AFFINE BUILDINGS OF TYPE $\tilde{C}_2$

BY

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Voorwoord

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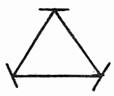
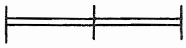
1. **Introduction**

By a celebrated result of J. Tits [13], all affine buildings of rank $\geqslant 4$ are classified and known to be classical, i.e. they all arise from some algebraic group over a local field (see Bruhat-Tits [2]). Affine buildings of rank 3 do not have this characterization as there are counterexamples (see Kantor [8], Ronan [10]). There are three classes of rank 3 affine buildings. They correspond to the respective diagrams



type $\tilde{A}_2$



type $\tilde{C}_2$

type $\tilde{G}_2$

In [14] and [15], we characterized all affine buildings of type $\tilde{A}_2$ and exhibited large classes of new, explicitly defined non-classical examples. Moreover, our construction made it possible to investigate the automorphism group of these buildings (see [7]). Hence the question : Does there exist an analogue for the other two types ? In this paper, we answer that question affirmatively for the type $\tilde{C}_2$. We will give definitions, main results and examples. Proofs will be published elsewhere, since this is beyond the intension of this abstract.

2. Definitions and notation

2.1. *Generalized quadrangles*

The main tool in this exposition is a generalized quadrangle, a notion introduced by J. Tits [11].

A *generalized quadrangle* is a point-line incidence geometry $S = (P(S),L(S,I)$, where $P(S)$ is the set of points, $L(S)$ is the set of lines and I is the symmetric incidence relation, all satisfying

(S.1) *Every line is incident with a constant number* $s+1$ *of points and two lines have at most one point in common* (s *is possibly infinite*).

(S.2) *Every point is incident with a constant number* $t+1$ *of lines and two points are on at most one common line* (t *is possibly infinite*).

(S.3) *For every* $P \in P(S)$ *and every* $L \in L(S)$ *with* P *not incident with* L, *there exists a unique pair* $(Q,M) \in P(S)$ *such that* $P \mathrm{I} M \mathrm{I} Q \mathrm{I} L$.

The pair (s,t) is called the *order of* S. Two points are called *collinear* if they are incident with a common line. Two lines are called *concurrent* if they are incident with a common point. More information on (finite) generalized quadrangles can be found in Payne-Thas [9]. We will only need the coordinatization theory of these objects.

2.2. *Coordinatization of generalized quadrangles*

Let us recall briefly from [5] how a generalized quadrangle $\mathbb{S} = (P(S),L(S), \mathrm{I})$ of order (s,t), $s > 1$, $t > 1$, is coordinatized.

Let R_1 and R_2 be two sets with the following properties. Their intersection consists of the distinct elements 0 and 1. They both do not contain the symbol ∞ and $|R_1| = s$, $|R_2| = t$. We choose an arbitrary but fixed ordinary quadrangle in S and label its elements (∞) I $[\infty]$ I (0) I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(0,0)$ I $[0]$ I (∞), where elements with round brackets denote points and elements with square brackets denote lines. We choose an arbitrary bijection from the set of points incident with $[\infty]$ except (∞) onto R_1 with the only condition that (0) corresponds to 0. If, according to that bijection, a point $P \mathrm{I} [\infty]$ corresponds to $a \in R_1$, then we label P by (a). Dually, every line L through (∞), $L \neq [\infty]$, will be labelled $[k]$, for some $k \in R_2$. The unique point on $[0,0,0]$ collinear with (a) is given coordinates $(a,0,0)$. The unique point on $[1]$ collinear with $(a,0,0)$ is given coordinates $(1,a)$. The unique point on $[0,0]$ collinear with $(1,a)$ is given coordinates $(0,0,a)$. The unique point on $[k]$, $k \in R_2$, collinear with $(0,0,a)$ is given coordinates (k,a). This way, all points collinear with (∞) are coordinatized. Dually, one coordinatizes all lines concurrent with $[\infty]$ by pairs $[b,\ell]$, $b \in R_1$, $\ell \in R_2$. Suppose now $P \in P(S)$ is not collinear with (∞), then P is incident with a unique line $[a,\ell]$ for some $a \in R_1$ and some $\ell \in R_2$, and P is collinear with a unique point $(0,a')$, $a' \in R_1$. We label P by (a,ℓ,a'). Dually, the unique line incident with (k,b) concurrent with $[0,k']$ is given coordinates $[k,b,k']$, $k,k' \in R_2$, $b \in R_1$. We now define two quaternary operations Q_1 and Q_2 as follows. Let $a,a',b \in R_1$ and $k,k',\ell \in R_2$, then

$$Q_1\,(k,a,\ell,a') = b \iff (a,\ell,a') \text{ is collinear with } (k,b),$$
$$Q_2\,(a,k,b,k') = \ell \iff [k,b,k'] \text{ is concurrent with } [a,\ell].$$

One can easily check that this is equivalent with

$$\left.\begin{array}{l} Q_1\,(k,a,\ell,a') = b \\ \\ Q_2\,(a,k,b,k') = \ell \end{array}\right\} \iff (a,\ell,a') \text{ is incident with } [k,b,k']$$

(see [5]). Moreover, one has the following theorem.

THEOREM (2.2.1.) (Hanssens-Van Maldeghem [5]). *If (R_1,R_2,Q_1,Q_2) is as above, then it has the following properties. For all $a,a',b \in R_1$ and for all $k,k',\ell \in R_2$:*

(0) $Q_1(k,0,0,a') = a' = Q_1(0,a,\ell,a')$.

($\bar{0}$) $Q_2(a,0,0,k') = k' = Q_2(0,k,b,k')$.

(1) $Q_1(1,a,0,0) = a$.

($\bar{1}$) $Q_2(1,k,0,0) = k$.

(A) *There exists a unique $x \in R_1$ such that $Q_1(k,a,\ell,x) = b$.*

($\bar{\text{A}}$) *There exists a unique $p \in R_2$ such that $Q_2(a,k,b,p) = \ell$.*

(B) *If $k \neq \ell$, then there exists a unique pair $(x,y) \in R_1^2$ such that*

$$Q_1(k,x,Q_2(x,k,a,k'),y) = a,$$
$$Q_1(\ell,x,Q_2(x,k,a,k'),y) = b.$$

($\bar{\text{B}}$) *If $a \neq b$, then there exists a unique pair $(p,q) \in R_2^2$ such that*

$$Q_2(a,p,Q_1(p,a,k,a'),q) = k,$$
$$Q_2(b,p,Q_1(p,a,k,a'),q) = \ell.$$

(C) *If*

$$Q_1(k,a,\ell,a') \neq b \qquad \text{(C1)}$$
$$Q_2(a,k,b,k') \neq \ell, \qquad \text{(C2)}$$

then there exists a unique quadruple $(x,x',p,p') \in R_1 \times R_1 \times R_2$ *such that*

$$Q_1(k,x,Q_2(x,k,b,k')) = b,$$
$$Q_1(p,x,Q_2(x,k,b,k'),x') = Q_1(p,a,\ell,a'),$$
$$Q_2(a,p,Q_1(p,a,\ell,a'),p') = \ell,$$
$$Q_2(x,p,Q_1(p,a,\ell,a'),p') = Q_2(x,k,b,k').$$

If exactly one of the statements (C1) *or (C2) holds, then there exists no quadruple* $(x,x',p,p') \in R_1 \times R_1 \times R_2 \times R_2$ *with the above properties.*

From now on, we call every quadruple (R_1,R_2,Q_1,Q_2) satisfying (0), $(\bar{0})$, (1), $(\bar{1})$, (A), $(\bar{A})$, (B), $(\bar{B})$, (C) and such that $R_1 \cap R_2 = \{0,1\}$ and neither R_1 nor R_2 contains ∞, a *quadratic quaternary ring,* or briefly a *QQR.*

THEOREM (2.2.2) (Hanssens-Van Maldeghem [5]). *Every QQR coordinatizes a unique generalized quadrangle or order* (s,t), $s > 1$, $t > 1$, *in the above sense and every generalized quadrangle is coordinatized by a QQR, not necessarily unique, but (only) depending on the choice of the elements* (∞), (0), $(0,0)$, $(0,0,0)$, (1) *and* $[1]$. *Hence every generalized quadrangle is uniquely determined by any of its coordinatizing QQRs.*

More information about QQRs can be found in [4], [5] and [6]. We now put a valuation on the structures thusfar defined.

2.3. *Generalized quadrangles with valuation*

Suppose $\mathbf{S} = (P(\mathbf{S}),L(\mathbf{S}),\mathrm{I})$ is a generalized quadrangle of order (s,t), $s > 1$, $t > 1$. Let u be a map,

$$u: \{(X,Y) \in P(\mathbf{S})^2 \cup L(\mathbf{S})^2 \mid X \text{ is collinear or concurrent with } Y\} \to \mathbb{N} \cup \{+\infty\}.$$

Then $(\mathbf{S},u)$ is called a *generalized quadrangle with valuation* if it satisfies (U.1) through (U.5).

(U.1) $u(X,Y) = +\infty$ *if and only if* $X = Y$.

(U.2) *If* X,Y,Z *are incident with a common point or line and* $u(X,Z) > u(Y,Z)$, *then* $u(X,Y) = u(Y,Z)$.

(U.3) $u/P(\mathbf{S})^2$ *and* $u/L(\mathbf{S})^2$ *are onto.*

(U.4) *There exist points* P,P_1,P_2,P_3,P_4 *and lines* L,L_1,L_2,L_3,L_4 *such that* $P_1 \,\mathrm{I}\, L_1 \,\mathrm{I}\, L_2 \,\mathrm{I}\, P_3 \,\mathrm{I}\, L3 \,\mathrm{I}\, P_4 \,\mathrm{I}\, P_1$, $P \,\mathrm{I}\, L_1$, $L \,\mathrm{I}\, P_1$ *and* $u(P_i,P_{i+1}) = u(L_i,L_{i+1}) = u(P_1,P) = u(P_2,P) = u(L_1,L) = u(L_4,L) = 0$, *for all* i (*mod* 4).

(U.5) *For all* P_1, P_2, P_3, $P_4 \in P(\mathbf{S})$ *and all* L_1, L_2, L_3, $L_4 \in L(\mathbf{S})$ *such that* $P_1 \,\mathrm{I}\, L_1 \,\mathrm{I}\, P_2 \,\mathrm{I}\, L_2 \,\mathrm{I}\, P_3 \,\mathrm{I}\, L_3 \,\mathrm{I}\, P_4 \,\mathrm{I}\, L_4 \,\mathrm{I}\, P_1$, *one has* $u(P_1,P_2) + u(L_1,L_2) + u(l_1,L_4) = u(P_3,P_4) + u(L_2,L_3) + u \cup (L_3,L_4)$.

This notion is introduced in [17]. Note that from (U.1) and (U.2) follows that u is symmetric.

2.4. *Quadratic quaternary rings with valuation*

Suppose (R_1,R_2,Q_1,Q_2) is a QQR and v is a map,

$$v:\ R_1 \times R_1 \cup R_2 \times R_2 \to \mathbb{Z} \cup \{+\infty\}.$$

Then we call (R_1,R_2,Q_2,v) a *QQR with valuation* if it satisfies :

(V.1) $v(x,y) = +\infty$ *if and only if* $x = y$, *for all* $(x,y) \in R_1^2 \cup R_2^2$.
(V.2) *If* $v(x,z) > v(y,z)$, *then* $v(x,y) = v(y,z)$, *for all* $(x,y,z) \in R_1^3 \cup R_2^3$.
(V.3) v/R_1^2 and v/R_2^2 are onto.
(V.4) *If*

$$\begin{aligned}
&Q_1(k_1,a_1,\ell_1,a_1') = Q_1(k_1,a_2,\ell_2,a_2') = b_1,\\
&Q_2(a_1,k_1,b_1,k_1') = Q_2(a_1,k_2,b_2,k_2') = \ell_1,\\
&Q_1(k_2,a_1,\ell_1,a_1') = Q_1(k_2,a_3,\ell_3,a_3') = b_2,\\
&Q_2(a_3,k_2,b_2,k_2') = Q_2(a_3,k_3,b_3,k_3') = \ell_3,\\
&Q_1(k_3,a_3,\ell_3,a_3') = Q_1(k_3,a_2,\ell_2,a_2') = b_3,\\
&Q_2(a_2,k_3,b_3,k_3') = Q_2(a_2,k_1,b_1,k_4') = \ell_2,
\end{aligned}$$

then

$$v(k_1,k_2) + v(k_1',k_4') = v(k_1,k_3) + v(k_2,k_3) + v(a_2,a_3).$$

(V.5) *If*

$$\begin{aligned}
&Q_1(k_1,a_1,\ell_1,a_1') = Q_1(k_1,a_2,\ell_2,a_2') = b_1,\\
&Q_2(a_1,k_1,b_1,k_1') = Q_2(a_1,k_2,b_2,k_2') = \ell_1,\\
&Q_1(k_2,a_1,\ell_1,a_3') = Q_1(k_2,a_2,\ell_2,a_2') = b_2,\\
&Q_2(a_2,k_1,b_1,k_3') = Q_2(a_2,k_2,b_2,k_2') = \ell_2,
\end{aligned}$$

then

$$\begin{aligned}
&v(k_1',k_3') = v(k_1,k_2) + v(a_1,a_2),\\
&v(a_1',a_3') = v(a_1,a_2) + v(k_1,k_2),
\end{aligned}$$

(V.6) *If*

$$\begin{aligned}
&Q_1(k,a,\ell,a_1') = b_1,\\
&Q_1(k,a,\ell,a_2') = b_2,
\end{aligned}$$

then

$$v(a_1',a_2') = v(b_1,b_2).$$

(V.7) *If*

$$\begin{aligned}
&Q_2(a,k,b,k_1') = \ell_1,\\
&Q_2(a,k,b,k_2') = \ell_2,
\end{aligned}$$

then

$$v(k_1',k_2') = v(\ell_1,\ell_2).$$

If (R_1,R_2,Q_1,Q_2,v) is a QQR with valuation, then we can define the metrics

$$\delta_i : R_i \times R_i \to \mathbb{R} : (x,y) \to e^{-v(x,y)},\ i = 1,2.$$

If (R_i,δ_i), $i = 1,2$, is a complete metric space, then we call (R_1,R_2,Q_1,Q_2,v) a *complete quadratic quaternary ring with valuation.*

One shows that, in every QQR with valuation v, v is symmetric (use (V.1) and (V.2)) and we abbreviate $v(x,0) = v(x) = v(0,x)$, for all $x \in R_1 \cup R_2$.

2.5. *Complete positive-valuated ternary systems*

Suppose (R_1,R_2,T_1,T_2,v) is a 5-tuple, where R_1 and R_2 are two distinct sets such that $\{0,1\} = R_1 \cap R_2$ and $\infty \notin R_1 \cup R_2$; T_1 and T_2 are two ternary operations,

$$T_1 : R_2 \times R_1 \times R_1 \to R_1,$$
$$T_2 : R_1 \times R_2 \times R_2 \to R_2,$$

and v is a map,

$$v: R_1 \times R_1 \cup R_2 \times R_2 \to \mathbb{N} \cup \{+\infty\}.$$

Then we call (R_1,R_2,T_1,T_2,v) a *complete positive-valuated ternary system* if it satisfies the following conditions.

(P0) $T_1(k,0,a') = a' = T_1(0,a,a')$, *for all* $k \in R_2$ *and all* $a,a' \in R_1$.
($\bar{\text{P0}}$) $T_2(a,0,k') = k' = T_2(0,k,k')$, *for all* $a \in R_1$ *and all* $k,k' \in R_2$.
(P1) $T_1(1,a,0) = a$, *for all* $a \in R_1$.
($\bar{\text{P1}}$) $T_2(1,k,0) = k$, *for all* $k \in R_2$.
(PA) *For all* $a,b \in R_1$, *for all* $k \in R_2$, *there exists an* $x \in R_1$ *such that* $T_1(k,a,x) = b$.
($\bar{\text{PA}}$) *For all* $k,\ell \in R_2$, *for all* $a \in R_1$, *there exists a* $p \in R_2$ *such that* $T_2(a,k,p) = \ell$.
(PB) *For all* $a,b \in R_1$ *and all* $k,\ell \in R_2$ *such that* $v(a,b) \geq 2v(k,\ell) \neq +\infty$, *there exists a pair* $(x,y) \in R_1 \times R_1$ *such that*

$$T_1(k,x,y) = a,$$
$$T_1(\ell,x,y) = b.$$

($\bar{\text{PB}}$) *For all* $a,b \in R_1$ *and all* $k,\ell \in R_2$ *such that* $v(k,\ell) \geq v(a,b) \neq +\infty$, *there exists a pair* $(p,q) \in R_2 \times R_2$ *such that*

$$T_2(a,p,q) = k,$$
$$T_2(b,p,q) = \ell.$$

(PC) *For all* $a,a',b \in R_1$ *and all* $k,k',\ell \in R_2$ *such that* $v(\ell, T_2(a,k,k')) \leq v(b,T_1(k,a,a')) \leq 2v(\ell, T_2(a,k,k')) \neq +\infty$, *there exists a quadruple* $(x,x',p,p') \in R_1 \times R_1 \times R_2 \times R_2$ *such that*

$$T_1(k,x,x') = b,$$
$$T_1(p,x,x') = T_1(p,a,a'),$$

$$T_2(a,p,p') = \ell,$$
$$T_2(x,p,p') = T_2(x,k,k'),$$

(PV.1) $v(a,b) = +\infty$ *if and only if* $a = b$, *for all* $a,b \in R_1$,
$v(k,\ell) = +\infty$ *if and only if* $k = \ell$, *for all* $k, \ell \in R_2$.

(PV.2) *If* $v(x,z) > v(y,z)$, *then* $v(x,y) = v(y,z)$, *for all* $(x,y,z) \in R_1^3 \cup R_2^3$.

(PV.3) v/R_1^2 *and* v/R_2^2 *are onto.*

(PV.4) *If*

$$T_1(k_1,a_1,a_1') = T_1(k_1,a_2,a_2'),$$
$$T_2(a_1,k_1,k_1') = T_2(a_1,k_2,k_2'),$$
$$T_1(k_2,a_1,a_1') = T_1(k_2,a_3,a_3'),$$
$$T_2(a_3,k_3,k_3') = T_2(a_3,k_2,k_2'),$$
$$T_1(k_3,a_3,a_3') = T_1(k_3,a_2,a_2'),$$
$$T_2(a_2,k_3,k_3') = T_2(a_2,k_1,k_4'),$$

then

$$v(k_1,k_2) + v(k_1',k_4') = v(k_1,k_3) + v(k_2,k_3) + v(a_2,a_3).$$

(PV.5) *If*

$$T_1(k_1,a_1,a_1') = T_1(k_1,a_2,a_2'),$$
$$T_1(k_2,a_1,a_3') = T_1(k_2,a_2,a_2'),$$

then

$$v(a_1',a_3') = v(a_1,a_2) + 2v(k_1,k_2).$$

(PV.6) *If*

$$T_2(a_1,k_1,k_1') = T_2(a_1,k_2,k_2'),$$
$$T_2(a_2,k_1,k_3') = T_2(a_2,k_2,k_2'),$$

then

$$v(k_1',k_3') = v(k_1,k_2) + v(a_1,a_2).$$

(PV.7) *If*

$$T_1(k,a,a_1') = b_1,$$
$$T_1(k,a,a_2') = b_2,$$

then

$$v(a_1',a_2') = v(b_1,b_2).$$

(PV.8) *If*

$$T_2(a,k,k_1') = \ell_1,$$
$$T_2(a,k,k_2') = \ell_2,$$

then

$$v(k_1',k_2') = v(\ell_1,\ell_2).$$

(PCO) *R_i is complete with respect to the metric*

$$\delta_i : R_i \times R_i \to \mathbb{R} : (x,y) \to e^{-v(x,y)},$$

for $i = 1,2$.

Again, v is symmetric. Moreover, one can show the uniqueness of the solutions in (PA), (P$\bar{\text{A}}$), (PB), (P$\bar{\text{B}}$) and (PC) (see [16]).

REMARK (2.5). Generalized quadrangles and QQRs possess a *point-line duality*, i.e., if $S = (P(S), L(S), I)$ is a generalized quadrangle, then $S^* = (P(S^*), L(S^*), I)$, where $P(S^*) = L(S)$ and $L(S^*) = P(S)$, is also a generalized quadrangle. If (R_1,R_2,Q_1,Q_2) is a QQR, then (R_2,R_1,Q_2,Q_1) as well. This is never the case for generalized quadrangles with valuation, QQRs with valuation or complete positive-valuated ternary systems. Indeed, suppose e.g. (R_1,R_2,Q_1,Q_2,v) and (R_2,R_1,Q_2,Q_1,v) are QQRs with valuation. Then the condition (V.5) readily implies that $v(a_1,a_2) = 0$, for every $(a_1,a_2) \in R_1^2$ and this contradicts (V.3). Similarly for a complete positive-valuated ternary system. For a generalized quadrangle with valuation, the proof is more tricky.

In section 3, we will connect the structures that we defined in 2.3, 2.4 and 2.5.

2.6. *Affine buildings of type $\tilde{C}_2$*

2.6.1. The standard apartment

Suppose **A** is the real affine plane endowed with the Euclidean distance $d_{\mathbf{A}}$ and let T be a fixed solid triangle with angles 45°, 45° and 90° respectively. Suppose the lengths of the sides of T are respectively $2^{1/2},1,1$. Denote by L_i,i = 1,2,3, the lines of **A** supporting the sides of T. Let **W** be the group of automorphisms of **A** generated by the reflexions about L_i = 1,2,3. The group **W** is called the *Weylgroup* of type $\tilde{C}_2$. The image of T under an arbitrary element of **W** is called a *chamber*. The set of all chambers determines a *tessellation* of **A** into triangles isometric to **T**. We call **A**, endowed with that tessellation, a *standard apartment of type* $\tilde{C}_2$. The images of the vertices, resp. sides, of T under the action of **W** are called *vertices* (resp. *panels*). Two vertices are called *adjacent* if they lie on a common panel. The images of L_i, $i = 1,2,3$, under the action of **W** are called *walls*. The set of walls coincides with the set of lines L of A for which the reflexion about L belongs to **W**. A vertex x is called *special* if for any wall L, there exists a wall L' parallel to L and containing x. This is equivalent to saying that x is on exactly **eight** panels or **four** walls. A *straight* wall is a wall not containing non special vertices (all panels have length $2^{1/2}$). A *diagonal* wall is a wall containing non special vertices (all panels have length 1). Let x be a special vertex and $\mathbf{L}_x$ the set of walls through x. The topological closure of a connected component (in the usual Euclidean sense) of $\mathbf{A} - \bigcup \mathbf{L}_x$ is called a *sector*. The topological closure of a connected component of $\cup \mathbf{L}_x - (x)$ is called a *sectorpanel* (*with source* x). A *straight* (resp. *diagonal*) sectorpanel is a sectorpanel contained in a straight (resp. diagonal) wall. Most of the above definitions are standard concepts and can be found in Bourbaki [1]. The standard apartment we just described is (a geometrical realization of) the *Coxeter complex of irreducible affine type* $\tilde{C}_2$.

2.6.2. *Affine buildings of type* $\tilde{C}_2$

An *affine building of type* $\tilde{C}_2$, also called a *discrete system of apartments of type* $\tilde{C}_2$, is a set Δ together with a family $\mathbf{F}$ of injections from $\mathbf{A}$ into Δ satisfying (SA.1), (SA.2), (SA.3) and (SA.4) stated below. Moreover, $(\Delta,\mathbf{F})$ is called symmetric, resp. complete, if $(\Delta,\mathbf{F})$ satisfies also (SA.5) resp. (SA.6). *Par abus de langage* we say that Δ is a (possibly *symmetric* or *complete*) *affine building of type* $\tilde{C}_2$. The image of $\mathbf{A}$ under anny element of $\mathbf{F}$ is called an *apartment* of Δ. The image of a chamber, panel, (special) vertex, adjacent vertices, (diagonal, resp. straight) wall, sector, (diagonal resp. straight) sectorpanel (with source x) under an arbitrary element f of $\mathbf{F}$ is also called respectively a *chamber, panel, (special) vertex, adjacent vertices, (diagonal,* resp. *straight) wall, sector, (diagonal,* resp. *straight*) *sectorpanel* (*with source* $f(x)$). The elements of Δ are called *points.* A *germ of sectors* in Δ is an equivalence class in the set of sectors of Δ with respect to the equivalence relation $\simeq$: $Q_1 \simeq Q_2$ if $Q_1 \cap Q_2$ contains a sector (see [13]).

(SA.1) $\mathbf{F} \circ \mathbf{W} = \mathbf{F}$.

(SA.2) *If* $f,f' \in \mathbf{F}$, *then the set* $B = (f^{-1} \circ f')(\mathbf{A})$ *is a (not necessarily finite) union of vertices, panels and chambers, it is closed and convex in* $\mathbf{A}$ *and there exists* $w \in \mathbf{W}$ *such that* $f/B = f' \circ w/B$,

(SA.3) *Any two points of* Δ *lie in a common apartment and every panel is contained in at least three chambers.*

(SA.4) *If* $f \in \mathbf{F}$ *and* $x \in f(\mathbf{A})$, *then there exists a retraction map (i.e. an indempotent surjection)* $\pi: \Delta \to f(\mathbf{A})$ *such that* $f^{-1} \circ \rho \circ f'$ *diminishes distances in* $\mathbf{A}$ *for every* $f' \in \mathbf{F}$, *and such that* $\rho^{-1}(x) = \{x\}$.

(SA.5) *Every two germs of sectors have respective representatives in a common apartment.*

(SA.6) *The set* $\{f(\mathbf{A}) \mid f \in \mathbf{F}\}$ *is a maximal set of apartments for* Δ *with respect to the properties* (SA.1), (SA.2), (SA.3) *and* (SA.4).

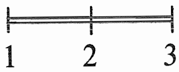
This system of axioms can be found in Tits [13]. In particular, the last axiom (SA.6) is justified by [13], théorème 1 and §5. We also recall from [13] that every complete affine building (of type $\tilde{C}_2$) is symmetric and that $d_{\mathbf{A}}$ induces a well defined *metric* d_Δ in Δ in the obvious way.

2.6.3. The diagram

Suppose x is a special vertex of some affine building Δ of type $\tilde{C}_2$. Define the incidence geometry $\mathbf{R}(x) = (P(x),L(x),\mathrm{I})$, called the *residue of* x (see Buekenhout [3]), as follows.

$P(x) = \{y \mid y$ is a special vertex in Δ adjacent to $x\}$,
$L(x) = \{z \mid z$ is a non special vertex in Δ adjacent to $x\}$,
If $y \in P(x)$ and $z \in L(x)$, then $y \mathrm{I} z$ if y and z are adjacent.

It is well known that $\mathbb{R}(x)$ is a generalized quadrangle. In fact, there is a type map *typ* from the set of vertices of Δ onto $\{1,2,3\}$ turning Δ into a rank 3 geometry with diagram



Actually, a similar map typ_{A} can be defined over the set of vertices of the standard apartment A and *typ* can be viewed as the image of typ_{A} under the mappings $f \in \mathrm{F}$. Such mapping *typ* is well defined by (SA.2). The vertices of type 1 and 3 are special. The residue of a non special vertex (a vertex of type 2) is a *generalized digon*, i.e. a rank 2 geometry where every variety of one type is incident with every variety of the other type (see Buekenhout [3]).

2.6.4. The geometry at infinity

Suppose Δ is a symmetric building of type $\tilde{\mathrm{C}}_2$. Two sectorpanels p and q are called *parallel* if they are on bounded distance from one another. This defines an equivalence relation (see Tits [13]) and we denote the equivalence class (which we call a *parallel class*) of a sectorpanel P by $c(p)$. In view of [13], proposition 17.3, one sees that every class contains either straight or diagonal sectorpanels.

We now define a point-line incidence geometry $\Delta_\infty = (P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$ as follows. The *points* (elements of $P(\Delta_\infty)$) are the parallel classes of straight sectorpanels. The *lines* (elements of $L(\Delta_\infty)$) are the parallel classes of diagonal sectorpanels. A point $c(p) \in P(\Delta_\infty)$ is incident with a line $c(\ell) \in L(\Delta_\infty)$ if there exist representatives $p' \in c(p)$ and $\ell' \in c(\ell)$ lying in a common sector of Δ.

THEOREM (2.6.4) (Tits [13]). *The geometry $\Delta_\infty = (P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$ of a symmetric affine building of type $\tilde{\mathrm{C}}_2$ is a(n infinite) generalized quadrangle.*

For every special vertex x of Δ, the generalized quadrangle Δ_∞ admits a natural epimorphism onto the generalized quadrangle $\mathbb{R}(x)$.

3. Main results

THEOREM (3.1). *Let* S *be a generalized quadrangle. Then the following conditions are equivalent.*

(i) S *is isomorphic to the geometry at infinity of some symmetric affine building Δ of type $\tilde{\mathrm{C}}_2$.*
(ii) S *can be coordinatized by a QQR with valuation.*

(iii) *Every coordinatizing QQR of* **S** *is a QQR with valuation.*
(iv) **S** *can be structured to a generalized quadrangle with valuation.*

If one of these conditions is satisfied, then the completeness of Δ *is equivalent to the completeness of any coordinatizing QQR with valuation of* **S**.

We spend some words on the proof of that theorem.

Suppose Δ is a symmetric affine building of type $\tilde{C}_2$. For every special vertex x of Δ, we construct the valuation u_x on Δ_∞ as follows. Let $P_1,P_2 \in P(\Delta_\infty)$ and suppose $P_i = c(p_i)$ for some sectorpanel p_i, $i = 1,2$. By Tits [13], proposition 5, we can choose p_i such that is has source x, $i = 1,2$. If P_1 and P_2 are collinear, then by definition $u_x(P_1,P_2)$ equals the number of panels in $p_1 \cap p_2$. For lines, we have to divide that number by two, since it is always even. Distinct special vertices x and x' give rise to distinct valuations u_x and $u_{x'}$ on Δ_∞. However, for every coordinatizing QQR with valuation (R_1,R_2,Q_1,Q_2,v) of Δ_∞, there exists a unique special vertex x such that

$$v(a,b) = u_x((a),(b)) - u_x((a),(\infty)) - u_x((b),(\infty)),$$
$$v(k,\ell) = u_x([k],[\ell]) - u_x([k],[\infty]) - u_x([\ell],[\infty]),$$

for all $a,b \in R_1$ and all $k,\ell \in R_2$.

Conversely, if we are given a QQR with valuation (R_1,R_2,Q_1,Q_2,v), then we construct explicitly the corresponding symmetric affine building of type $\tilde{C}_2$ and the vertex x such that the above equalities are satisfied. This construction however is too long to include here, even in part. Similarly to the case $\tilde{A}_2$, one constructs first a certain sequence of geometries with neighbour relation, here called *Hjelmslev quadrangles of level n,* $n \in \mathbb{N}$.

THEOREM (3.2). *Every complete positive-valuated ternary system* $(R_1^+,R_2^+,T_1,T_2,v^+)$ *can be canonically embedded in a complete QQR with valuation* (R_1,R_2,Q_1,Q_2,v) *such that*

$$R_i^+ = \{x \in R_i \mid v(x) \geq 0\},\ i = 1,2,$$
$$T_i(x,y,z) = Q_i(x,y,y',z),\ \textit{for all } y,z \in R_i^+ \text{ and all } x,y' \in R_{3-i}^+,$$
$$v(x,y) = v^+(x,y),\ \textit{for all } (x,y) \in R_i^{+2},\ i = 1,2.$$

Hence every complete positive-valuated ternary system defines a complete affine buildings of type $\tilde{C}_2$.
Conversely, we have :

THEOREM (3.3). *Suppose* (R_1,R_2,Q_1,Q_2,v) *is a complete QQR with valuation and suppose that* Q_1 *and* Q_2 *are both independent of their third argument. Define*

$$R_i^+ = \{x \in R_i \mid v(x) \geq 0\},\ i = 1,2,$$
$$T_i(x,y,z) = Q_i(x,y,9,z),\ \textit{for all } y,z \in R_i^+ \textit{ and all } x \in R_{3-i}^+,\ i = 1,2,$$
$$v^+(x,y) = v(x,y),\ \textit{for all } (x,y) \in R_i^{+2},\ i = 1,2.$$

Then $(R_1^+,R_2^+,T_1,T_2,v^+)$ *is a complete positive-valuated ternary system.*

The operations of theorems (3.2) and (3.3) are not mutually inverse. Indeed, examples (4.5) of the next section yield complete positive-valuated ternary systems which can be embedded in more than one complete QQR with valuation. Only one of them is the canonical embedding referred to in theorem (3.2).

REMARK (3.4). For every QQR (R_1,R_2,Q_1,Q_2), Q_1 is independent of its third argument if and only if the point (∞) is a *regular point*; Q_2 is independent of its third argument if and only if the line $[\infty[$ is a *regular line* (see [5]).

4. Examples

4.1. Let $R_1 = R_2 = \mathrm{GF}(q)((t))$, $q = 2^h$, $h > 1$. Let h_1 and h_2 be positive integers such that $(h, 1 + h_1 + h_2) = 1$. We first define a finite QQR $(\mathrm{GF}(q),\mathrm{GF}(q),Q_1^*,Q_2^*)$ as follows. Let Ψ_i denote the automorphism 2^{h_i}, then we define

$$Q_1^*(k,a,\ell,a') = k^{2\psi_1} . a + a',$$
$$Q_2^*(a,k,b,k') = a^{\psi_2} . k + k'.$$

One can indeed check that this defines a QQR (using the fact that $x \to x^{2\psi_1\psi_2 - 1}$ defines a permutation of $\mathrm{GF}(q)$). We now extend ψ_i to R_i, $i = 1, 2$, by putting

$$(\Sigma x_n t^n)^{\psi_i} = \Sigma x_n^{\psi_i} t^n, \ i = 1,2.$$

Note that ψ_i is a field automorphism of R_i, i = 1,2. We now define (R_1,R_2,Q_1,Q_2,v) by

$$Q_1(k,a,\ell,a') = k^{2\psi_1} . a + a',$$
$$Q_2(a,k,b,k') = a^{\psi_2} . k + k',$$
$$v(v,y) = v(x - y) = \text{smallest non-vanishing power of } t \text{ in } x - y.$$

One shows that (R_1,R_2,Q_1,Q_2,v) is a complete QQR with valuation (see also [16]). The corresponding complete affine building of type $\tilde{C}_2$ is classical if and only if $h_1 = h_2 = 0$. If $(h_1,h_2) \neq (0,0)$, then our building has residues coordinatized by $(\mathrm{GF}(q), \mathrm{GF}(q), Q_1^*,Q_2^*)$ as above, and they are generalized quadrangles of *Tits-type* $T_2(O)$.

4.2. Again, let $R_1 = R_2 = \mathrm{GF}(q)((t))$, $q = 2^h$, $h > 1$. We define (R_1,R_2,Q_1,Q_2,v) as follows.

$$Q_1(k,a,\ell,a') = k^2 . a + a',$$
$$Q_2(a,k,b,k') = a^{\psi} . k + k',$$
v as above,

where $(\Sigma x_n t^n)^{\psi} = \Sigma x_n \left(\frac{t}{1+t}\right)^n$

Again, ψ is a field automorphism and (R_1,R_2,Q_1,Q_2,v) is a complete QQR with valuation. The corresponding complete affine building of type $\tilde{C}_2$ is non classical, but it has classical residues isomorphic to the symplectic generalized quadrangle $W(q)$ (see [9],[4]).

4.3. Let $\mathbf{F}$ be any complete local field with valuation v. Let θ be any element of $\mathbf{F}$ with valuation 1, i.e. $v(\theta) = 1$. Let ψ be any field automorphism preserving the valuation v. Put $R_1 = \mathbf{F} \times \mathbf{F}$, $R_2 = \mathbf{F}$ and (with $x = (x_1,x_2) \in R_1$) :

$$Q_1(k,a,\ell,a') = (a_1k + a_1', a_2k^\psi + a_2'),$$
$$Q_2(a,k,b,k') = a_1^2k + \theta a_2^2k^\psi + k' - 2a_1b_1 - 2\theta a_2b_2.$$

We extend v to $\mathbf{F} \times \mathbf{F}$ by

$$v(a) = v(a_1^2 + \theta a_2^2), \text{ for all } a \in \mathbf{F} \times \mathbf{F}.$$

Again (R_1,R_2,Q_1,Q_2,v) is a complete QQR with valuation, coordinatizing a non classical generalized quadrangle if $\psi \neq 1$. Some (if not all) residues in the corresponding complete affine building of type $\tilde{C}_2$ are isomorphic to the classical symplectic generalized quadrangle defined over the residue field $\mathbf{f}$ of $\mathbf{F}$.

4.4. Let $\mathbf{F} = \mathbf{GF}(q)((t))$, q odd and $\theta \in \mathbf{GF}(q)$, $-\theta$ a non-square in $\mathbf{GF}(q)$. We define (R_1,R_2,Q_1,Q_2,v) as in example (4.3) (v is the natural valuation on $\mathbf{GF}(q)((t))$). This way, we obtain non classical complete affine buildings of type $\tilde{C}_2$ if and only if $\psi \neq 1$. At least one residue is isomorphic to a generalized quadrangle of *Kantor-type.*

4.5. We can apply theorem (3.3) to the examples (4.1), (4.2) and (4.3), $\mathbf{F}$ of characteristic 2. This way, we obtain again complete affine buildings of type $\tilde{C}_2$. Excluding the classical cases in (4.1),(4.2),(4.3), these buildings are again non classical and moreover, they are not isomorphic to one of the buildings above. Also the „*generalized quadrangles at infinity*" of these buildings are, although infinite, of a new type.

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