. For $\Gamma^*$ to be a generalized quadrangle, we need $p \triangleright L \in \mathcal{P}$ and $L \triangleright p \in \mathcal{L}$ for every $p \in \mathcal{P}$ and $L \in \mathcal{L}$. The latter of these, that $L \triangleright p \in \mathcal{L}$, follows from the fact that all $1+t$ lines of $\Gamma$ through $p$ belong to $\mathcal{L}$. For the former, let $a$ be the unique point of $\mathcal{O}^*$ on $L \triangleright p$. Then $p \triangleright L = a \triangleright L$, which is in $\mathcal{P}$ by the previous paragraph where we saw that every point of $\mathcal{O}^*$ is collinear with a point of $\mathcal{P}$ on $L$.

Thus $\Gamma^*$ is a generalized quadrangle with order $(s', t)$ or it is a dual grid if $s' = 1$ (see [8, p1]). Also, as the points in $\mathcal{P}$ are all on lines of $\mathcal{S}$ whose elements are pairwise opposite, the set $\mathcal{S}$ is a spread of $\Gamma^*$. Similarly, $\mathcal{O}^*$ is an ovoid. As a consequence, we have $s' < t^2$ since generalized quadrangles of order $(t^2, t)$ do not admit spreads. In order to show that $\Gamma^* = \Gamma$ and $\mathcal{O}^* = \mathcal{O}$, we have only to show that $s' = s$. To this end, we first show that $t|s'$. Notice that the claims that $t|s$ and $s < t^2$ will follow once we have shown that $s' = s$.

Let $a \in \mathcal{O}^*$ and let $L \neq a^\rho$ be a line through $a$. Let $\mathcal{B} = \{K \in \mathcal{S} \mid K \neq a^\rho \text{ and } K \text{ meets } L\}$ be the set of lines $K$ in $\mathcal{S}$ that meet $L$ in a point of $\mathcal{P}^*$. There are exactly $s'$ points of $\mathcal{P}^*$ on $L$ and each of these is on a unique line of $\mathcal{S}$. Thus $|\mathcal{B}| = s'$. Let $K \in \mathcal{B}$ and $g \in H(a)$. Then $a^\rho \neq K^g \in \mathcal{S}$ and $K^g$ meets $L$ since $L$ is fixed by $g$. Thus $H(a)$ fixes $\mathcal{B}$. Finally, since $H(a)$ is semiregular on $\mathcal{S} \setminus \{a^\rho\}$, it is semiregular on $\mathcal{B}$ and so the order $|H(a)|$ divides $|B|$. Hence $t|s'$ and so $\Gamma^*$ is a generalized quadrangle and not a dual grid.

We now rely heavily on the material in Section 2.1.

If $s' < s$ then $s' \leqq t < s$. Since $t|s'$, we then have $s' = t$ and hence $s = t^2$. So $\Gamma^*$ has order $t$ and $\Gamma$ has order $(t^2, t)$, and consequently, every point of $\mathcal{O} \setminus \mathcal{O}^*$ lies on a line of $\Gamma^*$. Let $a \in \mathcal{O} \setminus \mathcal{O}^*$ and let $L \in \mathcal{L}$ be such a line through $a$. As $\mathcal{O}^*$ is an ovoid of $\Gamma^*$, there is a point $b \in \mathcal{O}^*$ on $L$. But then $a$ and $b$ are distinct collinear points of $\mathcal{O}$, contrary to $\mathcal{O}$ being an ovoid of $\Gamma$. Therefore we cannot have $s' < s$ after all. It follows that $s' = s$ and so $\Gamma^* = \Gamma$ and $\mathcal{O}^* = \mathcal{O}$.

All that remains is to show that $G$ is transitive on $\mathcal{O}$.

Let $L \notin \mathcal{S}$ be a line through the point $u$ of $\mathcal{O}$. Let $w \in \mathcal{O}$ be an ovoid point whose corresponding spread line $w^\rho$ is not concurrent with $L$. Letting $K \in \mathcal{S}$ be the unique spread line passing through the point $w \triangleright L$, there is a collineation $g \in H(K)$ such that $u^g = w$. As there are $(st+1)-(s+1) = s(t-1)$ choices for the point $w$, the size of the orbit of $u$ under $G$ is at least $s(t-1)+1 = \frac{1}{2}(st+1) + \frac{1}{2}(s(t-2)+1) > \frac{1}{2}(st+1) = \frac{1}{2}|\mathcal{O}|$. Similarly, the size of the orbit $v^G$ is also greater than $\frac{1}{2}|\mathcal{O}|$, and so it follows that $G$ is transitive on $\mathcal{O}$, as required. □

In view of Lemma 3.1, through each point $a \in \mathcal{O}$ there is a line $a^\rho \in \mathcal{S}$ and the ovoid $\mathcal{O}$ is translation with respect to the flag $\{a, a^\rho\}$ with associated group $H(a)$.

**Lemma 3.2.** *If $t > 2$ is even then $G \cong \mathrm{Sz}(q)$, where $q = 2^{2e+1}$ for some $e \geqq 1$, and its action on $\mathcal{O}$ is the usual 2-transitive representation of $\mathrm{Sz}(q)$.*

Proof. First we show that $G$ acts primitively on $\mathcal{O}$. Then we show that $G$ does not have a transitive normal subgroup of odd order and that it cannot act on $\mathcal{O}$ as either of the groups $\mathrm{SL}(2, 2^h)$ or $\mathrm{PSU}(3, 2^h)$ in its usual 2-transitive representation. Finally, we apply the theorem of Hering [3] to arrive at the desired result.

As $G$ is transitive by Lemma 3.1, primitivity is equivalent to the stabilizer $G_u$ being a maximal subgroup (see [13, Theorem 8.2]). Let $g \in G \setminus G_u$ and consider the subgroup $K = \langle G_u, g\rangle$. Let $G_0 = \langle H(u), H(u^g)\rangle$. Substituting $u^g$ for $v$ in the foregoing, we have from Lemma 3.1 that $G_0$ is transitive on $\mathcal{O}$. Hence $H(v) \leqq G_0$ and so $G \leqq G_0 \leqq K \leqq G$. Thus $K = G$. As $g \in G \setminus G_u$ was chosen arbitrarily, it follows that $G_u$ is maximal and therefore $G$ is primitive.

Now we show that $G$ does not have a transitive normal subgroup of odd order by first supposing that it does and then arriving at a contradiction. Suppose then that $M \trianglelefteq G$ is transitive and $|M|$ is odd. Choose $N \leqq M$, $N \neq 1$, to be a minimal normal subgroup of $G$. As $|N|$ is odd, $N$ is solvable by the Feit-Thompson Odd Order Theorem [2]. By [13, Theorem 11.5], a solvable minimal normal subgroup of a primitive group is regular, so $N$ is regular. Thus $G = NG_u$ and for each $g \in G$ there is a unique $n \in N$ and $\bar{g} \in G_u$ such that $g = n\bar{g}$. As $H(u) \trianglelefteq G_u$, we have $NH(u) \trianglelefteq G$, and by the transitivity of $N$, we have $H(v) \trianglelefteq NH(u)$. Hence $G = \langle H(u), H(v)\rangle \trianglelefteq NH(u)$ and so $G = NH(u)$. By the uniqueness of the element $\bar{g} \in G_u$ for each $g \in G$, we conclude that $H(u) = G_u$. We will arrive at our intended contradiction by exhibiting a collineation $g \in G_u$ that does not fix the point $u$ linewise and so is not an element of $H(u)$. The reader is advised that one's sketch can become quite large while following the next step carefully.

Let $K$ and $L$ be lines through $u$, distinct from $U$ and distinct from each other. Let $w \neq u$ be a point on $K$, let $A \in \mathcal{S}$ be the unique spread line passing through $w$, let $z = A^\rho \triangleright L$, and let $C \in \mathcal{S}$ be the unique spread line passing through $z$. As $t > 2$, there is a line $D$ through $w$ that is distinct from $K$, $A$ and $C \triangleright w$. Let $E = D \triangleright z$ and let $f = z \triangleright D$ be the common point on the lines $D$ and $E$. Let $x \in \mathcal{O}$ be the unique ovoid point on the line $D$. As $u$ and $x$ are distinct points collinear with the common point $w$ on $A$, there is a non-trivial collineation $\alpha \in H(A)$ that maps $u$ to $x$. This $\alpha$ fixes the line $J = A^\rho z$, but as $H(A)$ is semiregular on $\mathcal{S}\setminus\{A\}$, it does not fix the line $C$ and so neither does it fix the point $z$ on $C$. Hence $x \triangleright J = u^\alpha \triangleright J = (u \triangleright J)^\alpha = z^\alpha \neq z = f \triangleright J$. Thus $x \neq f$. Let $y \in \mathcal{O}$ be the unique ovoid point on the line $E$. Then also $y \neq f$. Finally, let $B$ be the line of $\mathcal{S}$ through $f$. Now there is a $\beta \in H(B)$ such that $x^\beta = y$ and a $\gamma \in H(C)$ such that $y^\gamma = u$. The collineation $g = \alpha\beta\gamma$ is then in the stabilizer $G_u$. Furthermore, $K^g = K^{\alpha\beta\gamma} = D^{\beta\gamma} = E^\gamma = L \neq K$ and so $g \notin H(u)$. Thus we have arrived at the desired contradiction and we conclude that $G$ does not have a transitive normal subgroup of odd order.

Now we show that $G$ is neither $\mathrm{SL}(2, 2^h)$ nor $\mathrm{PSU}(3, 2^h)$ in its usual 2-transitive representation.

First, notice that the stabilizer $G_{u,v}$ fixes the point $w = v \triangleright U$. As there are other points of $\mathcal{O}$ collinear with $w$ and there are also points of $\mathcal{O}$ that are not collinear with $w$, it follows that $G_{u,v}$ is not transitive on $\mathcal{O} \setminus \{u, v\}$. Hence $G$ is not 3-transitive and so it is not the group $\mathrm{SL}(2, 2^h)$ with its usual 2-transitive action, since for even $q$ we have $\mathrm{SL}(2, q) \cong \mathrm{PSL}(2, q) \cong \mathrm{PGL}(2, q)$ and the last of these is sharply 3-transitive (for any $q$). Suppose then that $G \cong \mathrm{PSU}(3, q)$ with its usual 2-transitive action. Then $st+1 = |\mathcal{O}| = q^3+1$ and $G_{u,v}$ has two orbits of size 1, one of size $q-1$ and the remainder all have size $(q^2-1)/(q+1, 3)$ (see the end of Section 1 in [7]). In addition to $u$ and $v$, there are precisely $t-1$ other points of $\mathcal{O}$ collinear with $w$ and these are partitioned into orbits under $G_{u,v}$. As the size of each of these orbits is divisible by $q-1$, we have $q-1 | t-1$. Since $s < t^2$ by Lemma 3.1, we have $q^3 = st < t^3$, which implies that $q < t$, and so $st = q^3 < qt^2$. Thus $s/t < q$, where $s/t$ is an integer by Lemma 3.1. Next, $q^3 - 1 = st - 1 = (s/t - 1)t^2 + (t^2 - 1)$. Since $q - 1 | t - 1$, the numbers $q - 1$ and $t^2$ are coprime and we conclude that $q - 1 | s/t - 1$. Together with $s/t < q$, we then have $s = t$. Hence $q^3 = t^2$ so $q = z^2$ and $t = z^3$ for some integer $z > 1$. Now $q-1|t-1$ becomes $z^2-1|z^3-1$ and this implies that $z+1|z(z+1)+1$, which is absurd. Therefore $G$ is not the group $\mathrm{PSU}(3, q)$ with its usual 2-transitive action.

Finally, since $|H(u)| = t$ is even, the group $H(u)$ is a normal subgroup of $G_u$ of even order that is semiregular on $\mathcal{O} \setminus \{u\}$. Thus we may apply Theorem 2.2. Notice that the group $\langle Q^G \rangle$ in the statement of that theorem is here $\langle H(u)^G \rangle = \langle H(u), H(v) \rangle = G$. In view of what we have shown above about $G$, we conclude that $G \cong \mathrm{Sz}(q)$, where $q = 2^{2e+1}$ for some $e \geqq 1$, and that its action on $\mathcal{O}$ is the usual 2-transitive representation of $\mathrm{Sz}(q)$. □

Having established the nature of the group $G$ and its action on $\mathcal{O}$, we are now in a position to determine the structure of the generalized quadrangle $\Gamma$. With this, we complete the proof of the Main Theorem.

**Lemma 3.3.** *If $t$ is even then $\Gamma \cong Q(4, 2^{2e+1})$, for some $e \geqq 0$, and $\mathcal{O}$ is a Suzuki-Tits ovoid.*

Proof. To begin, consider the case that $t = 2$. Then $s = 2$ by Lemma 3.1, so $\Gamma$ has order 2. The result now follows from the fact that up to isomorphism there is a unique generalized quadrangle of order 2 and it, in turn, has a unique ovoid up to isomorphism (see [8, §6.1]).

Suppose then that $t > 2$. By Lemma 3.2, $G \cong \mathrm{Sz}(q)$ with $q = 2^{2e+1}$ for some $e \geqq 1$, and its action on $\mathcal{O}$ is the usual 2-transitive action of $\mathrm{Sz}(q)$. Hence $|\mathcal{O}| = q^2 + 1$ and we can endow $\mathcal{O}$ with the structure of an inversive plane $\mathcal{M}$ in such a way that the action of $G$ on $\mathcal{O}$ is as an automorphism group of $\mathcal{M}$.

Let $a, b, c \in \mathcal{O}$ be three distinct points of $\mathcal{O}$. Let $\mathcal{C}$ be the unique circle of $\mathcal{M}$ containing $a$, $b$ and $c$, let $Q$ be the unique Sylow 2-subgroup of $G_{\mathcal{C}}$ and let $w$ be the point in $\mathcal{C}$ that is fixed by $Q$. As $|H(w)| = t$ is even and $H(w) \leqq G_w$, the associated group $H(w)$ contains an involution that belongs to $G_w$ and so also to $Q$. Since $H(w) \trianglelefteq G_w$ and $G_w$ is transitive on $Q \setminus \{1\}$, it follows that $Q \leqq H(w)$. Taking the orders of these groups, we have $q \leqq t$. But $st + 1 = |\mathcal{O}| = q^2 + 1$ and $t \leqq s$ by Lemma 3.1, so we conclude that $s = t = q$. Hence $Q = H(w)$. As $Q$ is transitive on $\mathcal{C} \setminus \{w\}$ and $H(w)$ leaves the line $w^\rho$ fixed pointwise, it follows that the points of $\mathcal{C}$ are collinear with a common point on $w^\rho$. In particular, the three points $a$, $b$ and $c$ are collinear with a common point. As these points were chosen arbitrarily, it follows from Theorem 2.1 that the generalized quadrangle $\Gamma$ is isomorphic to $Q(4, q)$. Finally, the ovoid $\mathcal{O}$ is a Suzuki-Tits ovoid since it arises from a polarity by Theorem 3.1. □

## References

[1] P. Dembowski, Finite Geometries. Ergebnisse Math. Grenzgeb. **44**. Berlin 1968.

[2] W. Feit and J. G. Thompson, Solvability of groups of odd order. Pacific J. Math. **13**, 775–1029 (1963).

[3] C. Hering, On subgroups with trivial normalizer intersection. J. Algebra **20**, 622–629 (1972).

[4] D. HIGMAN, Partial geometries, generalized quadrangles and strongly regular graphs. In: Atti del Convegno di Geometria Combinatoria e sua Applicazioni. Perugia, pp. 265–293, 1971.
[5] H. LÜNEBURG, Translation Planes. New York 1980.
[6] A. OFFER and H. VAN MALDEGHEM, Spreads and ovoids translation with respect to disjoint flags. In: Proceedings of the Third Pythagorean Conference, Rhodes, Greece, 2003, Des. Codes Cryptogr. **32**, 351–367 (2004).
[7] M. O'NAN, Automorphisms of unitary block designs. J. Algebra **20**, 495–511 (1972).
[8] S. E. PAYNE and J. A. THAS, Finite Generalized Quadrangles. Res. Notes Math. **110**. London 1984.
[9] M. SUZUKI, A new type of simple groups of finite order. Proc. Nat. Acad. Sci. USA **46**, 868–870 (1960).
[10] M. SUZUKI, On a class of doubly transitive groups. Ann. of Math. **76**(2), 105–145 (1962).
[11] J. A. THAS, On 4-gonal configurations. Geom. Dedicata **2**, 317–326 (1973).
[12] J. TITS, Les groupes simples de Suzuki et de Ree. Sém. Bourbaki **210**, 1960/61.
[13] H. WIELANDT, Finite Permutation Groups. New York 1964.

Received: 14 May 2004

A. Offer, K. Thas and H. Van Maldeghem
Vakgroep Zuivere Wiskunde en Computeralgebra
Gebouw S22
Galglaan 2
B-9000 Gent
Belgium
aoffer@cage.ugent.be
kthas@cage.ugent.be
hvm@cage.ugent.be