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Split BN-pairs of rank at least 2 and the uniqueness of splittings

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Abstract. Let (G, B, N) be a group with an irreducible spherical BN-pair of rank at least 2, and let U be a nilpotent normal subgroup of B such that $B = U(B \cap N)$. We show that U is unique with respect to B. As a corollary, we obtain a complete classification of all irreducible spherical split BN-pairs of rank at least two.

1 Introduction

Let (G, B, N) be a group with a BN-pair (also called a *Tits system*). In this paper, we will only be interested in the *spherical case*, i.e. the case where the Weyl group $W = N/(B \cap N)$ is finite (and irreducible). The BN-pair is called *split* if there exists a normal nilpotent subgroup U of B such that $UH = B$, where $H = B \cap N$. There is a unique spherical building Ω associated with (G, B, N), and B is associated with a unique maximal flag, or *chamber*, C of Ω. The condition $UH = B$ now immediately translates into the property that U acts transitively on the set of chambers of Ω *opposite* C. Hence it makes sense to call U a *transitive normal nilpotent subgroup of* B. We remark that we always assume that B acts faithfully on Ω; this can be achieved by factoring out the kernel of the action. There is no point in classifying the transitive normal nilpotent subgroups of the Borel subgroup when G does not act faithfully on Ω (because one can take arbitrary direct products). The only interesting case in that respect is when G is a perfect central extension of the corresponding simple group. Our result easily implies that in this case the transitive normal nilpotent subgroup of B has to contain the standard unipotent subgroup U_+ (see below for definitions). To every Tits system corresponds a building. Spherical Tits systems correspond to spherical buildings, and when the (irreducible) rank is at least 3, then these buildings are classified. However, there may be many Tits systems related to one building, and a classification seems out of reach, as Tits pointed out in [10, (11.14)]. In the rank 2

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case, even the buildings are not classified [5], [11], but recently it was shown that a *split* BN-pair of spherical and irreducible rank 2 is essentially equivalent to the so-called *Moufang condition* for the associated generalized polygon (see [6], [7], [8]; the finite case was already treated back in the 1970s in [1], [2]). More precisely, if (G, B, N) defines a generalized n-gon Ω for $n > 2$, and if there is a normal nilpotent subgroup U of B such that $B = U(B \cap N)$, then Ω is a Moufang polygon and G contains all root groups. In the present paper we will show first that already U has to contain the appropriate root groups, i.e., U necessarily coincides with the standard unipotent subgroup U_+ of B, which is a product of root groups. Then it will follow that, if the (spherical and irreducible) rank of (G, B, N) is at least 3, and if this BN-pair splits, then G must contain all root groups, and U is again unique: it is necessarily the product of all root groups the corresponding roots of which contain the chamber C associated to B. So the splitting of the spherical BN-pair can be viewed as a group-theoretic Moufang condition. It is rather unexpected that a purely geometric transitivity condition translates in such full generality to a group-theoretic condition involving nilpotency of a certain subgroup (and in the arguments, nilpotency is essential). Hence the notions of an irreducible spherical split BN-pair of rank $\geqslant 2$, an irreducible spherical Moufang building of rank $\geqslant 2$, a (possibly twisted) Chevalley group of rank $\geqslant 2$ or semi-simple algebraic group of relative rank $\geqslant 2$ are essentially equivalent. We remark that the fact that the uniqueness of U for rank 2 implies the classification of split BN-pairs of higher rank was also noticed independently by Timmesfeld [9], who proved a result similar to ours for a rather restricted class of BN-pairs of rank 2, namely, only those appearing as proper residues in an irreducible spherical BN-pair of higher rank. Our proof will also yield uniqueness of a transitive normal nilpotent subgroup of the Borel subgroup of a split BN-pair of rank 1 in certain cases. We will not pursue this here. We content ourselves by mentioning that this easily follows from our proof for the Tits systems arising from skew-fields. For the other ones treated below, a recoordinatization will be carried out, and hence an additional argument is needed to obtain the desired result.

2 Notation and Main Result

Henceforth (B, N) is a BN-pair, or Tits system, in a group G. (For precise definitions and background, see [10, Chapter 3].) We denote by Ω the corresponding building. Most of the work will be in rank 2, so we concentrate for a moment on this case. Then the corresponding Weyl group $W = N/(B \cap N)$ is a dihedral group of order 6, 8, 12 or 16. Hence Ω is a generalized n-gon, $n \in \{3, 4, 6, 8\}$. That means that Ω can be viewed as a bipartite graph with bipartition $\{\Omega_0, \Omega_1\}$, and with girth $2n$ and diameter n. We will refer to Ω_0 as the point set of Ω, and to Ω_1 as its line set. Also, if a point and a line are adjacent in Ω, then we will also say that they are *incident* and we use the symbol $\sim$ for incident elements. We recall that G acts faithfully on Ω. We assume the existence of a transitive normal nilpotent subgroup U of B. By the results in [6], [7], [8], Ω is a Moufang polygon. To define this, we need some preparations. As the girth of Ω is $2n$, there are closed $2n$-paths in Ω, and we call these *apartments*. Any n-path contained in an apartment is a *root*, and the vertices different from the extreme ones are called *internal*. A root is called a *Moufang root* if the group of collineations (type-preserving automorphisms, i.e. graph automorphisms that preserve the sets Ω_0 and Ω_1) fixing every neighbor of every internal vertex of that root acts transitively on the set of apartments containing the root. Such collineations are called *root elations* and the corresponding groups are called *root groups*. If every root is a Moufang root, then Ω is a *Moufang polygon*. The *little projective group* G_+ of Ω is the group of collineations generated by all root elations. Note that G_+ is usually a simple group (the exceptions only occur in the small cases, i.e. when there are only three points on a line or three lines through a point; more precisely, the exceptions are the groups $B_2(2), G_2(2)$ and ${}^2F_4(2)$); it is always a Chevalley group of rank 2 (but sometimes a twisted one). A similar statement holds in the higher rank case. Now the Borel subgroup B fixes a unique chamber C of Ω and N stabilizes some apartment Σ containing C. Here, a chamber is just an edge, hence an incident point-line pair. Two chambers are said to be *opposite* if, as edges, they are at distance n from each other in the edge graph of Ω. It is well known that B acts transitively on the set of chambers opposite C. Let U_+ be the group generated by all elations related to roots in Σ which contain the vertices of C. Then it is well known that U_+ is a transitive normal nilpotent subgroup of B. In fact, U_+ acts sharply transitively on the set of chambers opposite C, and it is independent of the choice of the apartment Σ through C (or, equivalently, of N). We can now state our Main Result.

Main Result. *Let (G, B, N) be an irreducible Tits system of rank 2 acting faithfully on the corresponding generalized polygon, and suppose that U is a normal nilpotent subgroup of B such that $UH = B$, where $H = B \cap N$. Then $U = U_+$ as defined above.*

We say that a spherical irreducible Tits system (G, B, N) of rank at least 2, which acts faithfully on the corresponding spherical building, is *Moufang* if G contains all root groups, i.e. if $G \geqslant G_+$.

Corollary. *Let (G, B, N) be an irreducible Tits system of rank $\geqslant 2$ acting faithfully on the corresponding building, and suppose that U is a normal nilpotent subgroup of B such that $UH = B$, where $H = B \cap N$. Then $U = U_+$ as defined above, the corresponding building satisfies the Moufang condition and (G, B, N) is Moufang.*

A good introduction to buildings is [14]; also [10] can be used for more background.

3 Proof of the Main Result

3.1 Generalities. We keep the notation of the previous section. In particular, U is a transitive normal nilpotent subgroup of B, where (G, B, N) is an irreducible Tits system of rank 2, with Weyl group W of order 6, 8, 12 or 16. Since U and U_+ normalize each other, we may replace U by UU_+ and consequently we may assume that $U_+ \leqslant U$. But now, if $|W| = 16$, our Main Result follows directly from [6, (3.14)]. Note that we do not need the classification of Moufang octagons to derive our Main

Result for $|W| = 16$, unlike for the other values of $|W|$, where we shall use the classification of Moufang projective planes, Moufang quadrangles and Moufang hexagons as given in [12]. Our general aim is to show that $U = U_+$. For the rest of the proof we assume, by way of contradiction, that $U \neq U_+$. Then $U \cap N$ is non-trivial. Let φ be a non-trivial element of $U \cap N$. We will study the consequences of the existence of φ in some rank 1 Tits systems related to vertices of the apartment Σ, stabilized by N. We now introduce these rank 1 Tits systems. Let v^* be any vertex in Σ and let w_∞^* and w_0^* be the unique (distinct) vertices in Σ adjacent to v^* such that w_∞^* is closer to C than v^*. Let G^* be the quotient of the stabilizer in G of v^* by the group K^* that fixes every vertex adjacent to v^*; let B^* be the quotient of the stabilizer in G of v^* and w_∞^* by K^*, and let N^* be the quotient of the stabilizer in G of the pair $\{w_\infty^*, w_0^*\}$ by K^*. Then (G^*, B^*, N^*) is a Tits system of rank 1. Also, since Ω is a Moufang polygon and hence admits a highly symmetric group (in particular there are automorphisms fixing all neighbors of v^* and acting transitively on the set of chambers at the same distances to $\{v^*, w_\infty^*\}$ and to $\{v^*, w_0^*\}$ as C), it is easily seen that B^* is isomorphic to the stabilizer in B of v^* modulo the pointwise stabilizer in B of the set of vertices adjacent to v^*. If U^* denotes the subgroup of B^* induced by U in this way, then we see that U^* is a transitive normal nilpotent subgroup of B^* (and hence (G^*, B^*, N^*) is a split BN-pair of rank 1). We also write U_+^* for the group induced in B^* by U_+. Our assumptions imply that $U_+^* \trianglelefteq U^*$. We denote by φ^* the element of U^* induced by φ. The strategy of our proof will now be to examine some possible rank 1 groups and show that $U_+^* = U^*$ (and this will be achieved once we show that φ^* is necessarily the identity). For $n = 3, 6$, we will study all possibilities; for $n = 4$, we will only need half of them (see below for more details). The upshot of our investigations will in any case be that φ fixes every vertex adjacent to any vertex v^* of Σ (for $n \neq 4$), or to any vertex v^* of fixed type (for $n = 4$). Since all Moufang polygons are classified, there is a precise list of all possible rank 1 Tits systems (G^*, B^*, N^*) that we need to consider. We start with the ones related to skew-fields, and we fix $\varphi^* \in U^* \cap N^*$, which we assume to be non-trivial.

3.2 Skew-fields. Let $\mathbb{K}$ be a skew-field and consider the 2-transitive group $G^* = \mathrm{PSL}_2(V)$, with V a 2-dimensional right vector space over $\mathbb{K}$, acting on the vector lines of V. We identify w_∞^* with the Y-axis, and w_0^* with the X-axis. Then U_+^* can be identified with the additive group of $\mathbb{K}$. Also, we may identify the set of vertices adjacent to v^* with $\mathbb{K} \cup \{\infty\}$ in such a way that 0 corresponds to w_0^*, ∞ corresponds to w_∞^*, and a generic element t_a of U_+^* acts as $x \mapsto x + a$, $x, a \in \mathbb{K}$, with ∞ fixed. Since $[U_+^*, U^*] \leqslant U_+^*$, we see that there is a non-trivial element of U_+^* in the center of U^*. Without loss of generality, we may assume that it is t_1. Note that φ^* fixes both ∞ and 0. Let Z^* be the center of U^*. Then φ^* fixes the orbit 0^{Z^*} pointwise. Moreover Z^* is a normal subgroup of B^*. Now for $a \in \mathbb{K} \backslash \{0\}$ the mapping $\mu_a : x \mapsto axa$, $x \in \mathbb{K}$ (and ∞ fixed) belongs to B^*. Hence we deduce that $t_{a^2} \in Z^*$, for all $a \in \mathbb{K}$. If the characteristic of $\mathbb{K}$ is different from 2, then we deduce $t_a = t_{(1+a/2)^2} t_{-1} t_{-a^2/4} \in Z^*$. This implies that $\varphi^* = \mathrm{id}$. Hence $U^* = U_+^*$. Suppose now that $\mathbb{K}$ has characteristic 2. Let t_a', $a \in \mathbb{K}$ be the unique element of G^* that is contained in the conjugate of U_+^* which fixes 0 pointwise, and which maps ∞ to a. Note that the image of $x \in \mathbb{K}$ under t_a'

equals $x(1 + xa^{-1})^{-1}$. Now we consider the element $\psi := \varphi^{*-1} t'_a \varphi^*$. Since the conjugates of U_+^* are normal subgroups in the full stabilizer of their fixed element, we have that $\psi = t'_b$, for some $b \in \mathbb{K}$. Comparing images of ∞ under both $\varphi^{*-1} t'_a \varphi^*$ and t'_b, we see that $b = a^{\varphi^*}$. Now we compare the images of 1 under these two expressions of the same map, and we obtain that

$$((1 + a^{-1})^{-1})^{\varphi^*} = 1^{\psi} = 1^{t'_b} = (1 + (a^{\varphi^*})^{-1})^{-1}.$$

This holds for every $a \in \mathbb{K}$. From the first half of the proof of [13, Lemma 8.5.10] it follows that $(a^2)^{\varphi^*} = (a^{\varphi^*})^2$ and that φ^* is an automorphism or an antiautomorphism. Since we already know that $t_{a^2} \in Z^*$, we have $(a^2)^{\varphi^*} = a^2$. The result is now clear from the following lemma, which we state in full generality.

Lemma 3.1. *Let* $\mathbb{D}$ *be an arbitrary field or skew-field* (*possibly infinite-dimensional over its center*), *and let* ϕ *be an arbitrary automorphism or anti-automorphism of* $\mathbb{D}$ *such that* $(a^2)^{\phi} = a^2$ *for all* $a \in \mathbb{D}$. *Then* $\phi = \mathrm{id}$.

Proof. If the characteristic of $\mathbb{D}$ is different from 2, then $a = ((a+1)^2 - a^2 - 1)/2$, and hence $a^{\phi} = a$ for all $a \in \mathbb{D}$. So we may assume that the characteristic of $\mathbb{D}$ is 2. Let F be the set of fixed elements of ϕ in $\mathbb{D}$. Since $(a+b)^2 = a^2 + ab + ba + b^2$ for all $a, b \in \mathbb{D}$, we have $ab + ba \in F$ for all $a, b \in \mathbb{D}$. In particular, if we replace b by ab, we obtain that

$$a(ab + ba) = a(ab) + (ab)a \in F$$

for all $a, b \in \mathbb{D}$. Also, if $a \in F \setminus \{0\}$, then $(a^{-1})^{\phi} = (a^{\phi})^{-1} = a^{-1}$, and hence $a^{-1} \in F$ as well. Now suppose that $\phi \neq \mathrm{id}$, and let c be a fixed element of $\mathbb{D}$ such that $c^{\phi} \neq c$. Let $d := c + c^{\phi}$; then $d \neq 0$. Thus

$$cd = c^2 + cc^{\phi} = (c^2)^{\phi} + cc^{\phi} = dc^{\phi} \neq dc,$$

and hence $cd + dc \in F \setminus \{0\}$, so that $(cd + dc)^{-1} \in F$ as well. Also

$$c^2 d = c^3 + c^2 c^{\phi} = c^3 + (c^3)^{\phi} = c^3 + c^{\phi} c^2 = dc^2,$$

and it follows that $c(cd + dc) = (cd + dc)c$. Assume first that ϕ is an automorphism. Then

$$\begin{aligned} c^{\phi} &= (c \cdot (cd + dc) \cdot (cd + dc)^{-1})^{\phi} \\ &= (c(cd + dc))^{\phi} \cdot ((cd + dc)^{-1})^{\phi} \\ &= c(cd + dc) \cdot (cd + dc)^{-1} \\ &= c, \end{aligned}$$

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which contradicts the assumption that $c^\phi \neq c$. Now assume that ϕ is an anti-automorphism. Then

$$\begin{aligned} c^\phi &= (c \cdot (cd+dc) \cdot (cd+dc)^{-1})^\phi \\ &= ((cd+dc)^{-1})^\phi \cdot (c(cd+dc))^\phi \\ &= (cd+dc)^{-1} \cdot c(cd+dc) \\ &= (cd+dc)^{-1} \cdot (cd+dc)c \\ &= c, \end{aligned}$$

again contradicting our assumption. We conclude that ϕ must be the identity.

3.3 Alternative fields, hexagonal systems and indifferent sets in characteristic 2. The three algebraic structures in the title have one common property that will be responsible for U_+^* being unique. Indeed, they are all algebras defined over some field $\mathbb{K}$, which can be regarded as a subsystem of it, and every element x in such a structure is together with $\mathbb{K}$ contained in a subsystem $\mathbb{L}$, which is again a field. For alternative fields, this is a well-known property; it holds similarly for hexagonal systems by [4, Lemma (38.2)]. For indifferent sets in characteristic 2, we can argue as follows. Let $\mathbb{K}$ be the field of squares of some field $\mathbb{K}'$ of characteristic 2. An indifferent set L is a subset of $\mathbb{K}'$ which is at the same time a vector space over $\mathbb{K}$, and which is closed under taking (multiplicative) inverses (see [12, Chapter 10]). If $x \in L$, then we claim that, for any two polynomials f and g over $\mathbb{K}$, the quotient $f(x)/g(x)$ belongs to L (granted that $g(x) \neq 0$), which implies immediately that $\mathbb{K}(x) \subseteq L$. Suppose not, and let $f(x)/g(x)$ be a counter-example with $\deg f + \deg g$ minimal. We may assume that $\deg f < \deg g$, and that $f(x) \neq 0$. Then $g(x)/f(x) \notin L$. Now we can write $g(x)/f(x) = q(x) + r(x)/f(x)$, and hence $r(x)/f(x)$ is a counter-example with $\deg r + \deg f < \deg f + \deg g$, a contradiction. The claim is proved. Each of the three classes of algebraic structures in the title of the current subsection defines in a unique way a Tits system of rank 1. This Tits system can be viewed as a permutation group acting on the union of the algebraic structure and the singleton $\{\infty\}$. The group U_+^* can be identified with the additive group of the algebra. Also, up to equivalence, one may choose the element 1 arbitrarily (this amounts to recoordinatizing in the generalized polygon Ω). Hence, with the above notation, if φ^* belongs to a transitive normal nilpotent subgroup U^* of B^*, then we may assume that φ^* fixes 0 and 1 (and of course also ∞), because we may without loss of generality assume that the center Z^* of U^* contains the element $x \mapsto x+1$, as before. Now let a be an arbitrary element of the structure, and consider the field $\mathbb{F}$ generated by $\mathbb{K}$ and a. Then, together with ∞, this defines a Tits subsystem, implying that, for all $b \in \mathbb{F}$, the mapping $x \mapsto xb^2$ in $\mathbb{F}$ is induced by an element of the big Tits system. The arguments in the previous subsection now imply easily that φ^* fixes $\mathbb{F}$ pointwise, and hence a. But a was arbitrary. So φ^* is the identity and we conclude that $U^* = U_+^*$.

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3.4 Quadratic form sets. In this case, the Tits system (G^*, B^*, N^*) of rank 1 contains an orthogonal group defined by a quadratic form of Witt index 1 (over some field $\mathbb{K}$), and acts faithfully on the set of corresponding singular 1-spaces. We may identify w_0^* and w_∞^* with two arbitrary singular 1-spaces. This defines the root group $U_\infty^* \trianglelefteq B^*$ fixing w_∞^* and acting sharply transitively on the other singular 1-spaces. We denote by U^* a transitive normal nilpotent subgroup of B^* and we assume that $\varphi^* \in U^*$ fixes both w_0^* and w_∞^*. As before, the center of U^* contains an element of U_∞^* mapping w_0^* to some 1-space w^*. It follows that φ^* fixes w^*. Now consider any singular 1-space x^*. Then the four 1-spaces w_0^*, w_∞^*, w^* and x^* generate either a 3-space or a 4-space and we consider the restriction of the Tits system to that space. In case of a 3-space, the Tits system corresponds to the one defined by the field $\mathbb{K}$, and hence by our results above, φ^* fixes x^*. In case of a 4-space the corresponding quadratic form now defines a field extension of $\mathbb{K}$ and hence our results above again yield that φ^* fixes x^*. Note that the field extension could be inseparable (in characteristic 2), but this does not make a difference for the arguments. We conclude that φ^* must be the identity; hence $U^* = U_\infty^*$.

3.5 The cases $n = 3,6$. Suppose that Ω is a generalized n-gon with $n \in \{3, 6\}$ corresponding to a split Tits system (G, B, N) of rank 2, with some transitive normal nilpotent subgroup U of B. Recall that we may assume that $U_+ \leqslant U$. If $U \neq U_+$, then there is some non-trivial collineation $\varphi \in U$ fixing the standard apartment Σ pointwise. Let v^* be as before. If the corresponding (split) Tits system of rank 1 is defined by one of the structures dealt with in the previous three subsections, then φ fixes automatically all elements of Ω adjacent to v^*. If $n = 3, 6$, then clearly this is true for all vertices v^* in Σ (see [12, (17.2), (17.5)]); hence φ is the identity. Consequently $U = U_+$.

Remark. At this point we could also give an alternative proof for the case $n = 8$. Indeed, if $n = 8$, then for one bipartition class of the vertices v^*, the corresponding rank 1 Tits system is defined by a field (see [12, (17.7)]), and hence the fixed elements of Ω under φ form, up to duality, a thick ideal suboctagon (for terminology, see [13]). By [13, Proposition 5.9.13] (originally due to Joswig and the last author [3]), φ is the identity.

3.6 The case $n = 4$ concluded. It remains to consider the case $n = 4$. In this case, we put the set of vertices of Σ equal to $\{x_0, x_1, \ldots, x_7\}$, with subscripts modulo 8, and such that x_i and x_{i+1} are adjacent for all i. To fix ideas, we may think of x_0 as a point, and then x_1 is a line of Ω. We put $C = \{x_0, x_1\}$. By the classification of Moufang quadrangles (see [12]) we may suppose that the rank 1 Tits systems related to x_{2i+1} (for any integer i) are commutative. Also by the same classification result, we may assume that this rank 1 group corresponds either to a skew-field (Moufang quadrangles of *involution type*, of *quadratic form type*, and of *pseudo-quadratic form type*), or to a quadratic form of Witt index 1 (Moufang quadrangles of *exceptional types* E_6, E_7, E_8, F_4), or to an indifferent set in characteristic 2 (*indifferent* of *mixed* Moufang quadrangles). Hence, if $U \neq U_+$, and if φ is a non-trivial element of U fixing Σ pointwise, then φ fixes all points incident with one of x_{2i+1} (see Subsection 3.1, where

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x^* plays the role of x_{2i+1}). We now prove that φ is necessarily the identity, showing that $U = U_+$. Henceforth we assume $\varphi \neq 1$ and we seek a contradiction. We first introduce some (standard) notation. For three vertices a, b, c such that $a \sim b \sim c \neq a$, the root group fixing all elements incident with a, b, c is denoted by $G^{[1]}_{a,b,c}$. The group of collineations that fixes all elements at distance 2 from a certain vertex a is denoted by $G^{[2]}_a$. Every member of $G^{[2]}_a$ is called a *central elation* (with center a). The conjugate of U under a collineation that maps the chamber C to another chamber C' is denoted by $U[C']$ (and we have obviously $U = U[C]$). We denote by

$$\{1\} \trianglelefteq Z_1(U) \trianglelefteq Z_2(U) \trianglelefteq \cdots \trianglelefteq Z_l(U) = U$$

the ascending central series of U (where U has class l). We establish the Main Result in a number of small steps.

Step 1. *For every chamber* $C' = \{x_i, x_{i+1}\}$, *we have* $\varphi \in U[C']$. Indeed, let g be a nontrivial elation in $G^{[1]}_{x_1,x_2,x_3}$ which commutes with φ (g exists as

$$[\varphi, G^{[1]}_{x_1,x_2,x_3}] \leqslant G^{[1]}_{x_1,x_2,x_3} \leqslant U$$

and as U is nilpotent). Hence φ fixes x_7^g. Now let $h \in G^{[1]}_{x_5,x_6,x_7}$ be such that $x_1^h = x_7^g$. Since φ fixes x_1^h, we have $[\varphi, h] = 1$, and hence $[\varphi, hg^{-1}] = 1$. Consequently $\varphi = \varphi^{hg^{-1}} \in U^{hg^{-1}} = U[\{x_0, x_7\}]$. A similar argument now shows that $\varphi \in U[\{x_6, x_7\}]$. Continuing like this, the assertion follows.

Step 2. *Suppose that* $y \sim x_2$ *is not fixed by* φ, *and let* $u \in G^{[1]}_{x_3,x_4,x_5}$ *be such that* $y^u = y^\varphi$. *Then* $u \notin G^{[2]}_{x_4}$. Under the stated assumptions, choose $g \in G^{[1]}_{x_4,x_5,x_6} \setminus \{1\}$ arbitrarily. Then let $v \in G^{[1]}_{x_1,x_2,x_3}$ be such that $y^{\varphi g^{-1} v}$ is fixed under φ (this can be accomplished by putting $y^{\varphi g^{-1} v}$ equal to the unique vertex adjacent to $x_2^{\varphi g^{-1} v}$ at distance 2 from x_7). Then $\alpha := [\varphi^{-1}, g^v] \in G^{[1]}_{x_3,x_4,x_5}$. But evaluating y^α, we see that $y^\alpha = y^u$, and hence $\alpha = u$. So α is not the identity and hence φ cannot fix x_5^v (if it did, then α would fix all elements incident with x_6^v, a contradiction). This now implies that u cannot fix all elements incident with x_5^v, and so u is not a central elation.

Step 3. *Let* y *be an arbitrary vertex adjacent to* x_0 *but different from* x_1. *Let* $u \in Z_i(U)$ *be a non-central elation in* $G^{[1]}_{y,x_0,x_1}$ *with* i *minimal. Then* $[u, \varphi] = 1$. Indeed, it is clear that $[\varphi, u]$ is a central elation by minimality of i, and that $(x_3^u)^{[u,\varphi]} = (x_3^u)^\varphi$. If φ fixes x_3^u, then so does $[u, \varphi]$, and hence it is trivial. Otherwise, we apply Step 2 to obtain an immediate contradiction (noting that x_3^u is adjacent to x_2).

We can now finish the proof of the Main Result for $n = 4$. Let $u' \in G^{[1]}_{y',x_0,x_1}$ be non-central and contained in $Z_i(U)$, with i minimal, and with y' some line through x_0 different from x_1. We may assume that u' does not fix all points on x_7. Then $[u', \varphi] = 1$ by Step 3. We claim that we can re-choose y' in such a way that it is not fixed under φ. i.e., we claim that there exist $y \sim x_0$ and a non-central elation $u \in G^{[1]}_{y',x_0,x_1} \cap Z_i(U)$ such that $y^\varphi \neq y$. Indeed, suppose $z^\varphi = z$ for all $z \sim x_0$ which are fixed pointwise under

u'. Now let $y \sim x_0$ with $y^\varphi \neq y$ and let $v \in G^{[1]}_{x_1,x_2,x_3}$ with $(y')^v = y$. Note that $u' \notin G^{[1]}_y$, and hence $[u', v^{-1}] \notin G^{[1]}_{y'}$ and in particular $[u', v^{-1}] \neq 1$. Now $u = u'^v \in G^{[1]}_{y,x_0,x_1}$ is an elation belonging to $Z_i(U)$. It remains to show that u is non-central. Since u and u' have the same action on the set of vertices adjacent to x_2, but $u \neq u'$, they cannot both be elations in the same root group. Hence u does not fix every point on y', which is fixed under φ (and y' will play the role of x_7 below). The claim is proved. So we assume that $u \in G^{[1]}_{y,x_0,x_1}$ and $y^\varphi \neq y$. Let $y \sim y_2 \sim y_3 \sim x_4$, and let $w \in G^{[1]}_{y,y_2,y_3}$ be such that $x_1^w = x_7$. Then $u^w \in Z_i(U[x_7, x_0])$, and hence $[u^w, \varphi] = 1$ (using Steps 1 and 3). But then also $[[u, w], \varphi] = 1$. Notice that $[u, w] \in G^{[1]}_{y_2,y,x_0} \backslash \{1\}$, because the action on the points incident with x_1 is non-trivial. Since $y^\varphi \neq y$, it is hence impossible that $[[u, w], \varphi] = 1$. This contradiction proves the Main Result for $n = 4$.

4 Proof of the Corollary

For BN-pairs of rank 2, the corollary is contained in the Main Result and [6], [7]. So we now assume that (G, B, N) is an irreducible spherical BN-pair of rank at least 3. Let Ω be the corresponding building, let Σ be the apartment fixed by N and let C be the chamber fixed by B. Further, let Σ^+ be a half apartment in the apartment Σ containing C. Let φ be an arbitrary element of U fixing all chambers contained in Σ^+. Let P be a panel in the interior of Σ^+ and let R be a flag of corank 2 contained in P. Consider the stabilizer B_R of R in B, and the stabilizer U_R of R in U. Clearly $U_R \trianglelefteq B_R$ and U_R is nilpotent. Let C_R be the unique chamber containing R nearest to C (the *projection* of C onto R in building language), and let C_R^* be the chamber in Σ containing R opposite C_R in the residue of R. Let C_R' be any chamber containing R opposite C_R in the residue of R. There is an apartment Σ' containing C_R' and C (by the very definition of a building), and hence there exists $u \in U$ mapping Σ to Σ'. Clearly u fixes R and maps C_R^* onto C_R'. Hence U_R is transitive. So, if we denote by K the kernel of the action of G_R on the residue of R, then $U_R K/K$ is a splitting of the rank 2 BN-pair $(G_R/K, B_R K/K, N_R K/K)$ (with obvious notation). It follows from our Main Result that φ fixes all chambers containing P, i.e., φ is a root elation by definition. Hence U coincides with the standard unipotent subgroup U_+ and the Corollary is proved.

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