A characterization of the Split Cayley Generalized Hexagon $\mathbf{H}(q)$ using one subhexagon of order $(1, q)$

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Abstract

In this paper we prove that, if a generalized hexagon Γ of order q contains a subhexagon Γ' (of order $(1, q)$) isomorphic to the incidence graph of the Desarguesian plane $\mathbf{PG}(2, q)$, and if the automorphism group of Γ stabilizing Γ' induces all elations in $\mathbf{PG}(2, q)$, then Γ must be isomorphic to the split Cayley hexagon $\mathbf{H}(q)$.

Key words: Split Cayley hexagons, Moufang hexagons, subhexagons
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1 Introduction

It is well known that the split Cayley hexagon $\mathbf{H}(q)$ has a subhexagon Γ' of order $(1, q)$, which is isomorphic to the incidence graph of the Desarguesian plane $\mathbf{PG}(2, q)$. In fact, there is a unique such subhexagon containing any two given opposite points of $\mathbf{H}(q)$. The Moufang condition with respect to "roots" contained in Γ' implies that the collineation group of $\mathbf{H}(q)$ generated by all root elations (the so-called "little projective group", see [9]) induces in Γ' the little projective group $\mathbf{PSL}_3(q)$ (acting in a natural way on $\mathbf{PG}(2, q)$). One might wonder whether there are other, for instance non-classical, hexagons Γ of order q admitting a subhexagon Γ' of order $(1, q)$ isomorphic to the incidence graph of $\mathbf{PG}(2, q)$ and such that the automorphism group of Γ stabilizing Γ' induces $\mathbf{PSL}_3(q)$ in $\mathbf{PG}(2, q)$. We prove that the answer to that question is negative.

The motivation to treat such a question is threefold. Firstly, this question arises from an attempt to construct finite non-classical generalized hexagons, and hence should be recorded in order to avoid repetitions of attempts. Secondly, the conditions are a significant weakening of the Moufang property and make it easier to recognize $\mathbf{H}(q)$ in concrete situations. Thirdly, the proof we present is rather geometric and introduces an object (called "spheres") that has interesting combinatorial properties, and hence could be useful in other situations as well.

In Section 2 we give precise definitions and state our Main result. In Section 3, we prove our Main Result. Finally, in Section 4, we describe the analogue for generalized quadrangles (which yields a new characterization of the symplectic quadrangles).

2 Preliminaries

A *generalized hexagon* Γ (of order (s,t)) is a point-line geometry for which the incidence graph has diameter 6 and girth 12, every line is incident with $s+1$ points, and every point is incident with $t+1$ lines. If $s=t$, then we also say that Γ has *order* s. Note that, if $\mathcal{P}$ is the point set of Γ and $\mathcal{L}$ is the line set of Γ, then the *incidence graph* is the (bipartite) graph with vertices $\mathcal{P}\cup\mathcal{L}$ and adjacency given by incidence. The definition implies that, given any two elements a,b of $\mathcal{P}\cup\mathcal{L}$, then either these elements are at distance 6 from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path from a to b. In particular, in the latter case there is a unique element $\mathrm{proj}_a b$ incident with a and nearest to b. If a and b are distance 4 apart, then there is a unique element at distance 2 from both of them, and we denote it by $a \bowtie b$. If a and b are opposite elements, then we denote by a^b the set of elements which are at distance 2 from a and at distance 4 from b. From now on, if we mention distances of points and lines of a generalized hexagon, then it is measured in the incidence graph of the generalized hexagon.

In this paper we are mostly interested in the class of finite split Cayley hexagons $\mathbf{H}(q)$, which have order q, and which can be constructed as follows (the hexagons and their construction are due to Tits [8]).

Choose coordinates in the projective space $\mathbf{PG}(6,q)$ in such a way that a parabolic nonsingular quadric $\mathbf{Q}(6,q)$ has equation $X_0X_4+X_1X_5+X_2X_6=X_3^2$, and let the points of $\mathbf{H}(q)$ be all points of $\mathbf{Q}(6,q)$. The lines of $\mathbf{H}(q)$ are the lines on $\mathbf{Q}(6,q)$ whose Grassmann coordinates $(p_{01},p_{02},\ldots,p_{06},p_{12},\ldots,p_{56})$ satisfy the six relations $p_{12}=p_{34}$, $p_{56}=p_{03}$, $p_{45}=p_{23}$, $p_{01}=p_{36}$, $p_{02}=-p_{35}$ and $p_{46}=-p_{13}$.

The generalized hexagon $\mathbf{H}(q)$ has the following property (see [9], 1.9.17 and 2.4.15). Let x,y be two opposite points and let L,M be two (opposite) lines at distance 3 from both x,y. All points at distance 3 from both L,M are at distance 3 from all lines at distance 3 from both x, y. Hence we obtain a set $\mathcal{R}(x, y)$ of $q+1$ points every member of which is at distance 3 from any member of a set $\mathcal{R}(L, M)$ of $q+1$ lines. We call $\mathcal{R}(x, y)$ a *point regulus*, and $\mathcal{R}(L, M)$ a *line regulus*. Any regulus is determined by two of its elements. The two above reguli are said to be *complementary*, i. e. every element of one regulus is at distance 3 from every element of the other regulus. This is the reason why this property is sometimes called *distance*-3-*regularity*.

The generalized hexagon $\mathbf{H}(q)$ is a *Moufang hexagon*. We will not need a precise definition of the latter, but we content ourselves by mentioning that this is implied by the existence of the following collineations. For any line L and any opposite line M, the group of collineations $G_L^{[3]}$ fixing all elements at distance at most 3 from L acts transitively (and hence regularly) on the set $\mathcal{R}(L, M) \setminus \{L\}$. Each element of $G_L^{[3]}$ is called an *axial elation* (with axis L). Also, for every point x, and every point y opposite x, the group of all collineations fixing all points collinear with x and fixing all lines incident with a point of x^y, acts transitively (hence regularly) on $\mathcal{R}(x, y) \setminus \{x\}$; such collineations are called *root elations*. The group generated by all root elations and axial elations is called the *little projective group of* $\mathbf{H}(q)$. For $q > 2$, it is Dickson's simple group $\mathbf{G}_2(q)$; for $q = 2$, it is isomorphic to $\mathbf{P\Gamma U}_3(3)$ and hence has a simple subgroup of index 2.

The hexagon $\mathbf{H}(q)$ has a lot of subhexagons of order $(1, q)$. Each of these is isomorphic to the so-called *double of the Desarguesian projective plane* $\mathbf{PG}(2, q)$ denoted by $2\mathbf{PG}(2, q)$, i. e., the hexagon obtained from the incidence graph of $\mathbf{PG}(2, q)$ by taking as points the vertices of that graph, and as lines the edges (with natural incidence relation). Let us denote one of the subhexagons of order $(1, q)$ of $\mathbf{H}(q)$ by Γ'. Every elation of $\mathbf{PG}(2, q)$ is, via Γ', induced by a collineation of $\mathbf{H}(q)$, and in fact by an axial elation. All these axial elations do not generate the little projective group of $\mathbf{H}(q)$, and so it is reasonable to suspect that the existence of one subhexagon Γ', together with the fact that all elations of $\mathbf{PG}(2, q)$ are induced by collineations of $\mathbf{H}(q)$, is not responsible for $\mathbf{H}(q)$ being a Moufang hexagon. In this paper we show that this is, nevertheless, the case. More exactly, we show:

Main Result. *Let Γ be a generalized hexagon of order q admitting a subhexagon Γ' isomorphic to $2\mathbf{PG}(2, q)$, with q a power of some prime p. Suppose that all elations of $\mathbf{PG}(2, q)$ are induced (via Γ') by collineations of Γ. Then $\Gamma \cong \mathbf{H}(q)$ and every elation of $\mathbf{PG}(2, q)$ is induced by a unique collineation of Γ of order p, which is necessarily an axial elation.*

Some more terminology for projective planes: a *flag* of a projective plane is an incident point-line pair. Two flags $\{a, A\}$ and $\{b, B\}$, with a, b points and A, B lines, are *opposite*, if a is not incident with B and b is not incident with A. An *elation* is a collineation fixing all points of a certain line A and fixing all lines through a certain point c incident with that line. If the elation is not trivial, then the line A and the point c are unique and are called the *axis* and *center* of the elation, respectively.

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3 Proof of the Main Result

Throughout, we assume that Γ is a generalized hexagon of order q, admitting a subhexagon $\Gamma' \cong 2\mathbf{PG}(2,q)$. We also assume that all elations of $\mathbf{PG}(2,q)$ are induced by collineations of Γ. We denote by G the group of collineations of Γ generated by those collineations that induce elations in $\mathbf{PG}(2,q)$. Also, p is the unique prime dividing q. In order to be able to jump from Γ' to $\mathbf{PG}(2,q)$ without confusion, we shall denote, for every object X in Γ' the corresponding object in $\mathbf{PG}(2,q)$ by $\overline{X}$ (with respect to a fixed chosen isomorphism between Γ' and $2\mathbf{PG}(2,q)$). For instance, if L' is a line of Γ', then $\overline{L'}$ is a flag of $\mathbf{PG}(2,q)$. This notation will also be used for collineations.

For the sake of convenience, we will call lines of Γ' briefly Γ'-*lines* and lines of Γ that do not belong to Γ' briefly Γ-*lines*.

We will prove the Main Result by introducing and studying a geometric object which we will call a *sphere*. Let Δ be any generalized hexagon, and let L be a line of Δ. Then a *sphere with center* L is a set of lines of Δ, all opposite L, partitioning the set of points of Δ at distance 5 from L. If Δ has order (s,t), then it is readily verified that a sphere contains exactly s^2t^2 elements. It is also easy to see that the center of a sphere is unique. If $\mathcal{S}$ is a sphere, then we denote by $C(\mathcal{S})$ its center.

Lemma 1 *Let L be a Γ-line. Then L is concurrent with a unique Γ'-line L'. Furthermore, if $\mathcal{S}$ is the set of Γ'-lines opposite L' and at distance 4 from L (in Γ), then $\mathcal{S}$ is a sphere in Γ' with center L'.*

PROOF. The number of Γ-lines concurrent with some (non-fixed) Γ'-line is equal to $(1+q)(1+q+q^2)(q-1)q = q^5+q^4-q^2-q$. Adding the number of Γ'-lines we obtain $q^5+q^4+q^3+q^2+q+1$, which is the total number of lines of Γ. Hence every Γ-line L is concurrent with a (unique) Γ'-line L'. The rest of the lemma now follows from the fact that Γ contains neither quadrangles nor pentagons. $\square$

Hence, by the preceding lemma, every Γ-line L defines a unique sphere $\mathcal{S}(L)$ of Γ', with some center L'. Translated to $\mathbf{PG}(2,q)$, $\overline{\mathcal{S}(L)}$ is a set of flags opposite the flag $\{c,C\} := \overline{L'}$, with c a point and C a line of $\mathbf{PG}(2,q)$, and such that every point x of $\mathbf{PG}(2,q)$ not incident with C is contained in a unique member of $\overline{\mathcal{S}(L)}$, and likewise for lines of $\mathbf{PG}(2,q)$ not incident with c.

Our next aim is to prove some structural properties of $\mathcal{S}(L)$.

We begin by showing that every elation of $\mathbf{PG}(2,q)$ is uniquely induced by a collineation of Γ of order p. At this point, we cannot show yet that it must necessarily be an axial elation, but we show a slightly weaker property.

Lemma 2 *Let γ be an elation in $\mathbf{PG}(2,q)$ with center c and axis A. Let L' be the line of Γ' with $\overline{L'} = \{c,A\}$. Then there exists a unique collineation α of Γ stabilizing Γ' such that $\gamma = \overline{\alpha_{|\Gamma'}}$ and such that α has order p. Moreover, α fixes all points of Γ that are incident with a Γ'-line that is concurrent with L' (that includes L' itself).*

PROOF. Let K be the subgroup of G fixing Γ' pointwise. We claim that $(|K|,q) = 1$. Indeed, if not, then there is a collineation β of Γ fixing Γ' pointwise and having order p. It follows that β fixes at least three points on every line of Γ'. Hence β fixes a subhexagon of order (s',q), with $1 < s' \leq q$, implying by [6] that $s' = q$. The claim follows. Hence the subgroup E (of order $q|K|$) of G consisting of those collineations of Γ that induce an elation in $\mathbf{PG}(2,q)$ with center c and axis A has a Sylow p-subgroup P of order q, and this subgroup acts regularly on $\mathcal{R}(L',M') \setminus \{L'\}$, for every line M' of Γ' opposite L' (because $P \cap K$ is trivial).

Now, P is a(n abelian) group of order q. We claim that all nontrivial elements of P are conjugate in G. Indeed, since all elations of $\mathbf{PG}(2,q)$ are conjugate in $\mathbf{PSL}_3(q)$, it suffices to show that K acts transitively by conjugation the Sylow p-subgroups of the group $\langle P,K\rangle$. But $\langle P,K\rangle = PK$ and so if P^* is a Sylow p-subgroup of PK, then there exist $a \in P$ and $k \in K$ such that $P^* = P^{ak} = P^k$, which proves the claim. Hence all nontrivial elements of P fix the same number $n+2$ of points on the line L'. Suppose P has $t+2$ orbits on the set of points of Γ incident with L'. Since we already know that 2 of these orbits are trivial (they correspond to the points of Γ' on L'), Burnside's orbit counting theorem asserts that $tq = q-1+n(q-1) = (q-1)(n+1)$. Since $t \leq q-1$, and $q-1$ must divide t, it follows that $t = q-1$, and consequently $n+2 = q+1$. Hence P fixes all points of Γ on L'.

Clearly, there are at most q collineations of Γ fixing L' pointwise, stabilizing Γ', and fixing every line of Γ' meeting L', and the conjugate of such a collineation has the same properties again. It now follows that $P \trianglelefteq E$.

Suppose now that α is a collineation of Γ satisfying (i) α stabilizes Γ', (ii) α has order p, and (iii) $\gamma = \overline{\alpha_{|\Gamma'}}$. Then α is contained in a Sylow p-subgroup of E, and since P is the only such subgroup, we necessarily have $\alpha \in P$. Now, P (and hence also G) contains a unique element α satisfying (i), (ii) and (iii).

To show that P also fixes all points at distance ≤ 3 from L' and incident with a line of Γ', we remark that, if α, β are nontrivial elements of P, then $\alpha^g = \beta$, for some $g \in G$, and g can be chosen such that it fixes an arbitrary Γ'-line M' concurrent with L' (implying that all nontrivial elements of P have the same number of fixed points on M'). Then we can apply the same arguments as above and the assertions follow. □

We call the collineation α of the previous lemma a *long root elation with axis* L'.

We now show a very important intermediate result.

Lemma 3 *The group G acts transitively on the set of Γ-lines.*

PROOF. Let M and N be two concurrent Γ-lines with the additional property that their intersection point x is not incident with any Γ'-line. We first show that M and N are contained in the same G-orbit. Let L be a third line through x (so $M \neq L \neq N$). Put $L' = C(\mathcal{S}(L))$ (so L' is the unique Γ'-line concurrent with L) and let y be the intersection point of L and L'. Let w, W be the point and line, respectively, of $\mathbf{PG}(2, q)$ such that $\overline{L'} = \{w, W\}$.

Now consider the group $Q \leq G$ generated by all long root elations fixing all points of the line L'. Then $\overline{Q}$ is an automorphism group of order q^3 of $\mathbf{PG}(2, q)$ fixing $\{w, W\}$ and it is generated by all elations with center w and by all elations with axis W. This group acts sharply transitively on the set of flags of $\mathbf{PG}(2, q)$ opposite $\{w, W\}$. It follows that $|Q| = |\overline{Q}| = q^3$ (use the fact that every element of Q fixes all points incident with L', and so no nontrivial element of Q can fix all elements of Γ'). If some element α of Q fixed a Γ-line at distance 3 from y and not meeting L' (and recall that Q fixes y), then the corresponding element $\overline{\alpha}$ of $\overline{Q}$ would fix a flag of $\mathbf{PG}(2, q)$ opposite $\{w, W\}$, a contradiction. Hence Q acts transitively on the set of q^3 Γ-lines at distance 3 from y and not meeting L'. It follows that any two such lines are in the same G-orbit, and so, in particular, M and N are.

Since the set of lines outside Γ' is connected with respect to the adjacency relation "being incident with a common point not on a Γ'-line" (see [1]; we may assume that there are at least 4 points on a line by [2]), the assertion follows. □

We next show that, for any Γ-line L, the sphere $\mathcal{S}(L)$ has a rather big automorphism group. Define $G^\dagger$ to be the subgroup of G generated by the long root elations, and put $K^\dagger = K \cap G^\dagger$, with K as in the proof of Lemma 2. Note that, in fact, we proved above that $G^\dagger$ acts transitively on the set of Γ-lines.

Lemma 4 *Let L be any Γ-line. The order of the automorphism group of $\overline{\mathcal{S}(L)}$ inside $\mathbf{PSL}_3(q)$ is a multiple of $q^2(q-1)|K^\dagger|/(q-1,3)$.*

PROOF. By transitivity of $G^\dagger$ on the set of Γ-lines, it follows that $\mathbf{PSL}_3(q)$ acts transitively on the set $\overline{\Sigma}$, where Σ is the set of spheres of Γ' subtended by Γ-lines. Hence, every sphere is equally many times subtended, say k times. Also, we note that $K^\dagger$ acts semiregularly on the set of Γ-lines which follows immediately from the fact that Γ does not admit subhexagons of order (s', q), with $1 < s' < q$. Hence $k = k^*|K^\dagger|$, with k^* a natural number, and so the number of subtended spheres in Γ' is equal to $q(q^3-1)(q+1)/(k^*|K^\dagger|)$. The assertion now follows from the orbit formula, which says that the order of the automorphism group of $\overline{\mathcal{S}(L)}$ inside $\mathbf{PSL}_3(q)$ is equal to the order of $\mathbf{PSL}_3(q)$ divided by the number of subtended spheres in Γ'. This quotient is k^* times the number in the statement of the lemma. $\square$

A sphere $\mathcal{S}$ in Γ' is called a *regulus sphere* if, whenever $M \in \mathcal{S}$, then the regulus $\mathcal{R}(C(\mathcal{S}), M)$ is, with the exception of $C(\mathcal{S})$, entirely contained in $\mathcal{S}$. Translated to $\mathbf{PG}(2,q)$, we see that a sphere with center $\{x, X\}$ is a regulus sphere if, whenever the flag $\{y, Y\}$ belongs to it, then also every flag $\{y', Y'\}$ with $Y \cap Y'$ on X and the line yy' through x.

Lemma 5 *For every Γ-line L, the subtended sphere $\mathcal{S}(L)$ is a regulus sphere. Moreover, it is stabilized by a group of axial elations of order q and with axis $C(\mathcal{S}(L))$.*

PROOF. It is convenient to argue in the projective plane $\mathbf{PG}(2,q)$. Let the flag $\{x, X\}$ be the center of the (subtended) sphere $\overline{\mathcal{S}(L)}$. Let $\{y, Y\}$ belong to $\overline{\mathcal{S}(L)}$. By Lemma 4 and Sylow's theorem, there is a subgroup of $\mathbf{PSL}_3(q)$ of order q^2 stabilizing $\overline{\mathcal{S}(L)}$. Hence there is a group of order q stabilizing $\overline{\mathcal{S}(L)}$ and fixing the intersection point $X \cap Y$. Since this group is inside $\mathbf{PSL}_3(q)$, it must fix X pointwise, and hence it consists of elations. Let θ be such an elation. Then the center of θ cannot be a point on X different from x since, otherwise (and assuming without loss of generality that this center is $X \cap Y$) $\{y, Y\}^\theta$ and $\{y, Y\}$ share the line Y, a contradiction. Hence the group Φ of elations with center x and axis X is completely contained in the automorphism group of $\overline{\mathcal{S}(L)}$. Taking the images of $\{y, Y\}$ under this group, the first assertion follows.

Now we interpret this in Γ. For every element φ of Φ, there is a unique long root elation α with axis L', with $\overline{L'} = \{x, X\}$, such that $\overline{\alpha} = \varphi$. We easily deduce that the group Θ of long root elations with axis L' has order q (is isomorphic to Φ) and stabilizes the sphere $\mathcal{S}(L)$. In particular, it fixes L, because no other line through the intersection point of L and L' subtends $\mathcal{S}(L)$. Let a be a point on L, not on L'. Let M be any line through a, different from L. The orbit M^{Θ} consists of lines concurrent with L and such that the centers of the spheres they subtend belong to a regulus containing L'. For every line T of the regulus, different from L' and M' (where M' is the unique Γ'-line concurrent with M), there exists a unique line T_1 (belonging to M^{Θ}) intersecting T and L. Interchanging the roles of L and M, we also see that there exists a unique line T_2 (belonging to the set $L^{\Theta'}$ for some suitable group Θ' conjugate to Θ) intersecting T and M. Since there are no triangles, quadrangles or pentagons in Γ, the lines T_1 and T_2 coincide and contain the point a. Hence all lines of M^{Θ} are incident with a, which must then be fixed under Θ.

The lemma is proved. $\square$

Let $\mathcal{S}$ be a regulus sphere in Γ' and let $\{c, C\}$ be the center of $\overline{\mathcal{S}}$, with c a point and C a line of $\mathbf{PG}(2, q)$. By the definition of regulus sphere, there is a bijection φ from the set of lines through c to the set of points of C mapping C to c and mapping any other line X through c to the point x if the lines of all flags of $\overline{\mathcal{S}}$ with their point on X are incident with x. If φ is a projectivity, then we call $\mathcal{S}$ *classical*.

We now show that all subtended spheres are classical.

Lemma 6 *Every subtended sphere is classical.*

PROOF. Let L again be a Γ-line. Let L' be the center of the sphere $\mathcal{S}(L)$. Let $M' \in \mathcal{S}(L)$ be arbitrary, and put $M = L \bowtie M'$. Let $\Theta(L)$ and $\Theta(M)$ be the groups of axial elations with axis L' and M', respectively. Let x be the intersection of L and M, which is then fixed under $H := \langle \Theta(L), \Theta(M) \rangle$. We now translate to $\mathbf{PG}(2, q)$. Put $\overline{L'} = \{c, C\}$, and $\overline{M}' = \{a, A\}$. The group $\overline{H}$ fixes ac and $A \cap C$. Let $\{z, Z\}$ be a flag of the sphere, with z not on ac. We coordinatize $\mathbf{PG}(2, q)$ such that $c = (0, 0, 1)$, $a = (1, 0, 0)$, $A \cap C = (0, 1, 0)$, z is on the line $[1, 1, 0]$ and Z contains the point $(0, 1, 1)$. Then, for any $k \in \mathbf{GF}(q) \setminus \{0\}$, the collineation $\overline{h}_k$ induced by the linear transformation with matrix

$$\begin{pmatrix} k^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

belongs to $\overline{H}$. Hence there is an element $h_k \in H$ that induces $\overline{h}_k$ in $\mathbf{PG}(2,q)$. Now note that each element of H fixes every element of the point regulus complementary to the line regulus $\mathcal{R}(L', M')$. Hence h_k fixes all lines incident with a point of the foregoing point regulus and meeting L'. In particular, h_k fixes L and hence $\overline{h}_k$ stabilizes $\overline{\mathcal{S}(L)}$, maps $[1,1,0]$ to $[k,1,0]$, and maps $(0,1,1)$ to $(0,1,k)$. Hence we have $[k,1,0]^\varphi = (0,1,k)$, with φ as above in the definition of classical sphere, which is a projectivity, and we are done. □

The above proof also shows that the stabilizer $G^\dagger_L$ in $G^\dagger$ of a Γ-line L has at least order $q^2(q-1)$. On the other hand, it is easily seen that the stabilizer in $\mathbf{PSL}_3(q)$ of any classical sphere has order exactly $q^2(q-1)$. This implies that $|G^\dagger_L| = q^2(q-1)$. Hence $|\mathbf{PSL}_3(q)|.|K^\dagger| = |G^\dagger| = q^3(q^3-1)(q^2-1)$. Consequently $|K^\dagger| = (q-1,3)$.

Lemma 7 *The group $G^\dagger$ is isomorphic to $\mathbf{SL}_3(q)$.*

PROOF. Remember that $G^\dagger$ is generated by all axial elations that stabilize Γ'. Let L, M, N be three lines of Γ' such that L and N are at distance 4 from each other and $M = L \bowtie N$. Then it is easily seen that, if U_L (U_M, U_N, respectively) denotes the group of order q of axial elations with axis L (M, N, respectively), we have $[U_L, U_N] = U_M$. Hence $G^\dagger$ is a perfect group, i.e., $[G^\dagger, G^\dagger] = G^\dagger$.

Now let $\sigma \in K^\dagger$, and consider an arbitrary axial elation u with axis, say, L in Γ'. Since u fixes all lines concurrent with L, and since σ fixes L, the commutator $[u, \sigma]$ is the identity. So $G^\dagger$ is a perfect central extension of $\mathbf{PSL}_3(q)$ of order $(q-1,3)|\mathbf{PSL}_3(q)| = |\mathbf{SL}_3(q)|$. Since, by [3], the universal perfect central extension of $\mathbf{PSL}_3(q)$ has order $|\mathbf{SL}_3(q)|$ if $q \notin \{2,4\}$, and since $\mathbf{SL}_3(q)$ is a perfect central extension of $\mathbf{PSL}_3(q)$, we see that $G^\dagger$ is isomorphic to $\mathbf{SL}_3(q)$, if $q \notin \{2,4\}$. If $q = 2$, then $\mathbf{PSL}_3(2) \cong \mathbf{SL}_3(2)$ and the result follows. If, finally, $q = 4$, then, again according to [3], the universal perfect central extension of $\mathbf{PSL}_3(4)$ is isomorphic to $S.\mathbf{SL}_3(4)$, with S a group of order 16 isomorphic to the direct product of two cyclic groups of order 4. Hence $\mathbf{SL}_3(4)$ is the unique perfect central extension of order $|\mathbf{SL}_3(4)|$ and the proof of the lemma is complete. □

We now consider an arbitrary Γ-line L_0. Let $H^\dagger_0$ be the stabilizer of L_0 in $G^\dagger$, and let $g_0 \in G^\dagger$ be any axial elation with axis any element of $\mathcal{S}(L_0)$. Then every line L of Γ outside Γ' is uniquely determined by the right coset $H^\dagger_0 g$, with $L^g_0 = L$, $g \in G^\dagger$. We denote $g_L := g$. Moreover, the double coset $D_0 := H^\dagger_0 g_0 H^\dagger_0$ is the union of q^2 right cosets of $H^\dagger_0$, related to all Γ-lines that are concurrent with L_0.

Suppose two lines L and M are concurrent. Then the double coset $H_0^\dagger g_L g_M^{-1} H_0^\dagger$ coincides with D_0.

Conversely, if for two lines L and M the double coset $H_0^\dagger g_L g_M^{-1} H_0^\dagger$ coincides with D_0, then L and M are clearly the image of two concurrent lines, hence themselves concurrent.

Concurrency of lines of Γ with lines of Γ' is easy to see. Hence we have shown that the generalized hexagon Γ is completely determined, up to isomorphism. Hence $\Gamma \cong \mathbf{H}(q)$, and our Main Result is proved.

4 The symplectic quadrangle

The symplectic quadrangle $\mathbf{W}(q)$ is the generalized quadrangle arising from a symplectic polarity in $\mathbf{PG}(3, q)$. We refer to other papers in these proceedings for precise definitions and background on generalized quadrangles. The hexagon $\mathbf{H}(q)$ and the quadrangle $\mathbf{W}(q)$ are related in many ways (see for instance lots of similar characterizations in [9]). However, it is here more convenient to argue in the dual of $\mathbf{W}(q)$, which is the quadrangle $\mathbf{Q}(4, q)$ arising from a nonsingular quadric in $\mathbf{PG}(4, q)$.

The quadrangle $\mathbf{Q}(4, q)$ has a subquadrangle Γ' of order $(q, 1)$, a so-called *grid*. Let Γ' consist of the set of lines $\{L_0, L_1, \ldots, L_q, M_0, M_1, \ldots, M_q\}$, with L_i concurrent with M_j, for all $i, j \in \{0, 1, \ldots, q\}$. The stabilizer of that grid in the automorphism group of $\mathbf{Q}(4, q)$ contains the direct product $H := \mathbf{PSL}_2(q) \times \mathbf{PSL}_2(q)$, acting on Γ' as follows. The first factor of H fixes M_j, for all $j \in \{0, 1, \ldots, q\}$ and acts on the L_i, $0 \leq i \leq q$, permutation equivalent to the action of $\mathbf{PSL}_2(q)$ on the points of the projective line $\mathbf{PG}(1, q)$; the second factor fixes L_i, for all $i \in \{0, 1, \ldots, q\}$, and acts on the M_j, $0 \leq j \leq q$, permutation equivalent to the action of $\mathbf{PSL}_2(q)$ on the points of the projective line $\mathbf{PG}(1, q)$. We call an action of $\mathbf{PSL}_2(q) \times \mathbf{PSL}_2(q)$ on the $(q+1) \times (q+1)$-grid *natural* if it is equivalent to the above described action of H on the grid Γ.

We have the following result.

Theorem. *Let Γ be a generalized quadrangle of order q with a subquadrangle Γ' of order $(q, 1)$ such that one of the natural actions of $\mathbf{PSL}_2(q) \times \mathbf{PSL}_2(q)$ on Γ' is induced by the stabilizer of Γ' in an automorphism group G of Γ. Then Γ is isomorphic to $\mathbf{Q}(4, q)$.*

PROOF. Let $K \leq G$ be the pointwise stabilizer of Γ'. We claim $(|K|, q) = 1$.

indeed, if not, then there is a collineation β of γ fixing Γ' pointwise and having order p. It follows that β fixes at least three lines through every point of Γ'. By Theorems 2.2.1 and 2.4.1 of [5], it would then follow that β fixes every point of Γ, a contradiction.

With the above notation, let the first factor $H_L := \mathbf{PSL}_2(q)$ act naturally on the lines L_i, fixing M_j, $0 \leq i, j \leq q$. Let $P_0 \leq H_L$ be the group of order q fixing L_0 and acting regularly on the set $\{L_i : 1 \leq i \leq q\}$. Let E be the subgroup of G consisting of all collineations which stabilize Γ' and induce an element of P_0 in Γ'. This group has order $q|K|$ and hence has a Sylow p-subgroup P of order q. Since $P \cap K$ is trivial, every element of P_0 is induced by an element of P.

Similarly as in the second paragraph of the proof of Lemma 2, one can show that all nontrivial elements of P are conjugate if q is even, and that there are two conjugacy classes of nontrivial elements in P if q is odd.

Hence, if q is even, all nontrivial elements of P fix the same number of lines through the intersection point $L_0 \cap M_j$, for all $j \in \{0, 1, \ldots, q\}$. Using Burnside's orbit counting theorem again, as above, we conclude that P fixes all lines concurrent with L_0, hence L_0 is an *axis of symmetry.*

Suppose now that q is odd. Then there are two conjugacy classes in P, each of size $(q-1)/2$. Fix some $j \in \{0, 1, \ldots, q\}$. Let O be an orbit of P_0 in the set of lines through the intersection of L_0 and M_j. Then $|O| = p^n$, with p the unique prime dividing q, and n such that $1 \leq p^n < q$. If $|O|$ is nontrivial (meaning $|O| > 1$), then some element of P_0 must act fixed point freely, and hence $(q-1)/2$ elements must act fixed point freely on O. Burnside's orbit counting theorem now yields $q = p^n + k(q-1)/2$, which can never be satisfied for a natural number k. Hence also in this case, L_0 is an axis of symmetry.

We conclude that every line of Γ' is an axis of symmetry, hence Γ is span-symmetric and the result follows from [7], or from [4]. $\square$

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