Distance-j Ovoids and Related Structures in Generalized Polygons

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Abstract

Given a finite weak generalized polygon Γ with an order (s, t), we provide necessary conditions on the order for Γ to admit a distance-j ovoid with odd j. This leads to the introduction and study of similar structures involving flags, which we name *floveads.*

Key words: generalized polygon, ovoid, spread, flovead
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1 Introduction

A *weak generalized n-gon* is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines whose incidence graph has diameter n and girth $2n$. If each line of Γ is incident with exactly $s+1$ points and each point is incident with exactly $t+1$ lines, then Γ has *order* (s, t), and if $s = t$ then we may also say that Γ has *order* s. If both $s, t \geq 2$ then Γ is a *generalized n-gon.* By Feit and Higman [4], apart from ordinary n-gons, finite weak generalized n-gons with $n > 2$ and having an order (s, t) can exist only for $n \in \{3, 4, 6, 8, 12\}$, and if $n = 12$ then either $s = 1$ or $t = 1$.

The *distance* $\delta(u, v)$ between two elements u and v of Γ is the distance between them in the incidence graph. In particular, the value of $\delta(u, v)$ is at most n, the diameter of the incidence graph, and when $\delta(u, v) = n$ we say the elements u and v are *opposite*. For an element u of Γ, the set of elements at distance d from u is denoted $\Gamma_d(u)$. The sizes of the sets $\Gamma_d(u)$, $\mathcal{P}$ and $\mathcal{L}$ are given in [13, Lemma 1.5.4]. In particular, we will use $|\Gamma_{2i}(p)| = s^i t^{i-1}(t + 1)$, where p is a point and $0 < 2i < n$, and $|\mathcal{P}| = (1 + s)(1 + st + s^2t^2 + \cdots + s^{m-1}t^{m-1})$, when $n = 2m$ is even.

The *dual* Γ^D of a weak generalized n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the incidence structure $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathrm{I})$ obtained by interchanging the roles of points and lines. The dual Γ^D is then also a weak generalized n-gon, and if Γ has order (s,t) then Γ^D has order (t,s).

The *double* 2Γ of a weak generalized n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the incidence structure obtained by taking as points the points and lines of Γ, and as lines the *flags* $\{p, L\}$, $p \in \mathcal{P}$, $L \in \mathcal{L}$, of Γ, with incidence being symmetrized inclusion. This is really just the incidence graph of Γ with vertices and edges considered as points and lines. The double 2Γ is a weak generalized $2n$-gon, and if Γ has order s then 2Γ has order $(1,s)$. In fact, every finite weak generalized $2n$-gon with order $(1,s)$ arises as the double of a weak generalized n-gon of order s ([12], see also [13, 1.6.2]).

Let Γ be a weak generalized n-gon. For $1 \leq j \leq n/2$, a *distance-j ovoid* is a set $\mathcal{O}$ of points such that any two points of $\mathcal{O}$ are at least distance $2j$ apart and such that for every element p of Γ there is some element $q \in \mathcal{O}$ with $\delta(p,q) \leq j$. The dual notion is that of a *distance-j spread*. When $j = n/2$, we speak simply of *ovoids* and *spreads*.

There are several factors that motivate the study of distance-j ovoids in finite weak generalized n-gons. For instance, they give rise to perfect codes when j is odd (see [2]), and they are related to epimorphisms from n-gons to m-gons with $n \neq m$ (see [5], [6]). They also have relationships with such objects as 1-systems, semipartial geometries and strongly regular graphs (see [9]).

In Section 2, we give necessary conditions for the existence of certain distance-j ovoids.

2 Distance-j ovoids

In [10], it is shown that a finite weak generalized hexagon Γ of order (s,t) can have an ovoid $\mathcal{O}$ only if $s = t$. This is done there by a double counting argument, first by counting the points that lie in 'neighbourhoods' of the points of $\mathcal{O}$ and then by fixing a point p of $\mathcal{O}$ and counting the other points of $\mathcal{O}$ according to their positions relative to p. Applying this same idea to distance-j ovoids in other finite weak generalized $2m$-gons with an order (s,t) tells us nothing new when j is even as the two counts result in the same expression, but we do get restrictions on the order when j is odd. That is to say, in light of Feit and Higman [4], that this approach yields results when (j,m) is $(3,3)$, $(3,4)$, $(3,6)$ or $(5,6)$. As the first of these cases is treated by [10], here we treat the remaining ones and so prove the following theorem.

Theorem 1 *If a finite weak generalized octagon of order (s,t) admits a distance-3 ovoid then $s = 2t$. If a finite weak generalized dodecagon of order (s,t) admits a distance-3 ovoid then (s,t) is either $(1,1)$ or $(3,1)$. No finite weak generalized dodecagon with an order (s,t) has a distance-5 ovoid.*

Before proving this theorem, let us make some remarks. The only known finite generalized octagons are the Ree-Tits octagons, which have order (q,q^2) where $q = 2^{2e+1}$, and their duals (see [13, 2.5.3]). In view of the previous theorem, the Ree-Tits octagons do not have distance-3 ovoids and the dual Ree-Tits octagons do not have distance-3 ovoids when $q > 2$. It is an open question whether the dual Ree-Tits octagon of order $(4,2)$ admits a distance-3 ovoid. Moreover, it is an open question whether there exists a generalized octagon of order $(4,2)$ not isomorphic to the dual Ree-Tits octagon of that order. Also, there is a unique weak generalized octagon of order $(2,1)$, namely the dual (of the) double of the symplectic quadrangle of order 2, and this certainly does have a distance-3 ovoid. This is discussed in the next section.

A dodecagon of order $(1,1)$ is just an ordinary dodecagon and a distance-3 ovoid is illustrated by the three, six, nine and twelve o'clock positions on a clock face. For the final situation that arises in Theorem 1, there is no known weak generalized dodecagon of order $(3,1)$ with a distance-3 ovoid. Indeed, the only known weak generalized dodecagon of this order is the dual double of the split Cayley hexagon of order 3 and this dodecagon does not have a distance-3 ovoid. This is discussed in detail in the next section.

PROOF OF THEOREM 1. Let $\mathcal{O}$ be a distance-j ovoid of a weak generalized $2m$-gon of order (s,t), with j odd. We first determine the size of $\mathcal{O}$ by counting points in neighbourhoods of the points of $\mathcal{O}$.

Given a point p, let $S(p) = \{q \in \mathcal{P} \mid \delta(p,q) \leq j\}$ be the set of points within distance j of p. Then

$$|S(p)| = |\Gamma_0(p)| + |\Gamma_2(p)| + \cdots + |\Gamma_{j-1}(p)| = 1 + s(1+t) + \cdots + s^{\frac{j-1}{2}}t^{\frac{j-3}{2}}(1+t).$$

Since $\mathcal{O}$ is a distance-j ovoid, the sets $S(p)$, $p \in \mathcal{O}$, partition the set $\mathcal{P}$ of all points of Γ. Thus $|\mathcal{O}|\,|S(p)| = |\mathcal{P}|$ and so

$$|\mathcal{O}|(1+s(1+t)+\cdots+s^{\frac{j-1}{2}}t^{\frac{j-3}{2}}(1+t)) = (1+s)(1+st+s^2t^2+\cdots+s^{m-1}t^{m-1}). \quad (1)$$

We remark here that if j were even and one counted instead the lines within distance j of p, then one would arrive at the formula

$$|\mathcal{O}| = \frac{(st)^m - 1}{(st)^{j/2} - 1}. \quad (2)$$

For the second count, we fix a point $p \in \mathcal{O}$ and count the points of $\mathcal{O}$ by their positions relative to p. This is not so easily expressed in a general way, so we prefer to work case by case. Let us treat only the case of distance-3 ovoids of dodecagons in detail, the other cases being similar and simpler.

Partition $\mathcal{O}$ into sets $\mathcal{O} = \bigcup_{i=0}^{6} \mathcal{O}_{2i}$, where $\mathcal{O}_d = \{q \in \mathcal{O} \mid \delta(p,q) = d\}$ is the set of points of $\mathcal{O}$ at distance d from p. Notice that $\mathcal{O}_0 = \{p\}$ and $\mathcal{O}_2 = \mathcal{O}_4 = \emptyset$. For $d \in \{4,6,8,10\}$, let $\Gamma'_d \subset \Gamma_d(p)$ be the set of points at distance d from p that are collinear neither with a point of $\mathcal{O}_d$ nor with a point of $\mathcal{O}_{d-2}$. While $|\Gamma_d(p)| = s^{\frac{d}{2}} t^{\frac{d}{2}-1}(t+1)$, the number of points of $\Gamma_d(p)$ that are collinear with points of $\mathcal{O}_d$ is $s|\mathcal{O}_d|$ and the number of points of $\Gamma_d(p)$ that are collinear with points of $\mathcal{O}_{d-2}$ is $st|\mathcal{O}_{d-2}|$. Thus

$$|\Gamma'_d| = s^{\frac{d}{2}} t^{\frac{d}{2}-1}(t+1) - s|\mathcal{O}_d| - st|\mathcal{O}_{d-2}|, \text{ for } d \in \{4,6,8,10\}. \tag{3}$$

Since $\mathcal{O}$ is a distance-3 ovoid, each point $r \in \Gamma'_d$ is collinear with a unique point $q \in \mathcal{O}$. By the definition of Γ'_d, the point q belongs to $\mathcal{O}_{d+2}$. In the reverse direction, each point of $\mathcal{O}_{d+2}$ is collinear with a unique point of Γ'_d when $d \in \{4,6,8\}$, and with $t+1$ points of Γ'_d when $d = 10$. Thus

$$|\mathcal{O}_{d+2}| = |\Gamma'_d|, \text{ for } d \in \{4,6,8\}, \text{ and } |\mathcal{O}_{12}| = \frac{|\Gamma'_{10}|}{t+1}. \tag{4}$$

Summing the $|\mathcal{O}_d|$ and using (3) and (4), the size of the distance-3 ovoid $\mathcal{O}$ is

$$|\mathcal{O}| = 1+s^2t+s^2t^2-s^3t-s^3t^2+s^4t+2s^4t^2-s^4t^3+s^4t^4-s^5t+s^5t^2-s^5t^3+s^5t^4.$$

Upon putting this expression for $|\mathcal{O}|$ into (1) and simplifying, what remains is the equation $s^2 - 4st + 3t^2 = 0$, which has solutions $s = t$ and $s = 3t$. The part of the theorem that concerns distance-3 ovoids of dodecagons now follows from the fact that either $s = 1$ or $t = 1$ by [4].

The remaining two cases are treated similarly. In the case that $\mathcal{O}$ is a distance-3 ovoid of a finite weak generalized octagon of order (s,t), the second counting method leads to $|\mathcal{O}| = 1 + s^2t + s^2t^2 - s^3t + s^3t^2$. Putting this into (1) and simplifying leaves $s = 2t$, as stated in the theorem.

In the case that $\mathcal{O}$ is a distance-5 ovoid of a finite weak generalized dodecagon of order (s,t), we find $|\mathcal{O}| = 1 + s^3t^2 + s^3t^3 - s^4t^2 + s^4t^3$, and substitution into (1) leads to the relation

$$s^2t - 2st^2 - st + s - 2t = 0. \tag{5}$$

Hence $t|s$ and $s|2t$, so either $s = t$ or $s = 2t$. If $s = t$ then (5) gives $s^2+s+1 = 0$, which has no integral solutions, and if $s = 2t$ then (5) gives $t = 0$, which is not so. This proves the last part of the theorem. $\square$

3 Floveads

Consider a weak generalized n-gon Γ and its double 2Γ. The distance between two elements in Γ is half the distance between the corresponding points in 2Γ. In accordance with this, for flags F and G in Γ, we shall say that the distance $\delta(F,G)$ between them is half the distance between their corresponding lines in 2Γ. For instance, $\delta(F,G) = 1$ precisely when F and G have exactly one common element.

A distance-j ovoid $\mathcal{O}$ of $(2\Gamma)^D$ corresponds to a distance-j spread $\mathcal{S}$ of 2Γ, and this in turn corresponds to a set $\mathcal{F}$ of flags in Γ. Such a set $\mathcal{F}$ of flags satisfies: (i) any two flags of $\mathcal{F}$ are at least distance j apart; and (ii) given any flag F in Γ, there is a flag $G \in \mathcal{F}$ such that $\delta(F,G) \leq j/2$. A set $\mathcal{F}$ of flags of a weak generalized polygon satisfying these two properties shall be called a *distance-j flovead*, this being a combination of the words 'flag', 'ovoid' and 'spread'.

3.1 Odd j

By its very motivation, a distance-j ovoid of $(2\Gamma)^D$ gives rise to a distance-j flovead of Γ. It should be noted however that the converse is not necessarily so. This is due to the fact that for a distance-j ovoid $\mathcal{O}$ we require that for every element p there is a point $q \in \mathcal{O}$ with $\delta(p,q) \leq j$, but the second defining property of distance-j floveads corresponds to only requiring this when p is a point. However, these are equivalent when j is odd (see, for example, the proof of [13, 7.2.2]) and so we may state the following result.

Proposition 2 *For odd j, distance-j floveads of a weak generalized polygon Γ and distance-j ovoids of its dual double $(2\Gamma)^D$ are equivalent objects.*

Together with Theorem 1 and [10], we then have the following.

Proposition 3 *Let Γ be a weak generalized n-gon of order s that admits a distance-3 flovead. If $n = 3$ then $s = 1$. If $n = 4$ then $s = 2$. If $n = 6$ then $s = 1$ or $s = 3$. Also, no weak generalized hexagon of order s admits a distance-5 flovead.*

For $n = 3$ and $s = 1$, a distance-3 flovead corresponds to two opposite points of an ordinary hexagon; one is illustrated in the left picture of Figure 2. The existence of distance-3 floveads for the other values of n and s given in this proposition has been discussed after Theorem 1 in the context of distance-3 ovoids of the dual double polygons. In particular, for $n = 6$ and $s = 1$, a distance-3 flovead corresponds to a distance-3 ovoid of an ordinary dodecagon.

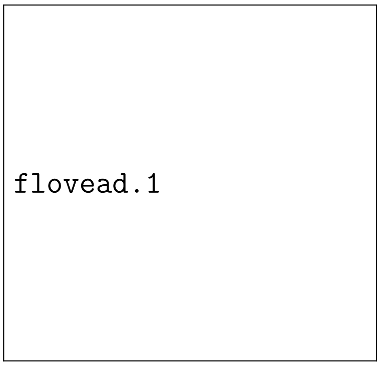

Fig. 1. A distance-3 flovead in the unique generalized quadrangle of order 2.

The remaining two cases need a little elaboration.

For the case $n = 4$ and $s = 2$, it is known that there is a unique generalized quadrangle of order 2, namely the symplectic quadrangle $W(2)$ (see [11, 5.2.3]). A common representation of this quadrangle is shown in Figure 1 with a distance-3 flovead illustrated by the solid lines and filled in points.

The last case, $n = 6$ and $s = 3$, is the only case from Proposition 3 for which we do not know of an example. There is only one known generalized hexagon of order 3, namely the split Cayley hexagon $H(3)$, and this hexagon does not admit a distance-3 flovead. It is an open question whether there is another hexagon of this order. We will now outline a proof of the nonexistence of distance-3 floveads in $H(3)$, but first, some terminology.

Given two opposite points, p_1 and p_2, of $H(3)$, there are exactly four lines L_1, L_2, L_3, L_4 and two further points p_3, p_4 such that $\delta(p_i, L_j) = 3$ for all $1 \leq i, j \leq 4$. The set $\{p_1, p_2, p_3, p_4\}$ of points is a *point regulus* and the set $\{L_1, L_2, L_3, L_4\}$ of lines is the complementary *line regulus*. The set of four points $q_i \mathrm{I} L_i$ collinear with p_1 is a *trace* on p_1. The point p_1, together with the points collinear with it, form a projective plane π of order 3, the lines being the lines of $H(3)$ through p_1 and the traces on p_1.

Suppose $\mathcal{F}$ is a distance-3 flovead of $H(3)$ and suppose p_1 is not in (any flag of) $\mathcal{F}$. There are exactly four points r_1, r_2, r_3, r_4 in $\mathcal{F}$ collinear with p_1, one on each line through p_1. If one of the other p_i is in $\mathcal{F}$ then one can show that exactly two of the q_i are also in $\mathcal{F}$ and so among the r_i. On the other hand, if none of the points p_i are in $\mathcal{F}$ then all of the q_i are, which implies that the r_i form a line in the plane π and so every trace on p_1 must contain either 1 or 4 points of $\mathcal{F}$. But there are points opposite p_1 that belong to $\mathcal{F}$ and so whose traces on p_1 contain exactly two points of $\mathcal{F}$. Hence the second case cannot occur and therefore every trace on p_1 contains exactly two points of $\mathcal{F}$. But then the set $\{p_1, r_1, r_2, r_3, r_4\}$ is a 5-arc in π, which cannot exist.

While Proposition 3 gives information about distance-j floveads in weak generalized polygons with order (s,t) when $s=t$, that is, when they correspond to distance-j ovoids in polygons with orders, we naturally ask ourselves now about the existence of these objects in a more general setting. We start with a result analogous to Theorem 1, obtained by counting as before but now considering distance-j floveads rather than distance-j ovoids. Notice that this theorem says that for odd j we gain nothing by allowing $s \neq t$, as we are quickly forced back to having $s=t$ anyway.

Theorem 4 *Let Γ be a finite weak generalized n-gon with order (s,t) that admits a distance-j flovead with j odd. Then $s=t$, $j=3$ and $(n,s) \in \{(3,1),(4,2),(6,1),(6,3),(12,1)\}$.*

PROOF. As in the proof of Theorem 1, we first count by neighbourhoods. We do this for general j (so j may be even) as we will want the results beyond just this proof.

Let $\mathcal{F}$ be a distance-j flovead in a generalized n-gon Γ with order (s,t). Let Φ be the set of flags in Γ, so

$$|\Phi| = (s+1)(t+1)\frac{(st)^{n/2}-1}{st-1}. \tag{6}$$

For each flag $F \in \Phi$, let n_F be the number of flags $G \in \mathcal{F}$ with $\delta(F,G) \leq j/2$. Then $n_F \in \{1,2\}$ with $n_F = 2$ if and only if the two flags $G, G' \in \mathcal{F}$ within distance $j/2$ from F are at precisely distance $j/2$ from F. In particular, if j is odd then $n_F = 1$ for all flags $F \in \Phi$. Partition Φ into two sets $\Phi = \Phi_1 \cup \Phi_2$, where Φ_i is the set of flags F for which $n_F = i$. Notice that $\Phi_2 = \emptyset$ when j is odd.

Let μ be the number of flags at distance at most $j/2$ from a given flag. Then

$$\mu = \begin{cases} (2+s+t)\frac{(st)^k-1}{st-1} - 1 + 2(st)^k & \text{for } \lfloor j/2 \rfloor = 2k \text{ even,} \\ (2+s+t)\frac{(st)^{k+1}-1}{st-1} - 1 & \text{for } \lfloor j/2 \rfloor = 2k+1 \text{ odd.} \end{cases} \tag{7}$$

Counting the flags at distance at most $j/2$ from each flag of $\mathcal{F}$, we count all flags of Γ since $\mathcal{F}$ is a distance-j flovead, with those in Φ_2 being counted twice. Thus

$$\mu|\mathcal{F}| = |\Phi| + |\Phi_2|. \tag{8}$$

This gives the formula $\mu|\mathcal{F}| = |\Phi|$ for j odd.

As in the proof of Theorem 1, the second count is done case by case and we only do one, namely that of distance-5 floveads in octagons, the other cases being similar.

Let $\mathcal{F}$ be a distance-5 flovead in a weak generalized octagon Γ with order (s, t) and fix a flag $F = \{p, L\}$ of $\mathcal{F}$ with p its point and L its line. Partition $\mathcal{F}$ into sets $\mathcal{F}_i$ whose elements are at distance i from F. Then $|\mathcal{F}_0| = 1$ and $\mathcal{F}_i = \emptyset$ for $i = 1, 2, 3, 4$.

Throughout, let $G = \{q, M\}$ be a flag of the flovead $\mathcal{F}$ with point q and line M.

If G is at distance 5 from F then either $\delta(p, q) = 4$ or $\delta(L, M) = 4$. The number of flags $G \in \mathcal{F}$ with $\delta(p, q) = 4$ is st^2 and the number with $\delta(L, M) = 4$ is s^2t. Thus $|\mathcal{F}_5| = s^2t + st^2$.

Suppose G is at distance 6 from F. Let H be the unique flag at distance 3 from both F and G. As $\mathcal{F}$ is a distance-5 flovead, there must be a flag K with $\delta(H, K) \leq 2$, but such a flag would have $\delta(F, K) \leq 4$ or $\delta(G, K) \leq 4$. Thus $|\mathcal{F}_6| = 0$.

If $G \in \mathcal{F}_7$ then either $\delta(p, q) = 6$ or $\delta(L, M) = 6$. Consider the former situation. There is a path $p \mathrm{~I~} L_1 \mathrm{~I~} p_1 \mathrm{~I~} L_2 \mathrm{~I~} p_2 \mathrm{~I~} L_3 \mathrm{~I~} q$ of length 6, and the flag G is uniquely determined by the p_i and L_i. For each of the three lines L_i there are t possibilities, and for the point p_1 there are s. There is a unique point $p' \mathrm{~I~} L_2$ that belongs to a flag $H \in \mathcal{F}_5$ and we must have $p_2 \neq p'$ as otherwise $\delta(G, H) \leq 4$, and so there are only $s - 1$ possibilities for the point p_2. Thus there are $st^3(s - 1)$ flags $G \in \mathcal{F}_7$ with $\delta(p, q) = 6$. Similarly, there are $s^3t(t - 1)$ with $\delta(L, M) = 6$ and so $|\mathcal{F}_7| = s^3t(t - 1) + st^3(s - 1)$.

Finally, similar reasoning as for $\mathcal{F}_6$ leads to $|\mathcal{F}_8| = 0$.

Adding the sizes of the $\mathcal{F}_i$ now gives

$$|\mathcal{F}| = 1 + s^2t + st^2 - s^3t - st^3 + s^3t^2 + s^2t^3.$$

Substituting this into (8), which in this case says $(2st+t+s+1)|\mathcal{F}| = (s+1) \times (t + 1)(st + 1)(s^2t^2 + 1)$, then simplifying and regrouping, leaves us with

$$(s + t)(s - t)^2 + st\Big(s^2(\tfrac{1}{2}t^2 - t + 1) + t^2(\tfrac{1}{2}s^2 - s + 1)\Big) = 0.$$

Since the discriminant of $\frac{1}{2}x^2 - x + 1$ is negative, the right term in the expression above is always positive. Together with the fact that the left term is always nonnegative for $s, t > 0$, it follows that there are no positive integer solutions. We conclude therefore that the octagon Γ has no distance-5 floveads.

We now briefly describe the details of the remaining cases.

When $n = 3$ we necessarily have $s = t$ and this has been addressed in Proposition 3.

When $n = 4$ and $j = 3$, we find $|\mathcal{F}| = 1 + 2st$. Substitution into (8) then gives $s + t = st$. From here, $s|t$ and $t|s$ so $s = t$ and the result follows.

When $n = 6$ and $j = 3$, we find $|\mathcal{F}| = 1 + 2st - s^2t - st^2 + 3s^2t^2$. Substitution into (8) then gives $2s^2t + 2st^2 = s^2 + t^2 + st + s^2t^2$. If both $s, t \geq 4$ then $s^2t^2 = (t/2)s^2t + (s/2)st^2 \geq 2s^2t + 2st^2$, so we must have either $s < 4$ or $t < 4$. Since the equation is symmetric in s and t, we may suppose that $t < 4$. If $t = 1$ then $s = 1$ since $s|t^2$. If $t = 2$ then $s = 3 \pm \sqrt{5}$, which are not integers. If $t = 3$ then $s = (15 \pm 9)/8$, from which we have $s = 3$.

When $n = 6$ and $j = 5$, we find $|\mathcal{F}| = 1 + s^2t + st^2$, which together with (8) gives $s^2 + t^2 + s^2t + st^2 = s^2t^2$. From here, $s^2|(s + 1)t^2$ and so $s^2|t^2$. Similarly, $t^2|s^2$. Therefore $s = t$ and so the equation becomes $2(1 + s) = s^2$, which has no integer solutions.

When $n = 8$ and $j = 3$, we find $|\mathcal{F}| = 1 + 2st - s^2t - st^2 + 4s^2t^2 + s^3t + st^3 - 3s^3t^2 - 3s^2t^3 + 4s^3t^3$. Substituting this into (8) then gives $s^3 + t^3 + s^2t + st^2 + 3s^3t^2 + 3s^2t^3 = 3s^3t + 3st^3 + 4s^2t^2 + s^3t^3$. If both $s, t \geq 6$ then the right hand side of the equation is strictly greater than $2s^3t + 2st^3 + (t/2)s^3t^2 + (s/2)s^2t^3$ which is in turn at least as large as the left hand side. Thus either $s < 6$ or $t < 6$. Since the equation is symmetric in s and t, we may suppose that $t \geq s$, so then $s < 6$. Since $t|s^3$ and $s \in \{1, 2, 3, 4, 5\}$, it follows that $t = as$ for some integer a such that $a|s^2$. Substituting $t = as$ into our equation, we discover that $s^4|s^3(1 + a + a^2 + a^3)$ and so $s|(1 + a + a^2 + a^3)$ which implies that $\gcd(a, s) = 1$. It now follows that $a = 1$ and therefore $s = t$. By [4] (see also [13, 1.7.1(v)]), if $s > 1$ then $s \neq t$ and so $s = t = 1$. But this is not a solution of our equation and so the result follows.

When $n = 8$ and $j = 7$, we find $|\mathcal{F}| = 1 + 2s^2t^2$ and the equation that we obtain is then $s + t + 2st + s^2t + st^2 = s^2t^2$. From here, $s|t$ and $t|s$ so $s = t$. As in the previous case, this implies that $s = t = 1$, but this is not a solution of the equation.

In the remaining cases, we have $n = 12$ and, dualizing if necessary, we may suppose that $s = 1$.

When $j = 3$, we find $|\mathcal{F}| = 1 + 2t - t^2 + 4t^3 + 2t^5$. Together with (8), this implies $(t - 1)(3t - 1) = 0$ and so $t = 1$ since t is an integer.

When $j = 5$, we find $|\mathcal{F}| = 1 + 3t^2 - 2t^3 + 3t^4$. Together with (8), this implies $(2t - 1)(t^2 - t - 1)(t^2 - t + 1) = 0$, which has no integer solutions.

When $j = 7$, we find $|\mathcal{F}| = 1 + 2t^2 - 2t^3 + 2t^4$. Together with (8), this implies $(t - 1)^2(2t + 1) = 0$ and so $t = 1$. An explicit check in an ordinary dodecagon reveals, however, that it does not have a distance-7 flovead.

When $j = 9$, we find $|\mathcal{F}| = 1 + 2t^3$. Together with (8), this implies $2t^4 - 2t^3 - 4t^2 + 1 = 0$, which has no integer solutions since the left expression is always odd.

Finally, when $j = 11$, we have $|\mathcal{F}| = 1 + 2t^3$, and this together with (8) leads to $(2t + 1)^2 = 0$, which does not have an integer solution. $\square$

Theorem 4 indicates that distance-j floveads with odd j in finite weak generalized polygons with orders, and so distance-j ovoids in the dual double polygons by Proposition 2, do not generally exist, with only a few exceptions in very small polygons. The following result continues in this spirit, showing the nonexistence of certain ones of these structures when the weak generalized polygon is not required to have an order nor even to be finite. It is interesting to notice that by [1], distance-2 ovoids always exist in certain infinite generalized polygons by means of a free construction, and yet despite the similarity with distance-j ovoids, these distance-j floveads, at least for certain values of j, can never succumb to such a construction.

Theorem 5 *Let Γ be a weak generalized n-gon. Let $j = n$ if n is odd and let $j = n - 1$ if n is even. If $n = 4$ then suppose that every line of Γ has at least four points on it, otherwise suppose that every line has at least three points on it. Then Γ does not have a distance-j flovead.*

PROOF. Suppose $\mathcal{F}$ is a distance-j flovead. As j is odd, put $j = 2k + 1$. Let $F = \{x, X\} \in \mathcal{F}$ be a flag of the flovead with x the point. Let $\tilde{z}$ be an element of Γ such that $\delta(\tilde{z}, X) = \delta(\tilde{z}, x) - 1 = k + 1$, and let $G = \{y, Y\}$ be the unique flag in $\mathcal{F}$, with y the point, such that $\delta(\tilde{z}, y) = \delta(\tilde{z}, Y) + 1 = k$. Then $\delta(F, G) = j$ and $\delta(X, Y) = j - 1$. Letting $d = \delta(x, y)$, we also have $d = j + 1 = n$ or $d = j - 1 = n - 1$, according as n is even or odd, or more succinctly, $d = j + (-1)^n = 2n - (j + 1)$.

Let $\tilde{y}$ be the unique point on Y that is at distance $k - 2$ from $\tilde{z}$. As there are at least three points on the line Y, we may choose a point y' incident with Y distinct from y and $\tilde{y}$. There is a path γ' of length $j + 1$ from x to y' that passes via X, $\tilde{z}$ and Y. Since Γ is a weak generalized n-gon, there is another path γ from x to y' which together with γ' makes a circuit of length $2n$. Notice that γ contains neither X nor Y, and its length is $d = \delta(x, y) = \delta(x, y')$.

Let u be the element in the path γ that is at distance $k + 1$ from x, and let $K \in \mathcal{F}$ be the unique flag of $\mathcal{F}$ at distance j from F and whose point w is at distance $k - 1$ from u. Similarly, let v be the element in γ at distance k from y', and let $K' \in \mathcal{F}$ be the unique flag of $\mathcal{F}$ at distance j from G whose line W is at distance $k - 1$ from v.

Now $\delta(u, y') = \delta(x, y') - \delta(x, u) = d - (k+1) = k + (-1)^n = \delta(v, y') + (-1)^n$, so u and v are incident. Also, $\delta(K, K') \leq \delta(w, u) + 1 + \delta(u, W) \leq \delta(w, u) + 1 + \delta(u, v) + \delta(v, W) = 2k < j$, and so $K = K' = \{w, W\}$ since $\mathcal{F}$ is a distance-j flovead. As $\{u, v\}$ is a flag and $\delta(u, w) = \delta(v, W) = k - 1 < n - 1$, it follows that $K = \{u, v\}$ and $k - 1 = 0$. Thus $k = 1$, $j = 3$ and $n = 3$ or 4.

If $n = 3$ then $\delta(x, v) = \delta(x, y') - \delta(v, y') = d - k = 1$ and $\delta(x, u) = 2$. But this implies that $\delta(F, K) = 2 < j$, contrary to $\mathcal{F}$ being a distance-j flovead.

If $n = 4$ then $v \mathrm{~I~} y'$, $\delta(x, u) = 2$ and $\delta(F, K) = 3$. Let y'' be a fourth point on the line Y distinct from y, y' and $\tilde{y}$. As before, there is a flag $K'' = \{u', v'\} \in \mathcal{F}$ with $v' \mathrm{~I~} y''$ and $\delta(x, u') = 2$. Let z be the unique point on v collinear with u' and let Z be the line zu'. As $\mathcal{F}$ is a distance-3 flovead, there is a flag E at distance at most 1 from the flag $\{z, Z\}$, and such a flag E contains either z or Z. But then E would be within distance 2 of either K or K'', which again is a contradiction. □

3.2 Even j

In this section we consider distance-j floveads in weak generalized polygons when j is even. This case is really more general than the odd j case, as is indicated by the following theorem.

Theorem 6 *A distance-$(2k+1)$ flovead is a distance-$2k$ flovead.*

PROOF. Let $\mathcal{F}$ be a distance-$(2k+1)$ flovead of a weak generalized polygon Γ. Let F and G be distinct flags of $\mathcal{F}$. Then $\delta(F, G) \geq (2k+1) > 2k$, so the flags of $\mathcal{F}$ are sufficiently far from each other. Next, let K be any flag of Γ. Then there is a flag $F \in \mathcal{F}$ such that $\delta(F, K) \leq (2k+1)/2$. As distances are integers, it follows that $\delta(F, K) \leq (2k)/2$. Hence $\mathcal{F}$ is a distance-$2k$ flovead of Γ. □

Let $\mathcal{F}$ be a distance-j flovead, j even, in a weak generalized n-gon Γ with order (s, t). We already have an expression for the size of $\mathcal{F}$ in (8), but this involves the size of a set Φ_2 which in general may depend upon the flovead. However, in the extremal case that $j = n$, the size of Φ_2 is fixed and so the size of $\mathcal{F}$ is as well. This leads to the following relationship between distance-j floveads and *ovoid-spread pairings*, which are sets of flags whose points form an ovoid $\mathcal{O}$ and whose lines form a spread $\mathcal{S}$. In generalized quadrangles, any ovoid $\mathcal{O}$ and spread $\mathcal{S}$ determine an ovoid-spread pairing, and in finite generalized hexagons, we know from [2] that polarities give rise to ovoid-spread pairings.

Theorem 7 *Let Γ be a finite weak generalized n-gon with n even and with order (s, t). If $\mathcal{F}$ is a distance-n flovead in Γ then $\mathcal{F}$ is an ovoid-spread pairing, and conversely.*

PROOF. First suppose that $\mathcal{F}$ is an ovoid-spread pairing. Given any two distinct flags $F = \{x, X\}$ and $G = \{y, Y\}$ in $\mathcal{F}$, with x and y the points, we have $\delta(x, y) = \delta(X, Y) = n$ and so $\delta(F, G) = n$. Now consider a flag $K = \{w, W\}$. Since the points of the flags of $\mathcal{F}$ form an ovoid, there is a flag $F = \{x, X\}$ in $\mathcal{F}$ with x the point such that $\delta(w, x) \leq n/2$. Similarly, there is a flag $G = \{y, Y\}$ with Y the line such that $\delta(w, Y) \leq n/2$. Now the closer of F and G to K is within distance $n/2$ of K. Thus $\mathcal{F}$ is a distance-n flovead. We remark that this part of the proof holds without the finiteness assumption.

Now suppose that $\mathcal{F}$ is a distance-n flovead. The points of the flags of $\mathcal{F}$ are then pairwise opposite and likewise for the lines. To show that $\mathcal{F}$ is an ovoid-spread pairing, it is sufficient now to show that $\mathcal{F}$ has the right size (see [13, 7.2.3]). We first determine the size of Φ_2 in (8) in terms of $|\mathcal{F}|$. Given any two distinct flags $F, G \in \mathcal{F}$ of the flovead, there are exactly two flags $K, K' \in \Phi_2$ at distance $n/2$ from both F and G. Furthermore, the flags F and G are the only flags of $\mathcal{F}$ at distance $n/2$ from K. Thus each pair $F, G \in \mathcal{F}$ determines a pair $K, K' \in \Phi_2$, and every flag of Φ_2 arises this way but with no flag of Φ_2 arising twice. Hence $|\Phi_2| = |\mathcal{F}|(|\mathcal{F}| - 1)$. Substituting this into (8) and solving the resulting quadratic equation in $|\mathcal{F}|$ gives $|\mathcal{F}| = (\mu+1)/2 \pm \frac{1}{2}\sqrt{(\mu+1)^2 - 4|\Phi|}$.

We consider the cases $4|n$ and $n = 6$ separately.

Suppose $4|n$ and put $m = n/4$. Set $b = (st)^m + 1$, which is the size of a hypothetical ovoid by (2). Then $|\mathcal{F}| \leq b$ since the points of $\mathcal{F}$ are mutually opposite. Setting also $a = (1 + s)(1 + t)((st)^m - 1)/(st - 1)$, we then have $|\Phi| = ab$ and $\mu + 1 = a + b$, and so $(\mu + 1)^2 - 4|\Phi| = (a - b)^2$. Thus $|\mathcal{F}| = (a + b)/2 \pm (a - b)/2 \geq b$. Hence $|\mathcal{F}| = b$, as required, so $\mathcal{F}$ is an ovoid-spread pairing.

Suppose now that $n = 6$. Dualizing if necessary, we may suppose that $s \leq t$. Set $\alpha = 1 + s^2t$. It is shown in [10] that α is the size of a hypothetical ovoid, so $|\mathcal{F}| \leq \alpha$ since the points of $\mathcal{F}$ are mutually opposite. Now $\mu + 1 = (2 + s + t) \times (1 + st) > 2 + s^2t + st^2 \geq 2\alpha$, thus $0 \leq (\mu + 1) - 2\alpha \leq (\mu + 1) - 2|\mathcal{F}| = \sqrt{(\mu + 1)^2 - 4|\Phi|}$. Squaring and simplifying, this leads to $(\mu + 1 - \alpha)\alpha \geq |\Phi|$. Substituting $\alpha = 1 + s^2t$ and the values for $|\Phi|$ and $\mu + 1$ given in (6) and (7), this inequality ultimately becomes $s \geq t$. Thus we conclude that $s = t$. Finally, putting this into our expression for $|\mathcal{F}|$ gives $|\mathcal{F}| = 1 + s^3 = \alpha$ and so $\mathcal{F}$ is an ovoid-spread pairing. $\square$

For other even values of j, we can get bounds on the size of a distance-j flovead $\mathcal{F}$ by determining bounds on the size of the set Φ_2 in (8). Let ν be the number of flags at distance equal to $j/2$ from a given flag. Then

$$\nu = \begin{cases} (s+t)(st)^k & \text{if } j/2 = 2k+1 \text{ is odd,} \\ 2(st)^k & \text{if } j/2 = 2k > 0 \text{ is even,} \\ 1 & \text{if } j = 0, \\ 0 & \text{if } j \text{ is odd.} \end{cases} \tag{9}$$

Counting the flags at distance $j/2$ from each flag of $\mathcal{F}$, the flags of Φ_2 are each counted twice so $2|\Phi_2| \leq \nu|\mathcal{F}|$. Together with (8), this gives the bounds

$$\frac{|\Phi|}{\mu} \leq |\mathcal{F}| \leq \frac{|\Phi|}{\mu - \frac{1}{2}\nu}. \tag{10}$$

When the upper bound is achieved, the distance-j flovead $\mathcal{F}$ takes the form of a *distance-$j/2$ ovoid-spread pairing*, which is a set of flags whose points form a distance-$j/2$ ovoid and whose lines form a distance-$j/2$ spread.

Theorem 8 *Let Γ be a finite weak generalized n-gon with order (s,t). If $\mathcal{F}$ is a distance-j flovead in Γ with $j < n$ and j even, whose size achieves the upper bound in* (10)*, then $\mathcal{F}$ is a distance-$j/2$ ovoid-spread pairing, and conversely.*

PROOF. First suppose that $\mathcal{F}$ is a distance-$j/2$ ovoid-spread pairing. Similar to the first paragraph of the proof of Theorem 7, the set $\mathcal{F}$ is also a distance-j flovead and this much holds without the assumption of finiteness. To see that $|\mathcal{F}|$ achieves the upper bound in (10), consider the cases that $j/2$ is odd or even separately. For $j/2$ even, the size of $\mathcal{F}$ is given by (2), and using (6), (7) and (9), this is equal to $|\Phi|/(\mu - \frac{1}{2}\nu)$. For $j/2$ odd, first notice that we have a distance-$j/2$ ovoid and a distance-$j/2$ spread with the same size, and so we equate (1) and its dual to find that $s = t$. The rest is as in the previous case.

Now suppose that $\mathcal{F}$ is a distance-j flovead that achieves the upper bound in (10) and let $\mathcal{O}$ and $\mathcal{S}$ be the sets of points and lines respectively of the flags of $\mathcal{F}$. As the flags of $\mathcal{F}$ are at least distance j apart, the points of $\mathcal{O}$ are at least distance $j-1$ apart from one another. As j is even, the points of $\mathcal{O}$ are therefore at least distance j apart and $\mathcal{O}$ is a partial distance-$j/2$ ovoid. Similarly, the set $\mathcal{S}$ is a partial distance-$j/2$ spread. It is sufficient now to check that $|\mathcal{F}| = |\Phi|/(\mu - \frac{1}{2}\nu)$ is equal to the size of a distance-$j/2$ ovoid (see the remark in [13, 7.3.9] regarding [13, 7.2.3]). If $j/2$ is even then the size of a distance-$j/2$ ovoid is given by (2) and the check is straightforward. If $j/2$ is odd then the size α of a distance-$j/2$ ovoid is given by (1). Putting $|\mathcal{F}| \leq \alpha$, we find $s \geq t$, but dualizing if necessary, we may suppose that $s \leq t$. Thus $s = t$ and $|\mathcal{F}| = \alpha$ as required. □

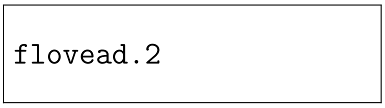

Fig. 2. Distance-2 floveads in an ordinary triangle.

Fig. 3. Distance-2 floveads in the projective plane of order 2.

For which values in the range (10) there exists a distance-j flovead of that size is not at all clear. Indeed, it is not easy to say if there are any distance-j floveads at all. However, in contrast to the situation for distance-j ovoids when j is odd, distance-2 floveads always exist in any weak generalized polygon; this follows from a free construction similar to that used in [1]. We now discuss the spectral problem a little further for the case of distance-2 floveads of projective planes.

Let Γ be a weak generalized triangle of order s. In this case, the bounds (10) become

$$\left\lceil \frac{(1+s)(1+s+s^2)}{1+2s} \right\rceil \leq |\mathcal{F}| \leq 1+s+s^2, \tag{11}$$

where the ceiling function has been used on the left since $|\mathcal{F}|$ is an integer. Notice that if $|\mathcal{F}|$ achieves the upper bound then $\mathcal{F}$ is a distance-1 ovoid-spread pairing (cf. Theorem 8), which is a partition of the set of all points and lines into disjoint flags. The bounds in (11) are sharp for $s = 1$ and $s = 2$, as illustrated in Figures 2 and 3 which show all distance-2 floveads for these cases up to isomorphism. However, we can say little about the sizes of distance-2 floveads of projective planes in general. Indeed, if Γ is a Desarguesian projective plane then the upper bound in (11) is always attainable, but the smallest distance-2 floveads of which we know have size $s^2 + 1$, which is about double the lower bound in (11). We now briefly describe constructions of distance-2 floveads of the sizes mentioned and a couple more.

Constructions

Let $\Gamma = \mathrm{PG}(2, q)$ be the Desarguesian plane of order q.

(1) The orbit of any given flag under a Singer cycle (see [7, 4.2]) is a distance-2 flovead with q^2+q+1 elements.

(2) Let $\mathcal{O}$ be an ovoid of a nondegenerate hyperbolic quadric $Q^+(5,q)$ in five dimensional projective space (see [8, AVI.2]). Let Γ and Γ' be two disjoint generators (planes) of $Q^+(5,q)$, let p and p' be the points of $\mathcal{O}$ in Γ and Γ', respectively, and let L be the unique line in Γ that is coplanar with p' in $Q^+(5,q)$. With each point $r \in \mathcal{O} \setminus \{p,p'\}$ we associate a flag of Γ as follows. There is a unique generator π_r containing r that meets Γ in a line L_r. Also, the plane π_r meets Γ' in a point p'_r and the line rp'_r meets Γ in a point p_r. The q^2-1 flags $\{p_r, L_r\}$ are then disjoint, they use all the lines of Γ except L and those through p, and they use all the points of Γ except p and those on L. Let $\mathcal{F}'$ be the set of these flags. Let the points on L be $u_0, u_1, \ldots, u_q$.

   If we now add to $\mathcal{F}'$ the flags $\{u_i, pu_i\}$, $i = 0,1,\ldots,q$, we have a distance-2 flovead with q^2+q elements.

   If instead we add to $\mathcal{F}'$ the flags $\{u_i, pu_i\}$, $i = 1,2,\ldots,q$, and the flags $\{p, pu_0\}$ and $\{u_0, L\}$, we have a distance-2 flovead with q^2+q+1 elements.

(3) The double 2Γ of Γ is a weak generalized hexagon of order $(1,q)$, which can be considered as a subhexagon of the split Cayley hexagon $H(q)$. Let K be a line of $H(q)$ that is not a line of 2Γ. Then K is concurrent with a unique line K' of 2Γ, and this corresponds to a flag $\{p, L\}$ of Γ. There are exactly q^2 lines of $H(q)$ that are concurrent with K but not K', and each of these is concurrent with some line of 2Γ. In this way we have a set of q^2 lines of 2Γ, and this in turn corresponds to a set $\mathcal{F}'$ of q^2 flags of Γ. The flags of $\mathcal{F}'$ are disjoint and use all the lines of Γ except those through p and all the points of Γ except those on L. Such a set $\mathcal{F}'$ is called a *sphere with centre* $\{p, L\}$ in [3] where these objects are studied in detail.

   If we now add to $\mathcal{F}'$ the flag $\{p, L\}$, we have a distance-2 flovead with q^2+1 elements.

   If instead we add to $\mathcal{F}'$ the flags $\{p, L'\}$ and $\{p', L\}$, for any line $L' \neq L$ through p and point $p' \neq p$ on L, we have a distance-2 flovead with q^2+2 elements.

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