Bislim Point and Line Transitive Geometries of Gonality 3: Construction and Classification

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Abstract

We consider point-line geometries having three points on every line, having three lines through every point (*bislim geometries*), and containing triangles. We give some (new) constructions and we prove that every point and line but not flag transitive such geometry

$(x_1, x_2)/c$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	total
0	-	4	6	4	16	24	6	24	36	120
1	4	16	24	16	56	72	24	72	76	360
2	6	20	24	20	52	44	24	44	20	254
3	4	8	4	8	8	0	4	0	0	36
4	1	0	0	0	0	0	0	0	0	1

Table 1: Counting of configurations in case of zero lines of three

1 Introduction

Let $\Gamma(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a connected point and line transitive bislim geometry of gonality 3. We suppose that Γ has a point and line transitive collineation group G which is not flag transitive. Let x be any point of Γ and L any line incident with x. Let x_1, x_2 be the two other points incident with L, and let L_1, L_2 be the two other lines incident with x. The points on L_i, $i = 1, 2$, different from x will be denoted by y_i and z_i. The *(local) configuration* or *local structure of a point* x is the subgeometry Γ_x of Γ with point set $x \cup \Gamma_2(x)$ and line set the lines of Γ incident with 2 or 3 of these points. Remark that this subgeometry is not necessarily bislim. Denote the lines of Γ_x not through x by Γ_x^l.

We start our research by searching for all possible non-isomorphic configurations. Therefore we first count the number of configurations with zero, one, two, three and four lines having three points of $\Gamma_2(x)$.
For the first case, consider the point x_1. There are either no lines (1 possibility), either one (4 possibilities) or either two lines (6 possibilities) in x_1. Idem for the point x_2. Remark that the lines we are talking about here are that lines belonging to the configuration. Now there are either zero ($(11 \times 11) - 1 = 120$ possibilities), either one (360 possibilities), either two (254 possibilities), either three (36 possibilities) or either four (1 possibility) lines on the point set $\{y_1, z_1, y_2, z_2\}$. For a counting method see table 1 where $(x_1, x_2) = (a, b)$ means that there are a lines in x_1 and b lines in x_2 and c stands for the number of lines between the points y_1, z_1, y_2 and z_2. For the total of 771 configurations we determine the non-isomorphic ones. The results are shown in table 2 and 3 and in the appendix.

Next we consider the case where there is one line containing three points of $\Gamma_2(x)$. There are eight possibilities for this line. In table 4 we count the number of configurations for which $x_1y_1y_2$ is a line. Totally there are $115 \times 8 = 920$ different configurations with one line of three points on $\Gamma_2(x)$. Since we are looking for non-isomorphic configurations we focus on the 115 configurations for which $x_1y_1y_2$ is the line of three and look there for the non-isomorphic configurations. The results can be found in table 5 and table 6 and in the appendix.

The third case is the case where there are two lines containing three points of $\Gamma_2(x)$. There are 16 possibilities for those two lines. We count the number of configurations for

♯ lines	♯ confs	♯ non-isom. confs	fig. appendix
1	12	1	1
2	66	4	$2-5$
3	196	8	$6-13$
4	297	12	$14-25$
5	180	7	$26-32$
6	20	3	$33-35$

Table 2: Non-isomorphic configurations in case of zero lines of three

which $x_1y_1y_2$ and $x_2z_1z_2$ are lines and the number of configurations for which $x_1y_1y_2$ and $x_1z_1z_2$ are lines (see table 7 and table 8). In total there are $(4 \times 18) + (12 \times 20) = 312$ different configurations with two lines of three points on $\Gamma_2(x)$. Since we are looking for non-isomorphic configurations we focus on the 18 configurations for which $x_1y_1y_2$ and $x_2z_1z_2$ are the lines of three and on the 20 configurations for which $x_1y_1y_2$ and $x_1z_1z_2$ are the lines of three. The non-isomorphic configurations amongst these can be found in table 9, table 10 and table 11 and in the appendix.

Next we consider the case where there are three lines containing three points of $\Gamma_2(x)$. There are $(8 \times 3 \times 2)/6 = 8$ different ways for choosing those three lines which are all isomorphic to each other. It is easily seen that this case gives rise to two non-isomorphic configurations (see table 12, table 13 and table 14 and appendix). (In total there are $(4 \times 8) = 32$ different configurations with three lines of three points on $\Gamma_2(x)$.)

Finally, there is only one configuration for which there are four lines containing three points of $\Gamma_2(x)$, giving the Fano geometry.

conf. appendix	♯ confs isomorphic to this conf.
1	12
2	12
3	24
4	6
5	24
	66
6	12
7	48
8	48
9	24
10	8
11	24
12	24
13	8
	196
14	3
15	48
16	6
17	12
18	24
19	12
20	24
21	24
22	48
23	24
24	24
25	48
	297
26	24
27	24
28	24
29	24
30	24
31	48
32	12
	180
33	12
34	4
35	4
	20

Table 3: Number of configurations isomorphic to a given configuration

$(x_1, x_2)/c$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	total
0	1	4	6	2	8	12	33
1	3	10	12	6	16	14	61
2	3	6	3	4	4	0	20
3	1	0	0	0	0	0	1

Table 4: Counting of configurations in case of $x_1y_1y_2$ only line of three

$\sharp$ lines	$\sharp$ confs	$\sharp$ non-isom. confs	fig. appendix
0	1	1	36
1	9	2	$37-38$
2	33	7	$39-45$
3	51	11	$46-56$
4	21	4	$57-60$

Table 5: Non-isomorphic configurations in case of $x_1y_1y_2$ only line of three

conf. appendix	♯ confs isomorphic to this conf.
36	1
37	6
38	3
	9
39	6
40	3
41	6
42	3
43	3
44	6
45	6
	33
46	3
47	6
48	6
49	6
50	6
51	6
52	6
53	6
54	2
55	3
56	1
	51
57	6
58	6
59	6
60	3
	21

Table 6: Number of configurations isomorphic to a given configuration

$(x_1, x_2)/c$	$(0,0)$	$(0,1)$	$(1,0)$	(1,1)	total
0	1	2	2	4	9
1	2	2	2	2	8
2	1	0	0	0	1

Table 7: Counting of configurations in case of $x_1y_1y_2$ and $x_2z_1z_2$ lines of three

$(x_1, x_2)/c$	$(0,0)$	$(0,1)$	$(0,2)$	total
0	1	4	6	11
1	2	4	2	8
2	1	0	0	1

Table 8: Counting of configurations in case of $x_1y_1y_2$ and $x_1z_1z_2$ lines of three

$\sharp$ lines	$\sharp$ confs	$\sharp$ non-isom. confs	fig. appendix
0	1	1	61
1	6	1	63
2	9	2	$66 - 67$
3	2	1	73

Table 9: Non-isomorphic configurations in case of $x_1y_1y_2$ and $x_2z_1z_2$ lines of three

$\sharp$ lines	$\sharp$ confs	$\sharp$ non-isom. confs	fig. appendix
0	1	1	62
1	6	2	$64 - 65$
2	11	5	$68 - 72$
3	2	1	74

Table 10: Non-isomorphic configurations in case of $x_1y_1y_2$ and $x_1z_1z_2$ lines of three

conf. appendix	$\sharp$ confs isomorphic to this conf.
61	1
62	1
63	6
64	4
65	2
	6
66	3
67	6
	9
68	2
69	2
70	2
71	4
72	1
	11
73	2
74	2

Table 11: Number of configurations isomorphic to a given configuration

$(x_1, x_2)/c$	$(0,0)$	$(0,1)$	total
0	1	2	3
1	1	0	1

Table 12: Counting of configurations in case of $x_1y_1y_2$, $x_1z_1z_2$ and $x_2y_1z_2$ lines of three

♯ lines	♯ confs	♯ non-isom. confs	fig. appendix
0	1	1	75
1	3	1	76

Table 13: Non-isomorphic configurations in case of $x_1y_1y_2$, $x_1z_1z_2$ and $x_2y_1z_2$ lines of three

conf. appendix	♯ confs isomorphic to this conf.
75	1
76	3

Table 14: Number of configurations isomorphic to a given configuration

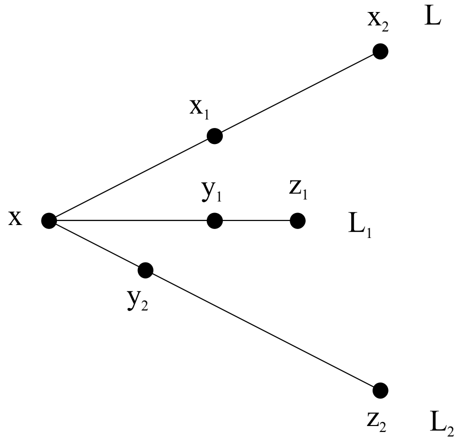

Figure 1: Nomination of points and lines

2 'Wrong' configurations

In this section we give an overview of all configurations which do not give rise to a bislim geometry with a point and line but not flag transitive collineation group. The configurations mentioned are found in the appendix. To make the following description easier we make some agreement about the assignment of names to the points and lines in the configurations 1 up till 76 as represented in the appendix. The top, middle and bottom line through x is called L, L_1 and L_2 respectively. Going from left to right the points on L, L_1 and L_2 are called x_1 and x_2, y_1 and z_1, y_2 and z_2 respectively (see figure 1).

Configurations 1 up to 35 do not contain a line sharing one point with each line through a point x of the geometry. In other words Γ_x^l has no lines with three points. Hence, because of point transitivity of the geometry there can be no geometry Γ with local structure $2, 6, 7, 9, 14, 15, 16, 17, 18, 22, 25, 26, 27, 28, 31, 32$ or 33.
Configuration 3 has two lines x_1y_1 and x_1z_2 in Γ_x^l. Because of line transitivity and no flag transitivity there is a collineation ϕ mapping L onto L_2 and hence taking x onto y_2. Then ϕ maps x_1 onto z_2 giving rise to a contradiction in the point z_2.
The set Γ_x^l of configuration 4 contains two lines x_1y_1 and x_2z_1. Point transitivity includes that x_2a is a line, with a the third point on the line x_1y_1. This gives a contradiction in the point x_2.
The lines of configuration 8 not through x are given by x_1y_1, x_2y_2 and x_2z_1. The collineation mapping the line L onto L_2 takes x onto y_2, x_1 onto x and x_2 onto z_2 giving a contradiction in z_2.
Configuration 10 has three lines x_1y_1, x_1y_2 and y_1y_2 in Γ_x^l. Mapping x onto x_2 gives that x_2a, x_2b and ab are lines, with a and b the third points on x_1y_1 and x_1y_2. This gives rise

to a contradiction in the point x_1.
For configuration 11 Γ_x^l consists of the three lines x_1y_1, x_1z_2 and z_1z_2. There can be no collineation mapping L onto L_1 which is in contradiction with the line transitivity.
The set Γ_x^l of configuration 12 contains the three lines x_1y_1, x_1y_2 and z_1z_2. There can be no collineation mapping the line L onto L_2, hence configuration 12 can not occur.
For configuration 19 x_1y_1, x_1y_2, x_2z_1 and x_2z_2 are the only lines in Γ_x^l. Since there can be no collineation mapping the line L onto L_2 configuration 19 can not occur.
Configuration 20 has four lines x_1y_1, y_1y_2, y_2x_2 and x_2z_1 in Γ_x^l. The collineation mapping L_2 onto L takes x onto x_2, z_2 onto x_1 and y_2 onto x, inducing that z_1a and y_2b are lines, with a the third point on x_2y_2 and b the third point on x_2z_1. Since there are now four lines through y_2 it follows that y_1y_2b is a line. Looking at the local structure in the point y_1 it is easily seen that x_1b is a line leading to a wrong local structure in x_2.
The lines of configuration 21 not through x are given by x_1y_1, y_1y_2, x_2z_1 and x_2z_2. Looking at the local structure in x_1 it is easily seen that y_1a is a line, with a a point on the third line through x_1. This includes that y_1y_2a is a line, leading to a wrong local structure in y_1.
The set Γ_x^l of configuration 23 contains the four lines x_1y_1, x_1y_2, y_1y_2 and x_2z_1. Looking at the local structure in x_2 it is easily seen that either x_1y_1a or either x_1y_2a is a line with a the third point on the line x_2z_1. In the first case x_1y_2b and ab are lines with b a point on the third line through x_2, leading to a wrong local structure in x_1. Similarly, in the second case x_1y_1b and ab are lines leading to a wrong local structure in x_1.
Configuration 29 has five lines x_1y_1, y_1y_2, x_1y_2, x_2z_1 and x_2z_2 in Γ_x^l. Looking again at the local structure in x_2 it is easily seen that either x_1y_1a or either x_1y_2a is a line with a the third point on the line x_2z_2. In the first case x_1y_2b and ab are lines with b the third point on x_2z_1, leading to a wrong local structure in x_1. Similarly, in the second case x_1y_1b and ab are lines leading to a wrong local structure in x_1.
The set Γ_x^l of configuration 30 consists of the five lines x_1y_1, y_1y_2, y_2x_2, x_2z_1 and z_1z_2. Considering the point x_1 we see that y_1y_2a, x_2z_1a, x_2y_2b and bc are lines with a and c the points on the third line through x_1 and b the third point on x_1y_1. Looking at the local structure in a it follows that z_1z_2c is a line. In the point z_1 it is impossible to obtain local structure 30.
The six lines of configuration 34 not through x are given by x_1y_1, y_1y_2, x_1y_2, x_2z_1, z_1z_2 and x_2z_2. Looking at the point x_2 we get that either x_1y_1b, x_1y_2a and ab or either x_1y_1a, x_1y_2b and ab are lines with a the third point on x_2z_1 and b the third point on x_2z_2. In the first case we get a contradiction considering y_2. In the second case the local structure in y_1 gives rise to the lines abc and z_1z_2c with c the third point on the line y_1y_2. We obtain the **Desargues geometry**.
Configuration 35 has six lines x_1y_1, y_1y_2, y_2x_2, x_2z_1, z_1z_2 and z_2x_1 in Γ_x^l. Looking at the local structure in the point x_1 it is easily seen that y_1y_2a, x_2z_1a, x_2y_2b and z_1z_2b are lines with a the third point on the line x_1z_2 and b the third point on x_1y_1. We obtain the **Pappus geometry**.

Considering the local structure in the point x_1 it is easily seen that we get a contradiction for configurations $36, 38, 39, 42$ and 44.
For configurations $37, 41, 43$ and 60 a contradiction arises in the point y_2.
Since we get a contradiction looking at the point y_1 in configurations $40, 45, 47, 55, 56$ and 59, those configurations can not occur.
Since there is no collineation mapping the line L onto the line L_2, configurations 46, 48, 49, 50, 52, 53 and 54 can not occur.
The lines not through x of configuration 57 are $x_1y_1, x_1z_1y_2, y_1x_2, x_2y_2$ and z_1z_2. Considering the point x_1 it is easily seen that z_1z_2a is a line with a the third point on the line x_1y_1. This gives two lines intersecting the three lines in z_1, a contradiction.
For configuration 58 Γ_x^l consists of the lines $x_1y_1y_2, x_1z_1, z_1x_2, x_2z_2$ and z_2y_1. Considering the point x_1 it follows that y_2a and x_2z_2a are lines with a the third point on the line x_1z_1. Looking now at the point x_2 we see that y_1z_2b and aby_2 are lines with b the third point on x_2z_1. We obtain a **bislim geometry on 9 points and 9 lines**.

Since there is no collineation mapping the line L onto the line L_2, configuration 72 can not occur.
Considering the local structure in the point x_1 it is easily seen that we get a contradiction for configurations 61, 63, 66, 68, 69, 70 and 74.
Since we get a contradiction looking at the point y_1 in configurations 62 and 67, those configurations can not occur.
For configurations 64, 65 and 71 a contradiction arises in the point y_2.
The lines not through x for configuration 73 are given by $x_1y_1y_2$, $x_2z_1z_2$, x_1z_1, y_1z_2 and y_2x_2. Considering the point x_1 it follows that x_2y_2a and y_1z_2a are lines with a the third point on the line x_1z_1. We get the **Möbius-Kantor geometry**.

Considering the point x_1 in the configurations 75 and 76 we easily see that both configurations can not give rise to a bislim point and line transitive geometry.

Configuration 77 is the **Fano geometry**.

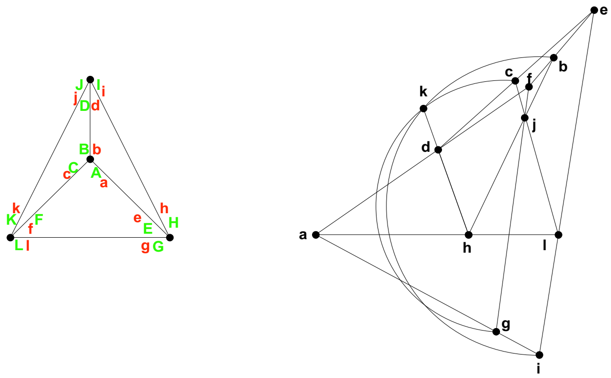

Figure 2: Graph of girth 3

3 Configuration 1

There is only one line in Γ_x^l, the line x_1y_1. It is easy to see that every point of the geometry Γ belongs to a unique triangle. For the point x this triangle is given by (x, x_1, y_1). This is also the unique triangle in the points x_1 and y_1. We consider a graph $\mathcal{G}_\Delta$ with vertex set the set of triangles of Γ. A vertex $v = (p_1, p_2, p_3)$ is adjacent to a vertex $w = (q_1, q_2, q_3)$ if $|\{p_1, p_2, p_3, q_1, q_2, q_3\}| = 6$ and there exist i, j and k in $\{1, 2, 3\}$ such that $p_i \mathrm{I} q_j q_k$ or $q_i \mathrm{I} p_j p_k$. Because of point and line transitivity of the geometry Γ, the graph $\mathcal{G}_\Delta$ is either 3- or either 6-regular. It is easy to see that a geometry with local structure 1 and having 9 or 12 points does not exist.

In the case of a 3-regular graph, point transitivity induces that the graph is edge-transitive (even transitive on the ordered edges).

Given a 3-regular graph $\mathcal{G}(V, E)$ admitting an automorphism group acting transitively on the set of ordered edges $(v, w) \in V \times V$ with $\{v, w\} \in E$. We define a geometry Γ in the following way. To every ordered edge (v, w) of $\mathcal{G}$ we attach a point and a line: the first half of the edge (v, w) is a line, the second half a point. We note them by $(v, w)^1$ and $(v, w)^2$. Given that v is a vertex of the graph adjacent to the vertices w_1, w_2 and w_3. Then, the line $(v, w_i)^1$ with $i \in \{1, 2, 3\}$ is incident with the points $(v, w_j)^2$ and $(v, w_k)^2$ where $\{i, j, k\} = \{1, 2, 3\}$ and with the point $(w_i, v)^2$. Analogously, the point $(v, w_i)^2$ with $i \in \{1, 2, 3\}$ is incident with the lines $(v, w_j)^1$ and $(v, w_k)^1$ where $\{i, j, k\} = \{1, 2, 3\}$ and with the line $(w_i, v)^1$. It is easily seen that the geometry Γ is bislim, without digons but containing triangles. Indeed, $((v, w_1)^2, (v, w_2)^2, (v, w_3)^2)$ is a triangle of Γ with sides $(v, w_1)^1$, $(v, w_2)^1$ and $(v, w_3)^1$. The geometry arising in that way from the 3-valent graph acting transitively on the ordered edges and of girth 3 has a wrong local structure, hence the girth of the 3-valent graphs is bigger than 3 (see fig. 2). It is easily seen that the geometries constructed from the graphs in the above mentioned way have local structure

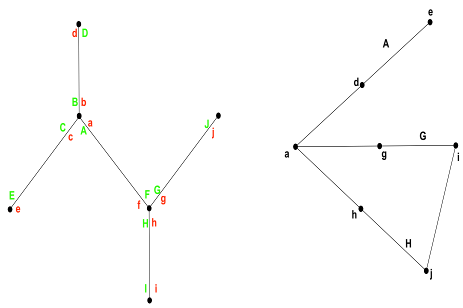

Figure 3: 3-regular graph of girth > 3

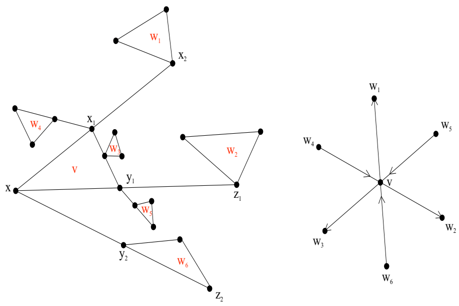

Figure 4: 6-regular graph

1 (see fig. 3). Transitivity on the ordered edges of the graph induces point and line transitivity of the geometry but no flag transitivity. This follows out of the construction method.

In the case of a 6-regular graph, it is clear that the graph $\mathcal{G}_\Delta$ is edge-transitive but not transitive on the ordered edges. Based on $\mathcal{G}_\Delta$ we define a new graph $\overrightarrow{\mathcal{G}_\Delta}$ with the same vertex and edge set as $\mathcal{G}_\Delta$ but the edges are given a direction: there is a directed edge (v, w) if a point of the triangle w is on a side of the triangle v. It is then easy to see that $\overrightarrow{\mathcal{G}_\Delta}$ is a 6-regular directed graph with 3 incoming edges and 3 outgoing edges in each vertex and admitting an automorphism group acting transitively on the set of directed edges. A vertex v hence has 2 orbits under the action of the stabilizer of that vertex. Suppose that v is a vertex of the graph and that $(v, w_1), (v, w_2), (v, w_3), (w_4, v), (w_5, v)$ and (w_6, v) are ordered or directed edges of the graph (see fig. 4). Because of point transitivity of the geometry there is a collineation g mapping x onto x_1. The point x_1 is then taken onto x (type 1) or onto y_1 (type 2). In the first case triangles v, w_1 and w_5 are fixed, while triangle w_2 and w_3 are mapped onto each other. The same holds for triangles w_4 and w_6. For the second case we have that w_1 is mapped onto w_3 onto w_2 onto w_1 and that w_5 is taken onto w_6 onto w_4 onto w_5. The triangle v is fixed. Remark that if G contains

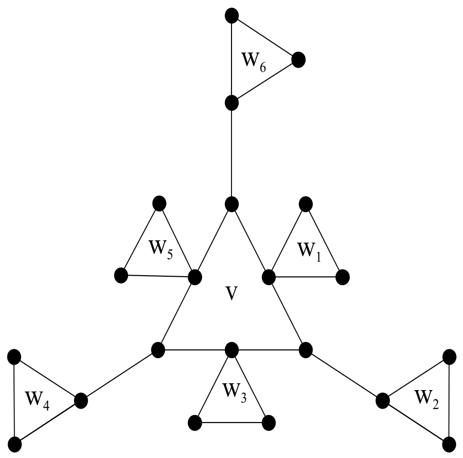

Figure 5:

a collineation of type 1 then also one of type 2. Indeed, there exists a collineation h in G mapping y_1 onto x. The point x is then taken onto x_1 or y_1. In the first case h is a collineation of type 2, in the second case gh is of type 2. If G contains a collineation of type 1 and hence also one of type 2 then the action of the stabilizer of the vertex v is the group $\mathcal{S}_3$ having two orbits on the set $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ and acting either trivial, either cyclically or either dihedrically onto both orbits. If G contains a collineation of type 2 but none of type 1, then the action of the stabilizer of the vertex v is the group C_3 having two orbits on the set $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ and acting trivial or cyclically onto both orbits. Because of point-transitivity we have the same situation in each vertex of the graph: either the stabilizer of any vertex acts as $\mathcal{S}_3$ onto its neighbors or either as C_3. In the first case we look wether or not a directed 3-cycle can occur in the graph associated with the geometry. Considering fig. 5 we see that we can distinguish two possibilities: (v, w_1, w_2, v) is a 3-cycle or (v, w_1, w_4, v) is a 3-cycle. First we consider the first possibility mentioned above: four different cases can occur (see fig. 6). It is clear that case (a) can not occur since the corresponding geometry has a wrong local structure. Since G contains a collineation mapping x and x_1 onto each other (type 1 see above) and hence fixing w_2 and w_5 and taking w_4 and w_6, resp. w_1 and w_3 onto each other, also case (b) and (d) can not occur. As mentioned before, there is also a collineation taking x onto x_1 onto y_1 onto x and hence taking w_4 onto w_6 onto w_2 onto w_4 and w_5 onto w_1 onto w_3 onto w_5. Applying those two collineations to w_1 and w_2 leads in case (c) to two lines sharing two points. Also for the second possibility we have four different cases to consider, for which it is easy to see that none of them leads to a contradiction using the same arguments as above (see fig. 7). But because of point transitivity, we have the same situation (either (a), either (b), either (c) or either (d)) in every triangle of the geometry. Based on this observation (and using the results of the previous case), the cases (a), (b) and (c) can

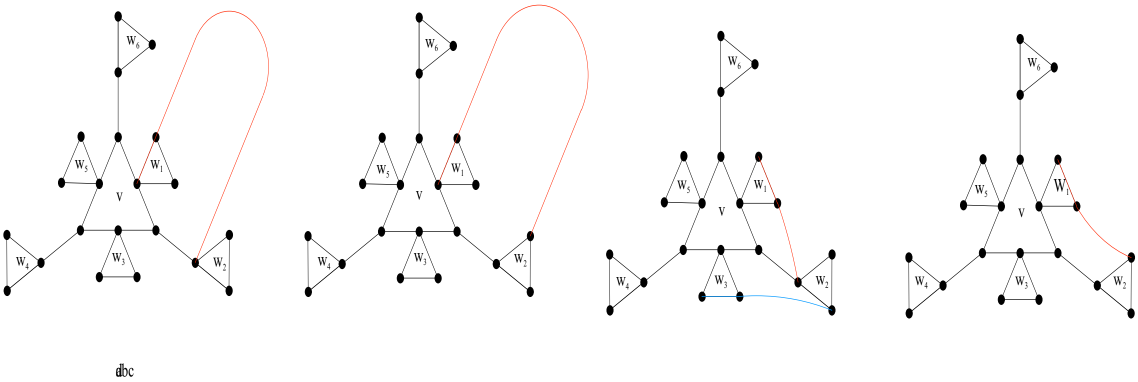

Figure 6: Case where (v, w_1, w_2, v) is a 3-cycle

not occur. We can conclude that if the geometry Γ gives rise to a directed 3-cycle in the associated graph then there are exactly three directed 3-cycles in every vertex of the graph.
Also in the second case where the action of G_v is given by C_3 we look wether or not a directed 3-cycle can occur in the graph associated with the geometry. Considering fig. 5 we see that we can distinguish two possibilities: (v, w_1, w_2, v) is a 3-cycle or (v, w_1, w_4, v) is a 3-cycle. First we consider the first possibility mentioned above: four different cases can occur (see fig. 8). It is clear that case (a) can not occur since the corresponding geometry has a wrong local structure. As mentioned before, there is a collineation taking x onto x_1 onto y_1 onto x and hence taking w_4 onto w_6 onto w_2 onto w_4 and w_5 onto w_1 onto w_3 onto w_5. Applying this collineation (and its inverse) leads to additional (blue) lines. To make the following description easier, we will call the seven possible types of 3-cycles type (1) up to type (7) as in fig. 9. First we focus on case (b). There are exactly three type (1) 3-cycles in v. Point transitivity includes that there are three type (1) 3-cycles in every triangle of the geometry. Triangle w_1 is adjacent to triangle v and w_2. The connection between triangle v and w_2 leads to a 3-cycle of type (2). Because of point transitivity and using the above mentioned collineations of G_v there are three 3-cycles of that type in triangle v (green lines). Now w_1 is adjacent to v, w_2 and w_6 and there is a type (5) connection between v and w_6. This leads to the three purple lines in fig. 10. This situation leads to nine directed 3-cycles in every vertex of the graph associated with the geometry. For case (c) there are exactly three type (2) 3-cycles in v. Point transitivity includes that there are three type (2) 3-cycles in every triangle of the geometry. Triangle w_1 is adjacent to triangle v and w_2. The connection between triangle v and w_2 leads to a 3-cycle of type (5). Because of point transitivity and using the above mentioned collineations of G_v there are three 3-cycles of that type in triangle v (green lines). Now w_1 is adjacent to v, w_2 and w_4 and there is a type (1) connection between v and w_4. This leads to the three purple lines in fig. 10. This situation leads to nine directed

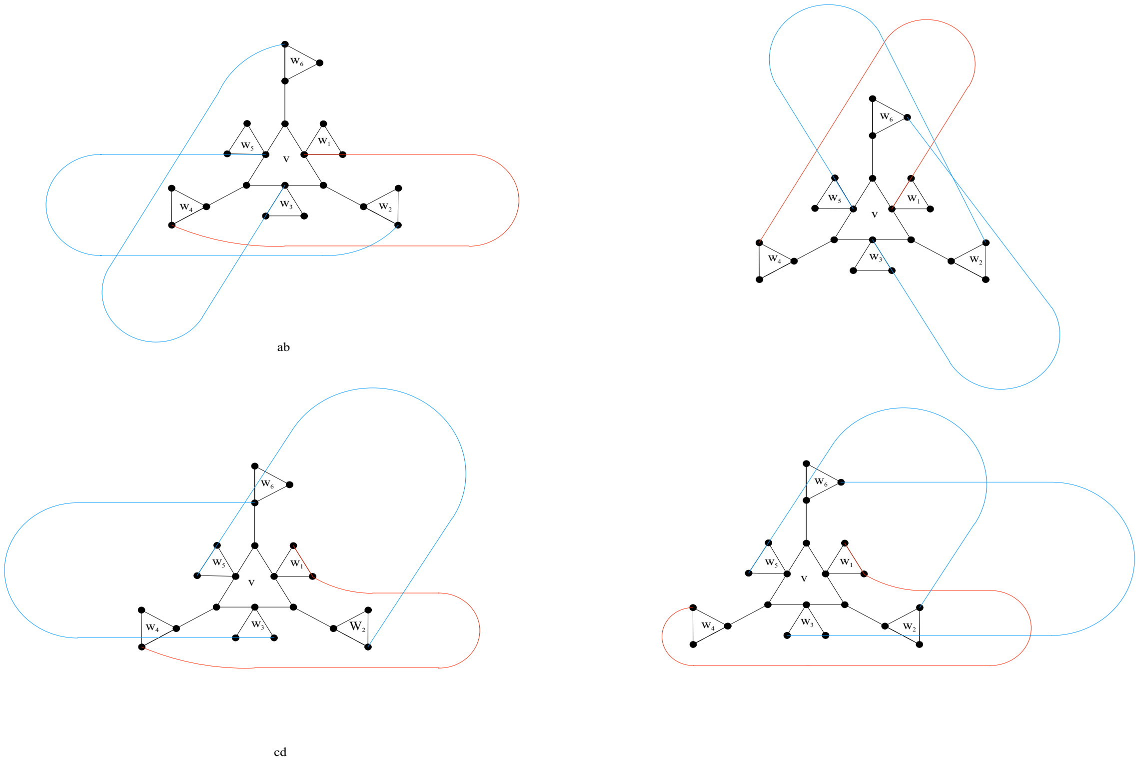

Figure 7: Case where (v, w_1, w_4, v) is a 3-cycle

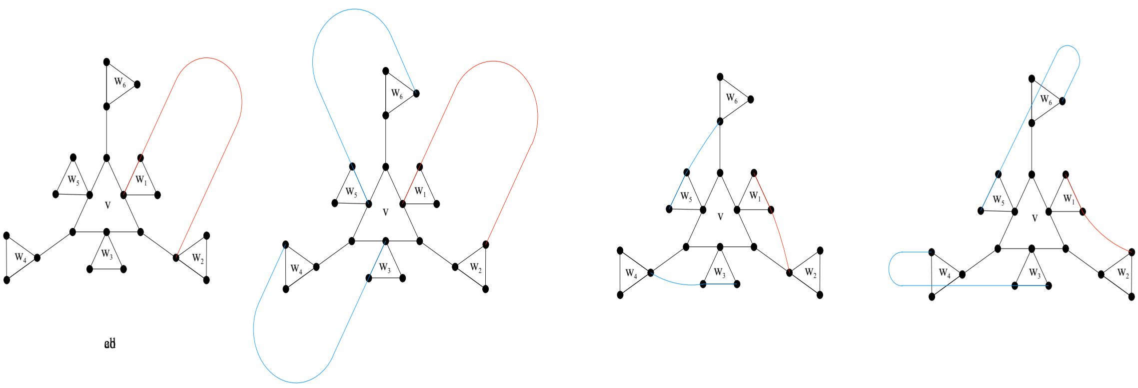

Figure 8: Case where (v, w_1, w_2, v) is a 3-cycle

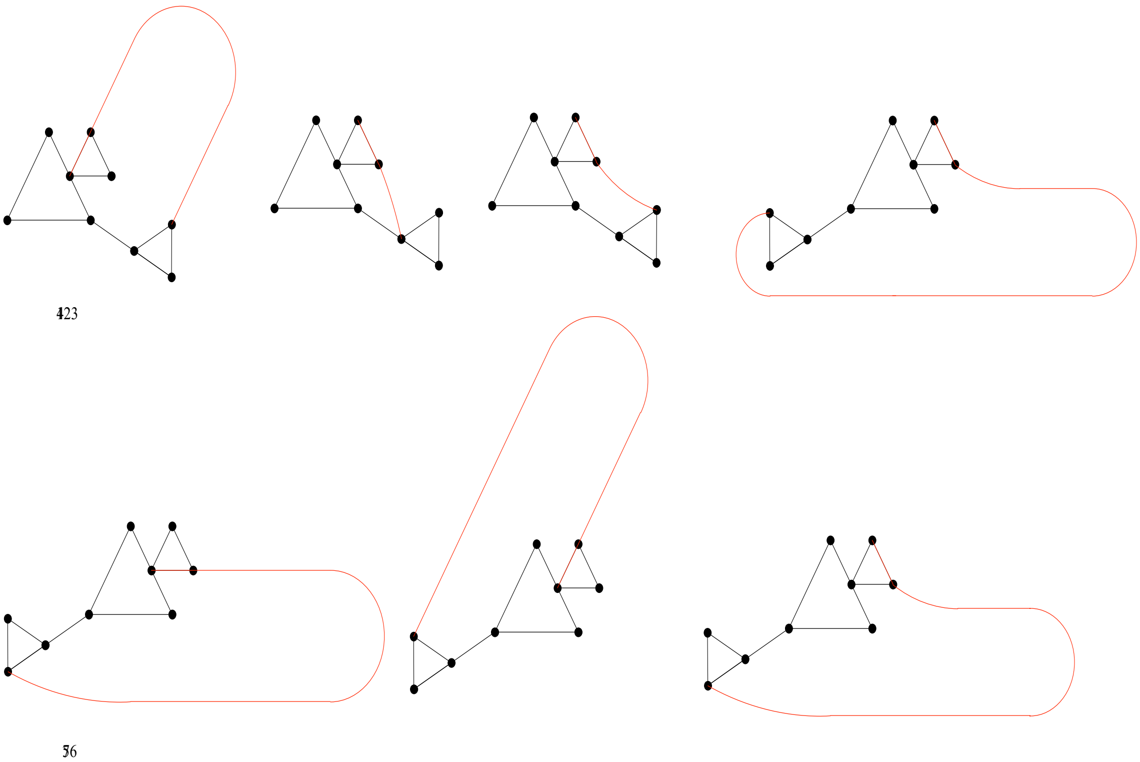

Figure 9: Types of 3-cycles

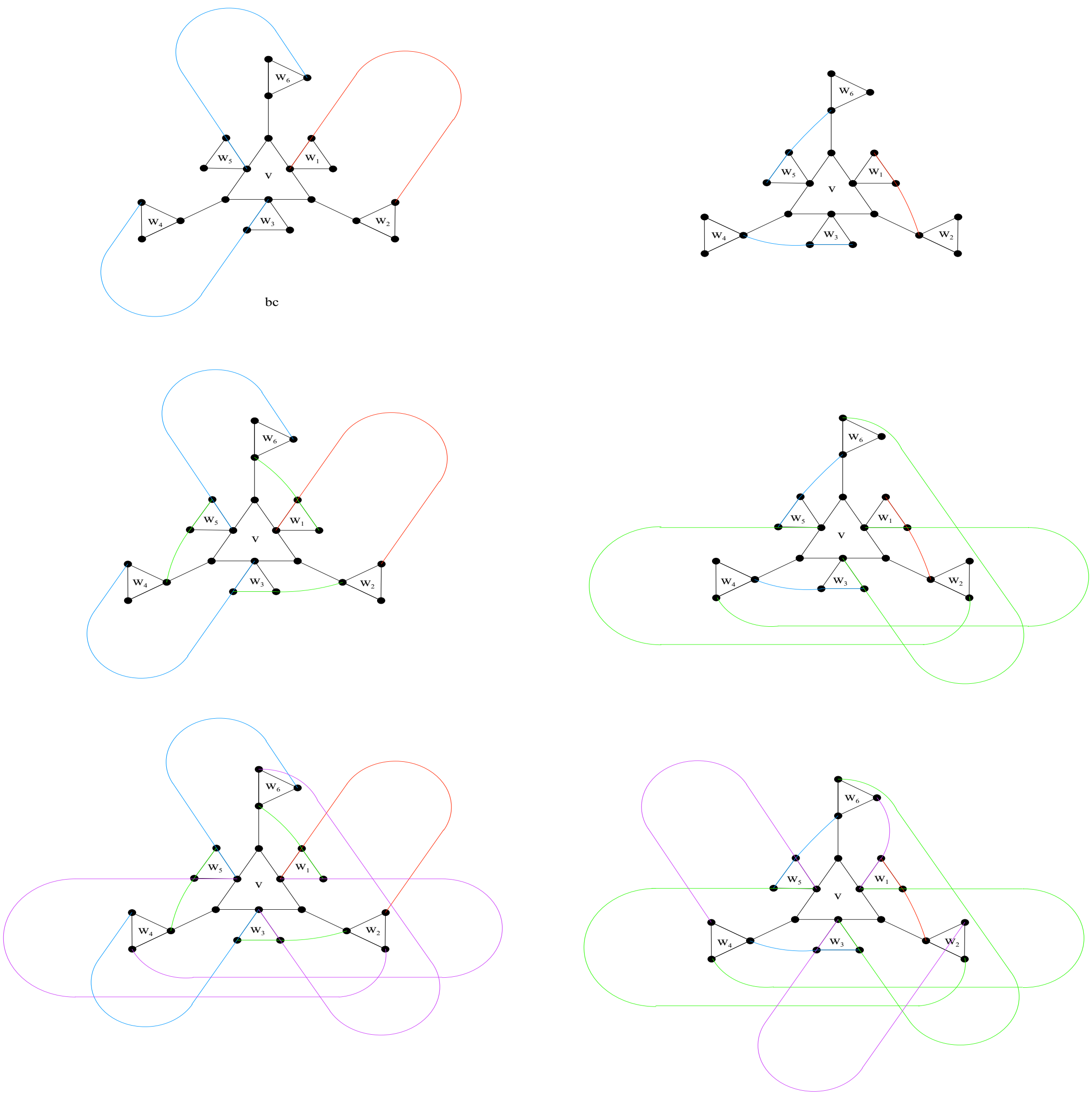

Figure 10: Cases (b) and (c)

3-cycles in every vertex of the graph associated with the geometry. It is easy to see that case (b) and (c) are isomorphic. For case (d) there are exactly three type (3) 3-cycles in v. Point transitivity includes that there are three type (3) 3-cycles in every triangle of the geometry. Triangle w_1 is adjacent to triangle v and w_2. The connection between triangle v and w_2 leads to a 3-cycle of type (7). Because of point transitivity and using the above mentioned collineations of G_v there are three 3-cycles of that type in triangle v, which is impossible. Also for the second possibility we have four different cases to consider (see fig. 7). First we focus on case (a). There are exactly three type (5) 3-cycles in v. Point transitivity includes that there are three type (5) 3-cycles in every triangle of the geometry. Triangle w_1 is adjacent to triangle v and w_4. The connection between triangle v and w_4 leads to a 3-cycle of type (1). Because of point transitivity and using the above mentioned collineations of G_v there are three 3-cycles of that type in triangle v (green lines). Now w_1 is adjacent to v, w_4 and w_6 and there is a type (2) connection between v and w_6. This leads to the three purple lines in fig. 11. This situation leads to nine directed 3-cycles in every vertex of the graph associated with the geometry and is again isomorphic to the two previous cases mentioned above. For case (b) there are exactly three type (6) 3-cycles in v. Point transitivity includes that there are three type (6) 3-cycles in every triangle of the geometry. Triangle w_1 is adjacent to triangle v and w_4. The connection between triangle v and w_4 leads to a 3-cycle of type (3). Because of point transitivity and using the above mentioned collineations of G_v there are three 3-cycles of that type in triangle v which is impossible. For case (c) there are exactly three type (7) 3-cycles in v. Point transitivity includes that there are three type (7) 3-cycles in every triangle of the geometry. Triangle w_1 is adjacent to triangle v and w_4. The connection between triangle v and w_4 leads to a 3-cycle of type (6). Because of point transitivity and using the above mentioned collineations of G_v there are three 3-cycles of that type in triangle v which is impossible. For case (d) there are exactly three type (4) 3-cycles in v. Point transitivity includes that there are three type (4) 3-cycles in every triangle of the geometry. Triangle w_1 is adjacent to triangle v and w_4. The connection between triangle v and w_4 leads to a 3-cycle of type (4). This situation leads to exactly three directed 3-cycles in every vertex of the graph associated with the geometry. We can conclude that if the geometry Γ gives rise to a directed 3-cycle in the associated graph then there are exactly three or nine directed 3-cycles in every vertex of the graph.

Given a 6-regular graph $\mathcal{G}$ admitting an automorphism group G which is vertex and edge transitive. We suppose that the action of the stabilizer G_v of a vertex v onto its neighbors has two orbits of length 3 and is given by either the symmetric group $\mathcal{S}_3$ or either the cyclic group C_3. We also assume that an element $g \in G_v$ acts either trivial, either cyclical or either dihedrical on both orbits.

First, we look at the case where the action of the stabilizer of a vertex v onto its neighbors is given by the symmetric group $\mathcal{S}_3$. An element of order 2 in $\mathcal{S}_3$ (there are 3 such elements) fixes one vertex in each orbit. Those two vertices are called opposite. The

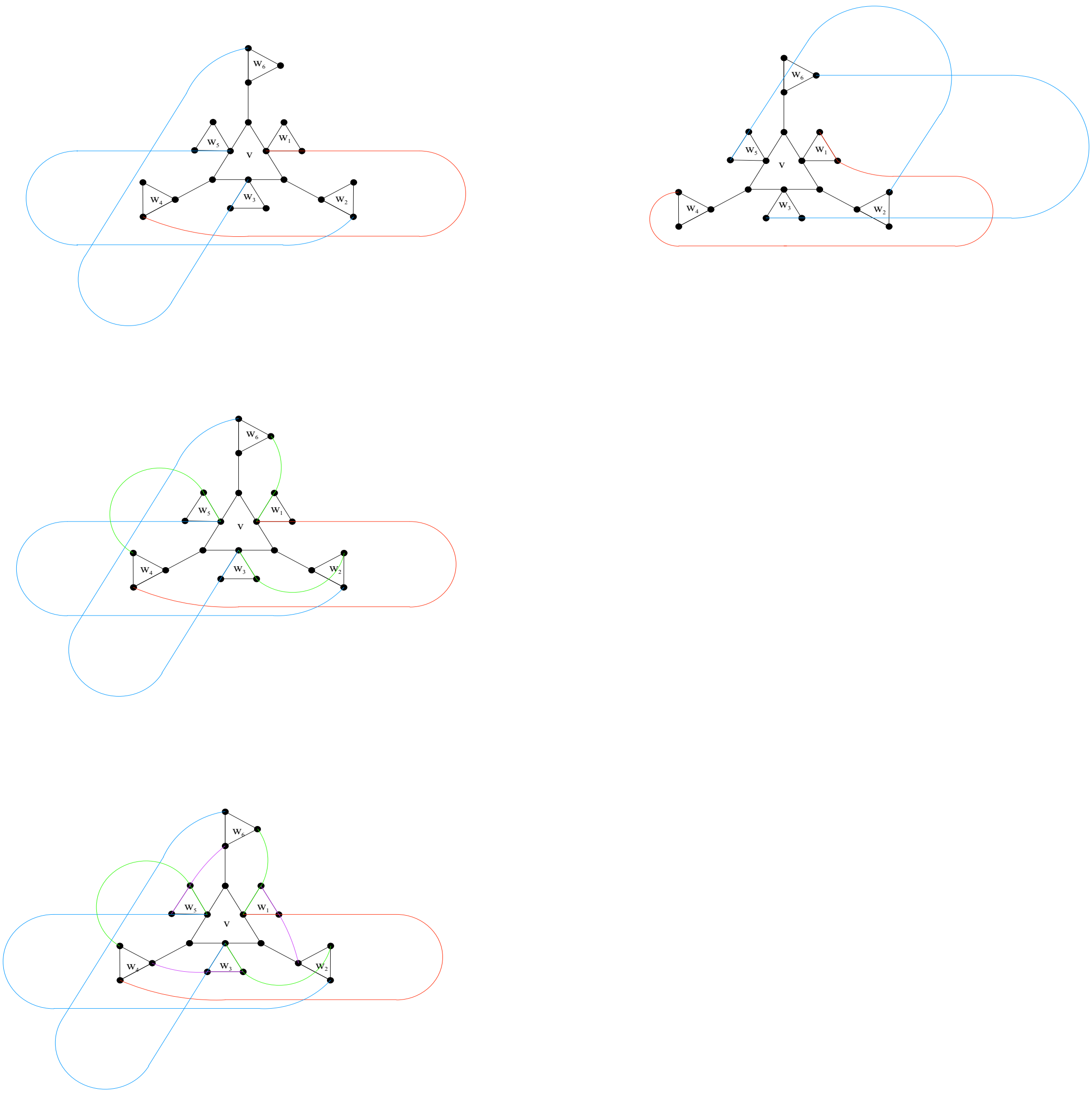

Figure 11: Case where (v, w_1, w_4, v) is a 3-cycle

opposite relation is clearly symmetric. Let $N = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be the set of neighbors of v. Say that the two orbits are given by $\{w_1, w_3, w_5\}$ and $\{w_2, w_4, w_6\}$. Is this opposite relation well defined? Or equivalent, are opposite vertices mapped onto opposite vertices under an automorphism in G? First remark that $G_{v^g} = (G_v)^g \stackrel{\text{def}}{=} g^{-1}G_v g$ with $g \in G$ and v a vertex of $\mathcal{G}$. Indeed, if $h \in G_{v^g}$ then $v^{gh} = v^g$ and hence $ghg^{-1} = k \in G_v$. Or $h = g^{-1}kg$ belongs to $(G_v)^g$. If $h \in (G_v)^g$ then there exists a $k \in G_v$ such that $h = g^{-1}kg$. It follows that $v^{gg^{-1}kg} = v^{gh} = v^{kg} = v^g$ or $h \in G_{v^g}$. Different elements of G_v give rise to different elements of G_{v^g}. Now suppose that v is mapped onto v^g under a collineation g in G. Let w_1 be opposite w_2. Then we prove that w_1^g is opposite w_2^g. The fact that w_1 and w_2 are opposite means that the action of $h \in G_v$ onto orbit $\{w_1, w_3, w_5\}$ is given by $(w_3\ w_5)$, the action onto orbit $\{w_2, w_4, w_6\}$ by $(w_4\ w_6)$. The orbit of w_1^g is given by the set $\{w_1^{gh} |\ h \in G_{v^g}\}$ or equivalently $\{w_1^{kg} |\ k \in G_v\}$. Hence, the orbit of w_1^g is the set $\{w_1^g, w_3^g, w_5^g\}$ and the orbit of w_2^g the set $\{w_2^g, w_4^g, w_6^g\}$. The action of $g^{-1}hg$ onto orbit $\{w_1^g, w_3^g, w_5^g\}$ is given by (w_3^g, w_5^g), the action onto orbit $\{w_2^g, w_4^g, w_6^g\}$ by (w_4^g, w_6^g) proving that w_1^g and w_2^g are opposite. Based on this opposite relation we make the undirected graph $\mathcal{G}$ directed. We choose $(v, w_1), (v, w_3), (v, w_5), (w_2, v), (w_4, v)$ and (w_6, v) as directed edges of the graph $\overrightarrow{\mathcal{G}}$. The direction of the other edges of $\mathcal{G}$ is determined by the vertex transitivity of G. Consider an arbitrary other vertex w of $\mathcal{G}$. Because of vertex transitivity, there is an automorphism mapping v onto $w = v^g$. The neighbors of v^g are then $\{w_1, w_2, w_3, w_4, w_5, w_6\}^g$. We then define the following directed edges: $(v^g, w_1^g), (v^g, w_3^g), (v^g, w_5^g), (w_2^g, v^g), (w_4^g, v^g)$ and (w_6^g, v^g). As mentioned before, the two orbits in v^g are given by the sets $\{w_1^g, w_3^g, w_5^g\}$ and $\{w_2^g, w_4^g, w_6^g\}$. Suppose that there also exists an automorphism $h \in G$ different from g for which $v^h = w$. The orbits arising from the stabilizer G_{v^h} are $\{w_1^h, w_3^h, w_5^h\}$ and $\{w_2^h, w_4^h, w_6^h\}$ and are the same as those arising from G_{v^g}. Suppose that $\{w_1^g, w_3^g, w_5^g\} = \{w_2^h, w_4^h, w_6^h\}$. Then for example $w_1^g = w_2^h$ or $w_1^{gh^{-1}} = w_2$ with $gh^{-1} \in G_v$. But then w_1 and w_2 belong to the same orbit which is impossible. Consequently, both g and h give rise to a same direction of the edges in v^g. It is easily seen that $\overrightarrow{\mathcal{G}}$ is transitive on the directed edges.

Now we define a geometry Γ based on this new directed graph. To every directed edge (v, w) of $\overrightarrow{\mathcal{G}}$ we attach a point and a line: the first half of the edge (v, w) is a line, the second half a point. We note them by $(v, w)^1$ and $(v, w)^2$. Given that v is a vertex of the graph adjacent to the vertices w_1, w_2, w_3, w_4, w_5 and w_6 and with orbits as mentioned above. Then, the line $(v, w_i)^1$ with $i \in \{1, 3, 5\}$ is incident with the points $(w_j, v)^2$ and $(w_k, v)^2$ where $\{j, k\} \subset \{2, 4, 6\}$ and w_j and w_k not opposite w_i and with the point $(v, w_i)^2$. Analogously, the point $(w_i, v)^2$ with $i \in \{2, 4, 6\}$ is incident with the lines $(v, w_j)^1$ and $(v, w_k)^1$ where $\{j, k\} \subset \{1, 3, 5\}$ and w_j and w_k not opposite w_i and with the line $(w_i, v)^1$. It is easily seen that the geometry Γ is bislim, without digons but containing triangles. Indeed, $((w_2, v)^2, (w_4, v)^2, (w_6, v)^2)$ is a triangle of Γ with sides $(v, w_1)^1$, $(v, w_3)^1$ and $(v, w_5)^1$.

Suppose that $\overrightarrow{\mathcal{G}}$ contains a directed 3-cycle (v, w_5, w_4, v) (see fig. 12). Since the vertices

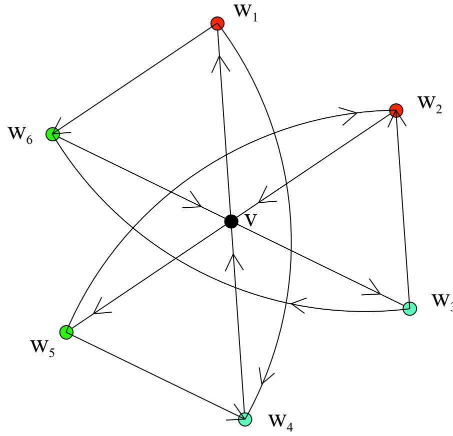

Figure 12: 6-regular graph with directed 3-cycle

w_1 and w_2, resp. w_3 and w_4, resp. w_5 and w_6 are opposite the automorphism $(w_3\ w_5)$ and $(w_4\ w_6)$, resp. $(w_1\ w_5)$ and $(w_2\ w_6)$, resp. $(w_1\ w_3)$ and $(w_2\ w_4)$ belongs to G_v. It follows that (w_5, w_2), (w_1, w_4), (w_3, w_2), (w_1, w_6) and (w_3, w_6) are directed edges of $\overrightarrow{\mathcal{G}}$. The second automorphism mentioned above, $(w_1\ w_5)$ and $(w_2\ w_6)$, corresponds to an automorphism fixing v and w_4, hence belonging to G_{w_4}. Since w_1 and w_5 are mapped onto each other and v is fixed it follows that v and w_5 or w_1 can not be opposite in the vertex w_4. Also v and w_4 or w_2 can not be opposite in the vertex w_5, neither v and w_4 or w_6 in w_1. Constructing a geometry in the above mentioned way leads to a geometry with the wrong local structure (see fig. 13. Hence, we have to assume that $\overrightarrow{\mathcal{G}}$ has no directed 3-cykels containing non-opposite vertices. Suppose that $\overrightarrow{\mathcal{G}}$ contains a directed 3-cycle (v, w_1, w_2, v) having two opposite vertices (see fig. 14). Since the vertices w_1 and w_2, resp. w_3 and w_4, resp. w_5 and w_6 are opposite the automorphism $(w_3\ w_5)$ and $(w_4\ w_6)$, resp. $(w_1\ w_5)$ and $(w_2\ w_6)$, resp. $(w_1\ w_3)$ and $(w_2\ w_4)$ belongs to G_v. It follows that (w_5, w_6) and (w_3, w_4) are directed edges of $\overrightarrow{\mathcal{G}}$. Due to the previous observation it follows that v and w_3 are opposite in w_4, v and w_4 are opposite in w_3, v and w_1 are opposite in w_2, v and w_2 are opposite in w_1, v and w_5 are opposite in w_6 and v and w_6 are opposite in w_5. Constructing a geometry in the above mentioned way leads in this special case to a geometry with the right local structure (see fig. 15. From now on we assume that $\overrightarrow{\mathcal{G}}$ has either no directed 3-cycles or either three directed 3-cycles in every vertex. In the second case it follows that these cycles are of type (v, w, u, v) with w and u opposite vertices in v. In both cases it is easily seen that the geometry constructed from the graph in the above mentioned way has local structure 1. Transitivity on the ordered edges of the graph induces point and line transitivity of the geometry but no flag transitivity. This follows out of the construction method.

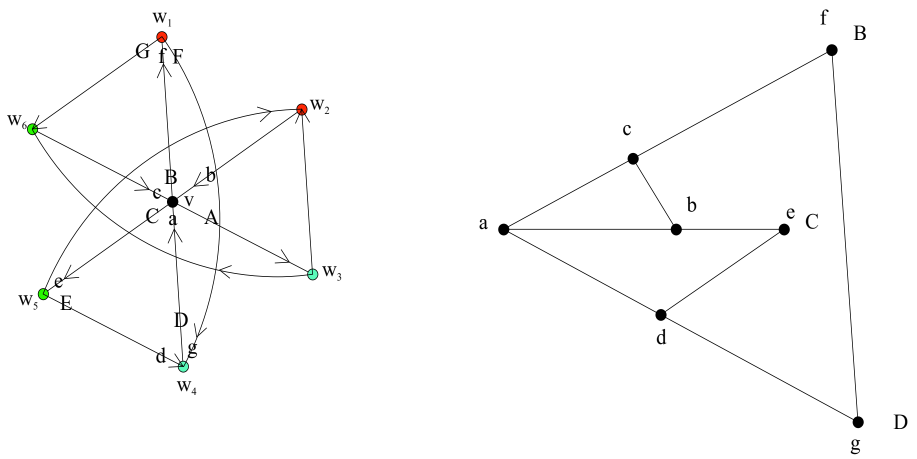

Figure 13: geometry associated with 6-regular graph with directed 3-cycle

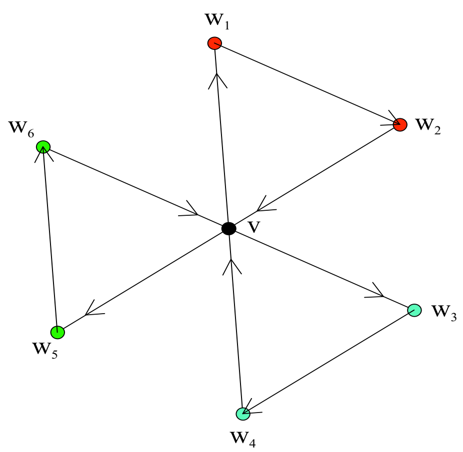

Figure 14: 6-regular graph with directed 3-cycle

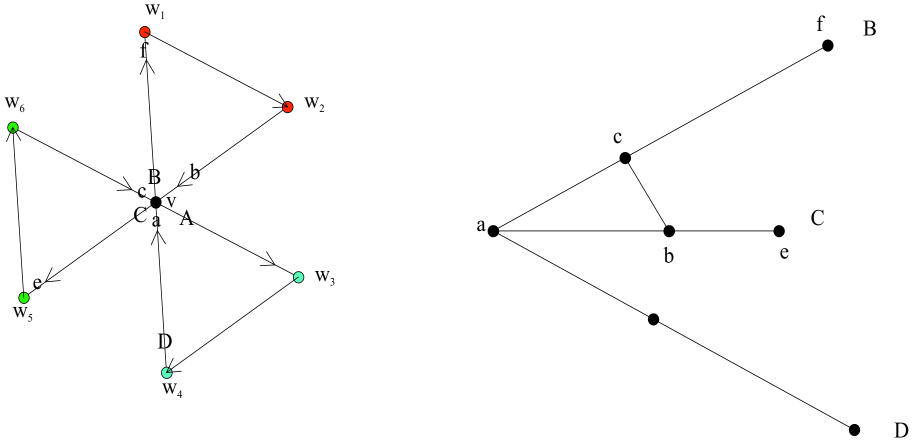

Figure 15: geometry associated with 6-regular graph with directed 3-cycle

Secondly, we look at the case where the action of the stabilizer of a vertex v onto its neighbors is given by the cyclic group $\mathcal{C}_3$. Let $N = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be the set of neighbors of v. Say that the two orbits are given by $\{w_1, w_3, w_5\}$ and $\{w_2, w_4, w_6\}$ and that the action of G_v onto N is either the identity, either $(w_1\ w_3\ w_5)$ and $(w_2\ w_4\ w_6)$ or either $(w_1\ w_5\ w_3)$ and $(w_2\ w_6\ w_4)$. We choose the vertices w_1 and w_2 to be opposite. The above mentioned automorphisms then define w_3 and w_4, resp. w_5 and w_6 to be opposite. The opposite vertices in another vertex w are then determined by vertex-transitivity. Suppose that g and h are both automorphisms of G mapping v onto w. Remind that $G_{v^g} = (G_v)^g \stackrel{\text{def}}{=} g^{-1}G_v g$ with $g \in G$ and v a vertex of $\mathcal{G}$. Different elements of G_v give rise to different elements of G_{v^g}. We have to prove that g and h define the same opposite relation onto the neighbors of w. The two orbits in w are given by $\{w_1^g, w_3^g, w_5^g\}$ and $\{w_2^g, w_4^g, w_6^g\}$. By definition w_1^g is opposite w_2^g, resp. w_3^g is opposite w_4^g, resp. w_5^g is opposite w_6^g. It is easy to see that $\{w_1^g, w_3^g, w_5^g\} = \{w_1^h, w_3^h, w_5^h\}$ and $\{w_2^g, w_4^g, w_6^g\} = \{w_2^h, w_4^h, w_6^h\}$. Suppose that $w_1^g = w_1^h$. Then gh^{-1} is the identity on the neighbors of v. It follows that in this case g and h define the same opposite relation onto the neighbors of w. Suppose that $w_1^g = w_3^h$. Then the action of gh^{-1} onto the neighbors of v is given by $(w_1\ w_3\ w_5)$ and $(w_2\ w_4\ w_6)$. It follows that $w_2^g = w_4^h$ and hence also in this case g and h define the same opposite relation in w. Suppose that $w_1^g = w_5^h$. Then the action of gh^{-1} onto the neighbors of v is given by $(w_1\ w_5\ w_3)$ and $(w_2\ w_6\ w_4)$. It follows that $w_2^g = w_6^h$ and hence also in this case g and h define the same opposite relation in w. Determining the opposite relation in one vertex of the graph, determines the opposite relation in all vertices of the graph by vertex transitivity. Opposite vertices are hence mapped onto opposite vertices, by definition. Based on this opposite relation we make the undirected graph $\mathcal{G}$ directed. We choose $(v, w_1), (v, w_3), (v, w_5), (w_2, v), (w_4, v)$ and (w_6, v) as directed edges of the graph $\overrightarrow{\mathcal{G}}$.

The direction of the other edges of $\mathcal{G}$ is determined by the vertex transitivity of G. The argumentation mentioned above proves that different automorphisms g and h mapping v onto w give the same directions of the edges in w. It is easily seen that $\overrightarrow{\mathcal{G}}$ is transitive on the directed edges. Consider a directed edge (v, w_1). An automorphism fixing this edge, belongs to G_v and acts trivial onto the neighbors of v. It is easy to see that every such isomorphism is equal to the identity and hence $\overrightarrow{\mathcal{G}}$ is sharply transitive on the directed edges.
Now we define a geometry Γ based on this new directed graph. To every directed edge (v, w) of $\overrightarrow{\mathcal{G}}$ we attach a point and a line: the first half of the edge (v, w) is a line, the second half a point. We note them by $(v, w)^1$ and $(v, w)^2$. Given that v is a vertex of the graph adjacent to the vertices w_1, w_2, w_3, w_4, w_5 and w_6 and with orbits as mentioned above. Then, the line $(v, w_i)^1$ with $i \in \{1, 3, 5\}$ is incident with the points $(w_j, v)^2$ and $(w_k, v)^2$ where $\{j, k\} \subset \{2, 4, 6\}$ and w_j and w_k not opposite w_i and with the point $(v, w_i)^2$. Analogously, the point $(w_i, v)^2$ with $i \in \{2, 4, 6\}$ is incident with the lines $(v, w_j)^1$ and $(v, w_k)^1$ where $\{j, k\} \subset \{1, 3, 5\}$ and w_j and w_k not opposite w_i and with the line $(w_i, v)^1$. It is easily seen that the geometry Γ is bislim, without digons but containing triangles. Indeed, $((w_2, v)^2, (w_4, v)^2, (w_6, v)^2)$ is a triangle of Γ with sides $(v, w_1)^1$, $(v, w_3)^1$ and $(v, w_5)^1$.
The graph $\overrightarrow{\mathcal{G}}$ has either zero, either three, either six or either nine 3-cycles in every vertex. We look at the different cases separately (see fig. 16). If there are zero 3-cycles in every vertex then the geometry constructed out of the graph has local structure 1. Case (b) where there are three 3-cycles between opposite vertices v, also gives rise to a geometry with the right local structure. Since in case (c) there are three 3-cycles only between non-opposite vertices in v it follows by transitivity that v and w_4 are not opposite in w_5 and v and w_5 are not opposite in w_4. Case (d) is completely similar. From now on we assume that $\overrightarrow{\mathcal{G}}$ has no three 3-cycles between non-opposite vertices. In case (e) transitivity induces that either v and w_4 are opposite in w_5 or either v and w_6 are opposite in w_5. If v and w_4 are opposite in w_5 then there is a unique automorphism mapping (w_5, w_6) onto (v, w_1). Then v is taken onto w_4 or w_6. In the first case there are more than six 3-cycles, hence the image of v is given by w_6. Hence w_4 is mapped onto w_5. Since w_5 and w_6 are opposite in v, it follows that v and w_1 are opposite in w_6 and hence v and w_5 are opposite in w_4. But then w_6 and v are opposite in w_5, a contradiction. Consequently v and w_4 are non-opposite in w_5, inducing that v and w_6 are opposite in w_5. If v and w_5 are opposite in w_4 then there is a unique automorphism mapping (w_3, w_4) onto (w_2, v). Then v is taken onto w_3 or w_5. In the second case there are more than six 3-cycles, hence the image of v is given by w_3. Hence w_5 is mapped onto w_4. Since w_3 and w_4 are opposite in v, it follows that w_2 and v are opposite in w_3 and hence v and w_4 are opposite in w_5, a contradiction. We conclude that v and w_6 are opposite in w_5 and that v and w_3 are opposite in w_4. Hence, v and w_4 are non-opposite in w_5 and v and w_5 are non-opposite in w_4. The geometry constructed from this graph has the wrong local structure. Case (f) is

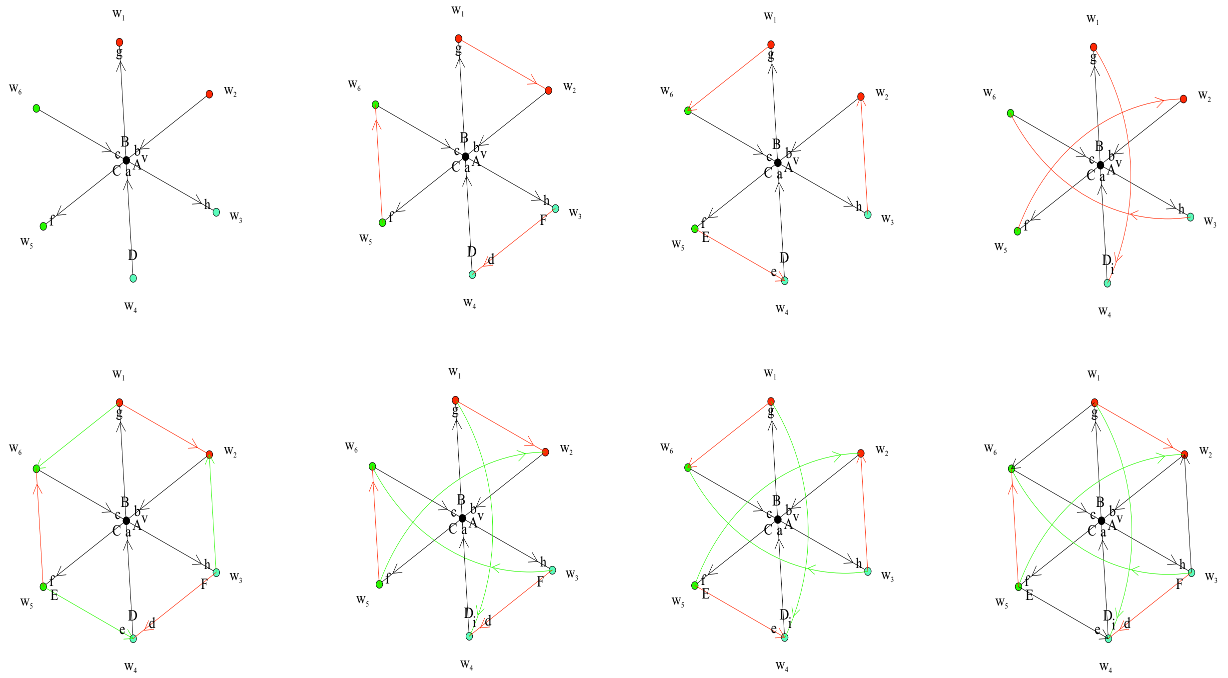

Figure 16:

analogously. Transitivity induces that either v and w_1 are opposite in w_4 or either v and w_3 are opposite in w_4. If v and w_1 are opposite in w_4 then there is a unique automorphism mapping (w_3, w_4) onto (w_2, v). Then v is taken onto w_5. Hence w_1 is mapped onto w_6. Since w_3 and w_4 are opposite in v, it follows that w_2 and v are opposite in w_5 and hence v and w_4 are opposite in w_1. But then w_5 and v are opposite in w_6, inducing that w_3 and v are opposite in w_4, a contradiction. Consequently v and w_1 are non-opposite in w_4, inducing that v and w_3 are opposite in w_4. If v and w_4 are opposite in w_1 then there is a unique automorphism mapping (w_1, w_2) onto (v, w_1). Then v is taken onto w_4. Hence w_4 is mapped onto w_3. Since w_1 and w_2 are opposite in v, it follows that v and w_1 are opposite in w_4, a contradiction. We conclude that v and w_3 are opposite in w_4 and that v and w_2 are opposite in w_1. Hence, v and w_1 are non-opposite in w_4 and v and w_4 are non-opposite in w_1. The geometry constructed from this graph has the wrong local structure. In case (g) transitivity induces that v and w_4 are not opposite in w_5, v and w_5 are not opposite in w_4, v and w_1 are not opposite in w_4 and v and w_4 are not opposite in w_1. The geometry constructed from this graph has the wrong local structure. We assume from now on that $\overrightarrow{\mathcal{G}}$ does not contain six directed 3-cycles. In the last case (h), either the vertices v and w_2, either v and w_4, either v and w_6 are opposite in w_5. This includes that v and w_4, resp. v and w_6, resp. v and w_2 are opposite in w_1. In w_4 either v and w_1, either v and w_3, either v and w_5 are opposite vertices. Only the cases where v and w_2 are opposite in w_5 and v and w_5 are opposite in w_4 (and hence v and w_1 are opposite in w_6) or where v and w_4 are opposite in w_5 and v and w_1 are opposite in w_4 (and hence v and w_3 are opposite in w_6) lead to geometries with the right local structure. We take a closer look at the situation where v and w_4 are opposite in w_5. There is a unique automorphism mapping (w_5, w_6) onto (v, w_1). The vertex v is then either taken onto w_4 or either onto w_6. In the first case the image of w_4 is given by w_3. Since w_5 and w_6 are opposite in v it follows that v and w_1 are opposite in w_4. In the second case the image of w_4 is given by w_5. Since w_5 and w_6 are opposite in v it follows that v and w_1 are opposite in w_6 and hence v and w_5 are opposite in w_4. But then w_6 and v are opposite in w_5, a contradiction. Next we take a closer look at the situation where v and w_2 are opposite in w_5. There is a unique automorphism mapping (w_5, w_6) onto (v, w_1). The vertex v is then either taken onto w_4 or either onto w_6. In the first case the image of w_2 is given by w_3. Since w_5 and w_6 are opposite in v it follows that v and w_1 are opposite in w_4 and hence v and w_5 are opposite in w_2. It follows that w_4 and v are opposite in w_3 and hence w_6 and v are opposite in w_5, a contradiction. In the second case the image of w_4 is given by w_5. Since w_5 and w_6 are opposite in v it follows that v and w_1 are opposite in w_6 and hence v and w_5 are opposite in w_4. Finally, let us look at the case where v and w_6 are opposite in w_5. The automorphism mapping (w_5, w_6) onto (v, w_5) takes v onto w_6. Since w_5 and w_6 are opposite in v it follows that v and w_5 are opposite in w_6. It is then easy to see that for every vertex a of the graph there are three 3-cycles (a, b, c) with the property, say property ($\Box$), that b and c are opposite in a, a and c are opposite in b and a and b are opposite in c. And hence six directed 3-cycles (a, d, e) with the property that d and e are not opposite in a, a and e are not opposite in d and a and d are not opposite in e. This is the only situation where nine 3-cycles in v give rise to a geometry with the wrong local structure.

We have to assume that $\overrightarrow{\mathcal{G}}$ contains no nine 3-cycles (a, b, c) in a vertex a for which three of them have property ($\Box$). We conclude that we assume from now on that $\overrightarrow{\mathcal{G}}$ either contains no directed 3-cycles, either three directed 3-cycles with property ($\Box$) or either nine directed 3-cycles in every vertex with the restriction that there are not three directed 3-cycles with property ($\Box$).

Sharp transitivity on the ordered edges of the graph induces sharp point and sharp line transitivity of the geometry but no flag transitivity. This follows out of the construction method.

We conclude by a final remark. If the geometry Γ contains a 3-cycle then there are either only three type 4 connections between a triangle and its neighbor triangles or either three type 1, three type 2 and three type 5 connections. Geometries of the former type having the same number of triangles (or points) are isomorphic (as mentioned above). The geometry defines an opposite relation in the graph in the following way: the triangle u in the third point of a side S of triangle v is opposite to the triangle w on the third line through the point of triangle v not on side S. We say that u and w have property

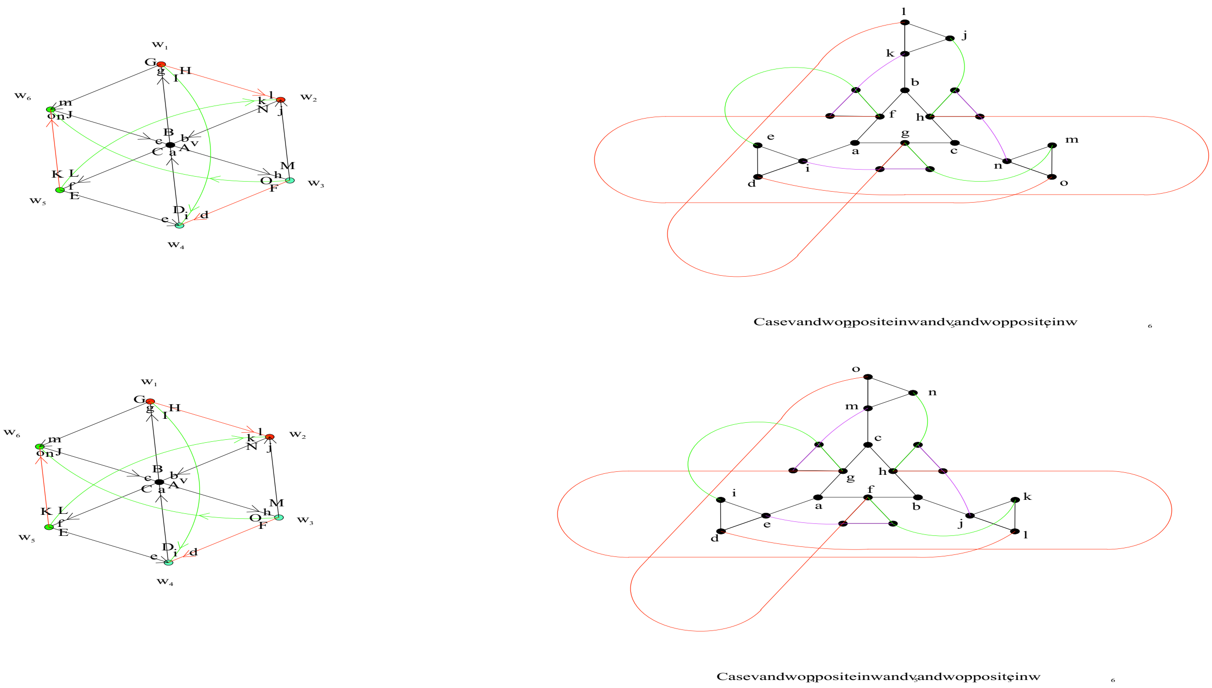

Figure 17:

($\triangle$) in v. Looking at fig. 5 triangle w_1 is opposite to triangle w_4. Given a graph of the above mentioned type. Say that u and w are opposite vertices in v. It is then easy to see that the triangles corresponding to u and w have property ($\triangle$) in v. If the graph contains nine directed 3-cycles, we assume that there are not any 3-cycles with property ($\square$) (see above). We proved that then v and w_2 are opposite in w_5 and v and w_5 are opposite in w_4 (and hence v and w_1 are opposite in w_6) or v and w_4 are opposite in w_5 and v and w_1 are opposite in w_4 (and hence v and w_3 are opposite in w_6). It is easy to see that both situations give rise to a geometry with three type 1, three type 2 and three type 5 connections with its neighbors. So if the graphs corresponding to each situation have the same number of vertices they define isomorphic geomtries and are hence isomorphic (see fig. 17).

The graph $\mathcal{G}_\Delta$ associated with the geometry Γ is either 6-regular or either 3-regular. In the case of a 6-regular graph, the stabilizer of a triangle G_Δ, with G the lifting of the collineation group of Γ, acts either as $\mathcal{S}_3$ or either as $\mathcal{C}_3$ on the six neighbors of the triangle Δ. We consider this last situation.

It is easily seen that in this case the collineation group G of Γ acts sharply point (and hence sharply line) transitive. Hence we can identify G with the points of Γ. We consider $.g$ as the action of G onto itself where . is the operation on G and g is an element of G. Let e, a and b be different elements of G on one line. Then a^{-1}, e and ba^{-1} are also on one line. The third line through e is then given by the points b^{-1}, ab^{-1} and e. The conditions arising from requiring that those lines contain seven different points are given by:

$$
\begin{array}{llll}
a \neq a^{-1} & \Leftrightarrow & \mathbf{a^2 \neq e} & \\
a \neq ba^{-1} & \Leftrightarrow & \mathbf{a^2 \neq b} & \\
b \neq a^{-1} & \Leftrightarrow & \mathbf{ab \neq e} & \\
b \neq ba^{-1} & \Leftrightarrow & a^{-1} \neq e & \Leftrightarrow \quad e \neq a \\
a \neq b^{-1} & \Leftrightarrow & ab \neq e & \\
a \neq ab^{-1} & \Leftrightarrow & b \neq e & \\
b \neq b^{-1} & \Leftrightarrow & \mathbf{b^2 \neq e} & \\
b \neq ab^{-1} & \Leftrightarrow & \mathbf{b^2 \neq a} & \\
a^{-1} \neq b^{-1} & \Leftrightarrow & a \neq b & \\
a^{-1} \neq ab^{-1} & \Leftrightarrow & b \neq a^2 & \\
ba^{-1} \neq b^{-1} & \Leftrightarrow & b \neq b^{-1}a & \Leftrightarrow \quad b^2 \neq a \\
ba^{-1} \neq ab^{-1} & \Leftrightarrow & \mathbf{ba^{-1}b \neq a} &
\end{array}
$$

Without loss of generality we assume that e and a belong to the same triangle. Then a and a^2 form another side of the same triangle. The point a^2 is hence collinear to e and the collineation $.a$ takes a^2 onto a^3 and it follows that $\mathbf{a^3 = e}$.

The collineation $.a$ takes the point e onto a. The points a, a^2 and ba are then on one line. Just as the points a, $b^{-1}a$ and $ab^{-1}a$. The seven points on the lines through a are all different. Requiring the right local structure, results in the following conditions:

$$
\begin{array}{llll}
ba \neq ba^2 & \Leftrightarrow & a \neq e & \\
ba \neq b^{-1} & \Leftrightarrow & \mathbf{b^2 \neq a^2} & \\
ba \neq ab^{-1} & \Leftrightarrow & \mathbf{a \neq bab} & \\
b^{-1}a \neq ba^2 & \Leftrightarrow & b^{-1} \neq ba & \Leftrightarrow \quad a^2 \neq b^2 \\
b^{-1}a \neq b^{-1} & \Leftrightarrow & a \neq e & \\
b^{-1}a \neq ab^{-1} & \Leftrightarrow & b^{-1}ab \neq a & \Leftrightarrow \quad \mathbf{ab \neq ba} \\
ab^{-1}a \neq ba^2 & \Leftrightarrow & ab^{-1} \neq ba & \Leftrightarrow \quad a \neq bab \\
ab^{-1}a \neq b^{-1} & \Leftrightarrow & ab^{-1} \neq b^{-1}a^2 & \Leftrightarrow \quad \mathbf{ba^2 \neq ab} \\
ab^{-1}a \neq ab^{-1} & \Leftrightarrow & a \neq e &
\end{array}
$$

The two other lines in the point b are given by a^2b, b, ba^2b and b, ab, b^2. To have the right local structure we need the following conditions:

$$
\begin{array}{llllllll}
a^2b \neq a^2 & \Leftrightarrow & b \neq e & & & & & \\
a^2b \neq ba^2 & \Leftrightarrow & a^2 \neq ba^2b^{-1} & \Leftrightarrow & a \neq bab^{-1} & \Leftrightarrow & ab \neq ba \\
a^2b \neq b^{-1} & \Leftrightarrow & a^2 \neq b^{-2} & \Leftrightarrow & a \neq b^2 & & \\
a^2b \neq ab^{-1} & \Leftrightarrow & ab \neq b^{-1} & \Leftrightarrow & ab^2 \neq e & \Leftrightarrow & b^2 \neq a^2 \\
ba^2b \neq a^2 & \Leftrightarrow & a^2 \neq b^{-1}a^2b^{-1} & \Leftrightarrow & a \neq bab & & \\
ba^2b \neq ba^2 & \Leftrightarrow & b \neq e & & & & \\
ba^2b \neq b^{-1} & \Leftrightarrow & a^2 \neq b^{-3} & \Leftrightarrow & \mathbf{a \neq b^3} & & \\
ba^2b \neq ab^{-1} & \Leftrightarrow & \mathbf{ba^2b^2 \neq a} & & & & \\
ab \neq a^2 & \Leftrightarrow & b \neq a & & & & \\
ab \neq ba^2 & & & & & & \\
ab \neq b^{-1} & \Leftrightarrow & ab^2 \neq e & \Leftrightarrow & b^2 \neq a^2 & & \\
ab \neq ab^{-1} & \Leftrightarrow & b^2 \neq e & & & & \\
b^2 \neq a^2 & & & & & & \\
b^2 \neq ba^2 & \Leftrightarrow & b \neq a^2 & & & & \\
b^2 \neq b^{-1} & \Leftrightarrow & \mathbf{b^3 \neq e} & & & & \\
b^2 \neq ab^{-1} & \Leftrightarrow & b^3 \neq a & & & &
\end{array}
$$

The line containing the points ba^2, a^2ba^2 and ba^2ba^2 and the line through the points aba^2, ba^2 and b^2a^2 are the two other lines in the point ba^2. Since the collineation $ba^2 \in G$ maps the three lines through e onto the three lines in ba^2, the seven points on the three lines through ba^2 are mutually different. To obtain the right local structure we have to require that the four points a^2ba^2, ba^2ba^2, aba^2 and b^2a^2 are different from the points b^{-1} and ab^{-1}. This results in the following conditions:

$$
\begin{array}{lllllllll}
a^2ba^2 \neq b^{-1} & \Leftrightarrow & a^2b \neq b^{-1}a & \Leftrightarrow & a \neq ba^2b & & & & \\
a^2ba^2 \neq ab^{-1} & \Leftrightarrow & a \neq a^2ba^2b & \Leftrightarrow & e \neq aba^2b & \Leftrightarrow & \mathbf{a^2 \neq ba^2b} & & \\
ba^2ba^2 \neq b^{-1} & \Leftrightarrow & b^2a^2ba^2 \neq e & \Leftrightarrow & a^2 \neq b^{-1}ab^{-2} & \Leftrightarrow & \mathbf{a \neq b^2a^2b} & \Leftrightarrow & a \neq ba^2b^2 \\
ba^2ba^2 \neq ab^{-1} & \Leftrightarrow & \mathbf{a^2ba^2 \neq b^{-1}ab^{-1}} & & & & & & \\
aba^2 \neq b^{-1} & \Leftrightarrow & baba^2 \neq e & \Leftrightarrow & a \neq bab & & & & \\
aba^2 \neq ab^{-1} & \Leftrightarrow & ba^2 \neq b^{-1} & \Leftrightarrow & a^2 \neq b^{-2} & \Leftrightarrow & a \neq b^2 & & \\
b^2a^2 \neq b^{-1} & \Leftrightarrow & a^2 \neq b^{-3} & \Leftrightarrow & a \neq b^3 & & & & \\
b^2a^2 \neq ab^{-1} & \Leftrightarrow & a^2 \neq b^{-2}ab^{-1} & \Leftrightarrow & a \neq ba^2b^2 & & & &
\end{array}
$$

The third line through a^2 contains the points $b^{-1}a^2$ and $ab^{-1}a^2$. Those two points need to be different from the points b^{-1} and ab^{-1}:

$$
\begin{array}{lllll}
b^{-1}a^2 \neq b^{-1} & \Leftrightarrow & a^2 \neq e & & \\
b^{-1}a^2 \neq ab^{-1} & \Leftrightarrow & ab \neq ba^2 & & \\
ab^{-1}a^2 \neq b^{-1} & \Leftrightarrow & aba^2 \neq b & \Leftrightarrow & ab \neq ba \\
ab^{-1}a^2 \neq ab^{-1} & \Leftrightarrow & a^2 \neq e & &
\end{array}
$$

We have found the following conditions on the group elements e, a and b to obtain the right local structure in the point e: element a has order 3 and b has order bigger than 3 and

$$
\begin{array}{lll}
a^2 \neq b & & \\
ab \neq e & \Leftrightarrow & b \neq a^2 \\
\mathbf{b^2 \neq a} & & \\
ba^{-1}b \neq a & \Leftrightarrow & \mathbf{a \neq ba^2b} \\
\mathbf{b^2 \neq a^2} & & \\
\mathbf{a \neq bab} & & \\
\mathbf{ab \neq ba} & & \\
\mathbf{ba^2 \neq ab} & & \\
\mathbf{a \neq b^3} & & \\
\mathbf{a \neq ba^2b^2} & & \\
\mathbf{a^2 \neq ba^2b} & & \\
a \neq b^2a^2b & \Leftrightarrow & a^2 \neq b^{-2}ab^{-1} \quad \Leftrightarrow \quad a \neq ba^2b^2 \\
\mathbf{a^2ba^2 \neq b^{-1}ab^{-1}} & &
\end{array}
$$

Remark that $a^2 \neq b$ follows from $b^3 \neq e$. Indeed suppose that $a^2 = b$ then $a^6 = e = b^3$. Because of transitivity of the group it follows that we have the right local structure in each point.

4 Configuration 5

4.1 Description of the geometries

There are two lines in Γ_x^l, the line x_1y_1 and the line x_2y_2. It is easy to see that every point of the geometry Γ belongs to two triangles with one line (through the point) in common. For the point x these triangles are given by (x, x_1, y_1) and (x, x_2, y_2) sharing the line $L = xx_1x_2$. The lines x_1y_1 and x_2y_2 can not intersect. In x_1 the line common to the two triangles in x_1 is either the line L or either the line x_1y_1. Suppose first that x_1 and x_2 belong to the same triangle. Then there exists a collineation g taking x_1 onto x and hence fixing the line L. Because of the fact that G is not flag transitive, it follows that x is mapped onto x_1. There also exists a collineation h taking x_2 onto x and hence mapping x onto x_2. But then gh takes x onto x_1 onto x_2 onto x, a contradiction. Consequently the line common to the two triangles in x_1 is the line x_1y_1. The line common to the two triangles in y_1 is either $L_1 = xx_1y_1$ or either x_1y_1. Suppose that a and y_1 belong to the same triangle with a the third point on the line x_1y_1. But then the collineation taking x_1 onto x, maps the line x_1y_1 onto L and then x_1 and x_2 belong to the same triangle, which is impossible.

4.2 Collineation group

We consider a collineation g fixing the point x. It follows that the line L is fixed. The point x_1 is then either fixed or either taken onto the point x_2. In the first case g is the identity. In the second case the collineation g is fully determined and is an involution. Hence, a collineation group for a geometry with local structure 5 is either sharply transitive or has either order of the stabilizer of a point equal to two.

There is some collineation taking x onto x_1. The line L is then mapped onto the line x_1y_1a, including that x_1 is either taken onto y_1 or either onto a. In the case of a sharply transitive collineation group only one collineation mapping x onto x_1 belongs to G. In the case of a non sharply transitive collineation group both collineations taking x onto x_1 belong to G.

Suppose that G is sharply transitive and the collineation g_1 mapping x onto x_1, maps x_1 onto y_1 and hence y_1 onto x. Then the collineation h_1 mapping x onto x_2 either takes x_2 onto y_2 or either onto b, with b the third point on the line x_2y_2. Suppose that x_2 is taken onto b then y_1 is taken onto x which is in contradiction with the sharp transitivity of the group. It follows that h_1 takes x_2 onto y_2.

Suppose that G is sharply transitive and the collineation g_1 mapping x onto x_1, maps x_1 onto a and hence y_2 onto x. Suppose that h_1 maps x_2 onto y_2 then y_2 is taken onto x, which is in contradiction with sharp point transitivity of G. It follows that h_1 maps x_2 onto b. Considering the image of x, it follows that $g_1h_1g_1$ is the identity. Since $x_2^{g_1h_1g_1} = x_1$ this case leads to a contradiction.

4.3 Sharply point transitive collineation group

We can identify G with the points of the geometry Γ. We consider $.g$ as the action of G onto itself where . is the operation on G and g is an element of G. Let e, a and b be different elements of G on one line. Then a^{-1}, e and ba^{-1} are also on one line. The third line through e is then given by the points b^{-1}, ab^{-1} and e. The conditions arising from requiring that those lines contain seven different points are given by:

$$
\begin{array}{lcll}
a \neq a^{-1} & \Leftrightarrow & \mathbf{a^2 \neq e} & \\
a \neq ba^{-1} & \Leftrightarrow & \mathbf{a^2 \neq b} & \\
b \neq a^{-1} & \Leftrightarrow & \mathbf{ab \neq e} & \\
b \neq ba^{-1} & \Leftrightarrow & a^{-1} \neq e & \Leftrightarrow \quad e \neq a \\
a \neq b^{-1} & \Leftrightarrow & ab \neq e & \\
a \neq ab^{-1} & \Leftrightarrow & b \neq e & \\
b \neq b^{-1} & \Leftrightarrow & \mathbf{b^2 \neq e} & \\
b \neq ab^{-1} & \Leftrightarrow & \mathbf{b^2 \neq a} & \\
a^{-1} \neq b^{-1} & \Leftrightarrow & a \neq b & \\
a^{-1} \neq ab^{-1} & \Leftrightarrow & b \neq a^2 & \\
ba^{-1} \neq b^{-1} & \Leftrightarrow & b \neq b^{-1}a & \Leftrightarrow \quad b^2 \neq a \\
ba^{-1} \neq ab^{-1} & \Leftrightarrow & \mathbf{ba^{-1}b \neq a} &
\end{array}
$$

Without loss of generality we assume that e and a belong to a triangle and e and b belong to a triangle. There is a unique collineation, $.a$ taking e onto a. The line containing the points e, a and b is then taken onto the line through the points a, a^2 and ba. The collineation $.a$ fixes the triangle through e and a. The point a is taken onto a^2 which is then collinear to e and $\mathbf{a^3} = \mathbf{e}$. The collineation $.b$ then fixes the triangle through e and b. Hence, b^2 is collinear to e and $\mathbf{b^3} = \mathbf{e}$.

The collineation $.a$ takes the point e onto a. The points a, a^2 and ba are then on one line. Just as the points a, b^2a and ab^2a. The seven points on the lines through a are all different. Requiring the right local structure, results in the following conditions:

$$
\begin{array}{lclcl}
ba \neq ba^2 & \Leftrightarrow & a \neq e & & \\
ba \neq b^2 & \Leftrightarrow & a \neq b & & \\
ba \neq ab^2 & \Leftrightarrow & \mathbf{a \neq bab} & & \\
b^2a \neq ba^2 & \Leftrightarrow & ba \neq a^2 & \Leftrightarrow & a \neq b \\
b^2a \neq b^2 & \Leftrightarrow & a \neq e & & \\
b^2a \neq ab^2 & \Leftrightarrow & b^2ab \neq a & \Leftrightarrow & \mathbf{ab \neq ba} \\
ab^2a \neq ba^2 & \Leftrightarrow & ab^2 \neq ba & \Leftrightarrow & a \neq bab \\
ab^2a \neq b^2 & \Leftrightarrow & ab^2 \neq b^2a^2 & \Leftrightarrow & \mathbf{ba^2 \neq ab} \\
ab^2a \neq ab^2 & \Leftrightarrow & a \neq e & &
\end{array}
$$

The two other lines in the point b are given by a^2b, b, ba^2b and b, ab, b^2. To have the right local structure we need the following conditions:

$$
\begin{array}{lclclcl}
a^2b \neq a^2 & \Leftrightarrow & b \neq e & & & & \\
a^2b \neq ba^2 & \Leftrightarrow & a^2 \neq ba^2b^2 & \Leftrightarrow & a \neq bab^2 & \Leftrightarrow & ab \neq ba \\
a^2b \neq ab^2 & \Leftrightarrow & ab \neq b^2 & \Leftrightarrow & a \neq b & & \\
ba^2b \neq a^2 & \Leftrightarrow & a \neq b^2ab^2 & \Leftrightarrow & bab \neq a & & \\
ba^2b \neq ba^2 & \Leftrightarrow & b \neq e & & & & \\
ba^2b \neq ab^2 & \Leftrightarrow & ba^2b^2 \neq a & \Leftrightarrow & ba^2 \neq ab & & \\
ab \neq a^2 & \Leftrightarrow & b \neq a & & & & \\
ab \neq ba^2 & & & & & & \\
ab \neq ab^2 & \Leftrightarrow & b \neq e & & & &
\end{array}
$$

The third line through a^2 contains the points b^2a^2 and ab^2a^2. Those two points need to be different from the points b^2 and ab^2:

$$
\begin{array}{lclcl}
b^2a^2 \neq b^2 & \Leftrightarrow & a^2 \neq e & & \\
b^2a^2 \neq ab^2 & \Leftrightarrow & ab \neq ba^2 & & \\
ab^2a^2 \neq b^2 & \Leftrightarrow & aba^2 \neq b & \Leftrightarrow & ab \neq ba \\
ab^2a^2 \neq ab^2 & \Leftrightarrow & a^2 \neq e & &
\end{array}
$$

The line containing the points ba^2, a^2ba^2 and ba^2ba^2 and the line through the points aba^2, ba^2 and b^2a^2 are the two other lines in the point ba^2. Since the collineation $ba^2 \in G$ maps the three lines through e onto the three lines in ba^2, the seven points on the three lines through ba^2 are mutually different. To obtain the right local structure we have to require that the four points a^2ba^2, ba^2ba^2, aba^2 and b^2a^2 are different from the points b^2 and ab^2.

This results in the following conditions:

$$\begin{array}{llllllll}
a^2ba^2 \neq b^2 & \Leftrightarrow & a^2b \neq b^2a & \Leftrightarrow & a \neq ba^2b & & \\
a^2ba^2 \neq ab^2 & \Leftrightarrow & ba^2 \neq a^2b^2 & \Leftrightarrow & ab^2 \neq ba & \Leftrightarrow & a \neq bab \\
ba^2ba^2 \neq b^2 & \Leftrightarrow & b^2a^2ba^2 \neq e & \Leftrightarrow & a^2 \neq b^2ab & \Leftrightarrow & ba^2 \neq ab \\
ba^2ba^2 \neq ab^2 & \Leftrightarrow & a^2ba^2 \neq b^2ab^2 & \Leftrightarrow & \mathbf{ab^2a \neq ba^2b} & & \\
aba^2 \neq b^2 & \Leftrightarrow & baba^2 \neq e & \Leftrightarrow & a \neq bab & & \\
aba^2 \neq ab^2 & \Leftrightarrow & a^2 \neq b & & & & \\
b^2a^2 \neq b^2 & \Leftrightarrow & a^2 \neq e & & & & \\
b^2a^2 \neq ab^2 & \Leftrightarrow & a^2 \neq bab^2 & \Leftrightarrow & a \neq ba^2b^2 & \Leftrightarrow & ab \neq ba^2
\end{array}$$

We have found the following conditions on the group elements e, a and b to obtain the right local structure in the point e: elements a and b have order 3 and

$$\begin{array}{lll}
\mathbf{a^2 \neq b} & & \\
ab \neq e & \Leftrightarrow & b \neq a^2 \\
b^2 \neq a & & \\
ba^{-1}b \neq a & \Leftrightarrow & \mathbf{a \neq ba^2b} \\
\mathbf{a \neq bab} & & \\
\mathbf{ab \neq ba} & & \\
\mathbf{ba^2 \neq ab} & & \\
\mathbf{ab^2a \neq ba^2b} & &
\end{array}$$

Because of transitivity of the group it follows that we have the right local structure in each point.

4.4 Non sharply point transitive collineation group

Then the order of the stabilizer in G of a point is equal to two. Consider a subgroup $\{e, a\}$ of G with a an involution. Then we can identify the right cosets of $\{e, a\}$ in G with the points of the geometry Γ. We consider $.g$ as the action of G onto its quotient group where . is the operation on G and g is an element of G. Let $\{e, a\}$, $\{b, ab\}$ and $\{c, ac\}$ be different elements of $G/\{e, a\}$ on one line. Therefore e, a, b and c should be mutually different and $\mathbf{c \neq ab}$. It follows that $\{b^{-1}, ab^{-1}\}$, $\{e, a\}$ and $\{cb^{-1}, acb^{-1}\}$ are also on one line. The third line through $\{e, a\}$ is then given by the points $\{c^{-1}, ac^{-1}\}$, $\{bc^{-1}, abc^{-1}\}$ and $\{e, a\}$. The conditions arising from requiring that those lines contain seven different points are given by:

$$
\begin{array}{lcl}
\{b,ab\} \neq \{b^{-1},ab^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}\mathbf{b^2 \neq e}\\ \mathbf{b^2 \neq a}\end{array}\right.\\
\{b,ab\} \neq \{cb^{-1},acb^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}\mathbf{b^2 \neq c}\\ \mathbf{b^2 \neq ac}\end{array}\right.\\
\{b,ab\} \neq \{c^{-1},ac^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}\mathbf{bc \neq e}\\ \mathbf{bc \neq a}\end{array}\right.\\
\{b,ab\} \neq \{bc^{-1},abc^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}c \neq e\\ \mathbf{bc \neq ab}\end{array}\right.\\
\{c,ac\} \neq \{b^{-1},ab^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}bc \neq e\\ \mathbf{cb \neq a}\end{array}\right.\\
\{c,ac\} \neq \{cb^{-1},acb^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}b \neq e\\ \mathbf{cb \neq ac}\end{array}\right.\\
\{c,ac\} \neq \{c^{-1},ac^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}\mathbf{c^2 \neq e}\\ \mathbf{c^2 \neq a}\end{array}\right.\\
\{c,ac\} \neq \{bc^{-1},abc^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}\mathbf{c^2 \neq b}\\ \mathbf{c^2 \neq ab}\end{array}\right.\\
\{b^{-1},ab^{-1}\} \neq \{c^{-1},ac^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}b \neq c\\ \mathbf{b \neq ca}\end{array}\right.\\
\{b^{-1},ab^{-1}\} \neq \{bc^{-1},abc^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}c \neq b^2\\ \mathbf{bab \neq c}\end{array}\right.\\
\{cb^{-1},acb^{-1}\} \neq \{c^{-1},ac^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}c^2 \neq b\\ \mathbf{b \neq cac}\end{array}\right.\\
\{cb^{-1},acb^{-1}\} \neq \{bc^{-1},abc^{-1}\} & \Leftrightarrow & \left\{\begin{array}{l}\mathbf{cb^{-1} \neq bc^{-1}}\\ \mathbf{cb^{-1} \neq abc^{-1}}\end{array}\right.
\end{array}
$$

Without loss of generality we assume that $\{e,a\}$ and $\{b,ab\}$ belong to a triangle and also $\{e,a\}$ and $\{c,ac\}$ belong to a triangle. Since the involution $.a$ takes $\{b,ab\}$ and $\{c,ac\}$ onto each other, it follows that $\{ba,aba\} = \{c,ac\}$. Since ba can not be equal to c we have that $ba = ac$. Also, since $\{ca,aca\} = \{b,ab\}$ we have that $ca = ab$. This is equivalent to $\mathbf{c = aba}$. The two collineations mapping the point $\{e,a\}$ onto the point $\{b,ab\}$ are $.b$ and $.ab$. The inverse images of $\{e,a\}$ under these two collineations give the third points of both triangles in $\{e,a\}$: $\{b^{-1},ab^{-1}\}$ and $\{b^{-1}a,ab^{-1}a\}$. We distinguish two possibilities: either $\{e,a\}$, $\{b,ab\}$ and $\{b^{-1},ab^{-1}\}$ belong to a triangle or either $\{e,a\}$, $\{b,ab\}$ and $\{b^{-1}a,ab^{-1}a\}$ belong to a triangle.

1. $\{e,a\}$, $\{b,ab\}$ and $\{b^{-1},ab^{-1}\}$ belong to a triangle.

   The second triangle in $\{e,a\}$ is then given by the points $\{e,a\}$, $\{aba,ba\}$ and $\{b^{-1}a,ab^{-1}a\}$. The collineation $.b$ takes $\{b^{-1},ab^{-1}\}$ onto $\{e,a\}$ onto $\{b,ab\}$ onto $\{b^2,ab^2\}$, which is equal to $\{b^{-1},ab^{-1}\}$. It follows that b^2 is either b^{-1} or either ab^{-1}.

In the second case b^3 is equal to a which is impossible since $.b^3$ is not an involution. Hence $\mathbf{b^3 = e}$. It is easily seen that also $c^3 = (aba)^3 = e$.

The collineation $.b$ takes the point $\{e, a\}$ onto $\{b, ab\}$. The points $\{b, ab\}$, $\{b^2, ab^2\}$ and $\{abab, bab\}$ are then on one line. Just as the points $\{b, ab\}$, $\{ab^2ab, b^2ab\}$ and $\{bab^2ab, abab^2ab\}$. The seven points on the lines through $\{b, ab\}$ are all different. Requiring the right local structure, results in the following conditions:

$$\{abab, bab\} \neq \{abab^2, bab^2\} \quad \Leftrightarrow \quad \begin{cases} bab \neq abab^2 & \Leftrightarrow \quad aba \neq bab \\ bab \neq bab^2 & \Leftrightarrow \quad e \neq b \end{cases}$$

$$\{abab, bab\} \neq \{ab^2a, b^2a\} \quad \Leftrightarrow \quad \begin{cases} bab \neq ab^2a & \Leftrightarrow \quad \mathbf{b^2ab^2 \neq aba} \\ bab \neq b^2a & \Leftrightarrow \quad \mathbf{aba \neq b} \end{cases}$$

$$\{abab, bab\} \neq \{bab^2a, abab^2a\} \quad \Leftrightarrow \quad \begin{cases} bab \neq bab^2a & \Leftrightarrow \quad e \neq ba \\ bab \neq abab^2a & \Leftrightarrow \quad \mathbf{babab \neq aba} \end{cases}$$

$$\{ab^2ab, b^2ab\} \neq \{abab^2, bab^2\} \quad \Leftrightarrow \quad \begin{cases} b^2ab \neq abab^2 & \Leftrightarrow \quad b^2ab^2 \neq aba \\ b^2ab \neq bab^2 & \Leftrightarrow \quad b \neq aba \end{cases}$$

$$\{ab^2ab, b^2ab\} \neq \{ab^2a, b^2a\} \quad \Leftrightarrow \quad \begin{cases} b^2ab \neq ab^2a & \Leftrightarrow \quad aba \neq bab^2 \quad \Leftrightarrow \quad bab \neq aba \\ b^2ab \neq b^2a & \Leftrightarrow \quad b \neq e \end{cases}$$

$$\{ab^2ab, b^2ab\} \neq \{bab^2a, abab^2a\} \quad \Leftrightarrow \quad \begin{cases} b^2ab \neq bab^2a & \Leftrightarrow \quad aba \neq b^2ab^2 \\ b^2ab \neq abab^2a & \Leftrightarrow \quad \mathbf{b^2abab \neq aba} \end{cases}$$

$$\{bab^2ab, abab^2ab\} \neq \{abab^2, bab^2\} \quad \Leftrightarrow \quad \begin{cases} bab^2ab \neq abab^2 & \Leftrightarrow \quad bab^2ab^2 \neq aba \quad \Leftrightarrow \quad aba \neq babab \\ bab^2ab \neq bab^2 & \Leftrightarrow \quad b \neq a \end{cases}$$

$$\{bab^2ab, abab^2ab\} \neq \{ab^2a, b^2a\} \quad \Leftrightarrow \quad \begin{cases} bab^2ab \neq ab^2a & \Leftrightarrow \quad b^2abab^2 \neq aba \quad \Leftrightarrow \quad aba \neq babab \\ bab^2ab \neq b^2a & \Leftrightarrow \quad aba \neq bab \end{cases}$$

$$\{bab^2ab, abab^2ab\} \neq \{bab^2a, abab^2a\} \quad \Leftrightarrow \quad \begin{cases} bab^2ab \neq bab^2a & \Leftrightarrow \quad b \neq e \\ bab^2ab \neq abab^2a & \Leftrightarrow \quad \mathbf{bab^2abab \neq aba} \end{cases}$$

The two other lines in the point $\{aba, ba\}$ are given by $\{ba, aba\}$, $\{b^2a, ab^2a\}$, $\{ababa, baba\}$ and $\{ba, aba\}$, $\{ab^2aba, b^2aba\}$, $\{bab^2aba, abab^2aba\}$. To have the right local structure we need the following conditions:

$$\begin{array}{lcl}
\{ababa, baba\} \neq \{b^2, ab^2\} & \Leftrightarrow & \left\{\begin{array}{lllll} baba \neq b^2 & \Leftrightarrow & aba \neq b & & \\ baba \neq ab^2 & \Leftrightarrow & aba \neq b^2ab^2 & & \end{array}\right. \\
\{ababa, baba\} \neq \{abab^2, bab^2\} & \Leftrightarrow & \left\{\begin{array}{lllll} baba \neq abab^2 & \Leftrightarrow & babab \neq aba & & \\ baba \neq bab^2 & \Leftrightarrow & a \neq b & & \end{array}\right. \\
\{ababa, baba\} \neq \{bab^2a, abab^2a\} & \Leftrightarrow & \left\{\begin{array}{lllll} baba \neq bab^2a & \Leftrightarrow & e \neq b & & \\ baba \neq abab^2a & \Leftrightarrow & aba \neq bab & & \end{array}\right. \\
\{ab^2aba, b^2aba\} \neq \{b^2, ab^2\} & \Leftrightarrow & \left\{\begin{array}{lllll} b^2aba \neq b^2 & \Leftrightarrow & b \neq e & & \\ b^2aba \neq ab^2 & \Leftrightarrow & aba \neq bab^2 & \Leftrightarrow & bab \neq aba \end{array}\right. \\
\{ab^2aba, b^2aba\} \neq \{abab^2, bab^2\} & \Leftrightarrow & \left\{\begin{array}{lllll} b^2aba \neq abab^2 & \Leftrightarrow & b^2abab \neq aba & & \\ b^2aba \neq bab^2 & \Leftrightarrow & aba \neq b^2ab^2 & & \end{array}\right. \\
\{ab^2aba, b^2aba\} \neq \{bab^2a, abab^2a\} & \Leftrightarrow & \left\{\begin{array}{lllll} b^2aba \neq bab^2a & \Leftrightarrow & ba \neq ab & & \\ b^2aba \neq abab^2a & \Leftrightarrow & b^2a \neq abab & \Leftrightarrow & b^2ab^2 \neq aba \end{array}\right. \\
\{bab^2aba, abab^2aba\} \neq \{b^2, ab^2\} & \Leftrightarrow & \left\{\begin{array}{lllll} bab^2aba \neq b^2 & \Leftrightarrow & aba \neq bab & & \\ bab^2aba \neq ab^2 & \Leftrightarrow & aba \neq bab^2ab^2 & \Leftrightarrow & babab \neq aba \end{array}\right. \\
\{bab^2aba, abab^2aba\} \neq \{abab^2, bab^2\} & \Leftrightarrow & \left\{\begin{array}{lllll} bab^2aba \neq abab^2 & \Leftrightarrow & bab^2abab \neq aba & & \\ bab^2aba \neq bab^2 & \Leftrightarrow & b \neq e & & \end{array}\right. \\
\{bab^2aba, abab^2aba\} \neq \{bab^2a, abab^2a\} & \Leftrightarrow & \left\{\begin{array}{lllll} bab^2aba \neq bab^2a & \Leftrightarrow & b \neq a & & \\ bab^2aba \neq abab^2a & \Leftrightarrow & bab^2a \neq abab & \Leftrightarrow & aba \neq babab \end{array}\right.
\end{array}$$

The third line through $\{b^2, ab^2\}$ contains the points $\{ab^2ab^2, b^2ab^2\}$ and $\{bab^2ab^2, abab^2ab^2\}$. Those two points need to be different from the points $\{ab^2a, b^2a\}$ and $\{bab^2a, abab^2a\}$:

$$\begin{array}{lcl}
\{ab^2ab^2, b^2ab^2\} \neq \{ab^2a, b^2a\} & \Leftrightarrow & \left\{\begin{array}{lllll} b^2ab^2 \neq ab^2a & \Leftrightarrow & bab \neq aba & & \\ b^2ab^2 \neq b^2a & \Leftrightarrow & b^2 \neq e & & \end{array}\right. \\
\{ab^2ab^2, b^2ab^2\} \neq \{bab^2a, abab^2a\} & \Leftrightarrow & \left\{\begin{array}{lllll} b^2ab^2 \neq bab^2a & \Leftrightarrow & bab^2 \neq ab^2a & \Leftrightarrow & bab \neq aba \\ b^2ab^2 \neq abab^2a & \Leftrightarrow & babab \neq aba & & \end{array}\right. \\
\{bab^2ab^2, abab^2ab^2\} \neq \{ab^2a, b^2a\} & \Leftrightarrow & \left\{\begin{array}{lllll} bab^2ab^2 \neq ab^2a & \Leftrightarrow & aba \neq b^2abab & & \\ bab^2ab^2 \neq b^2a & \Leftrightarrow & ab^2ab^2 \neq ba & \Leftrightarrow & b^2ab^2 \neq aba \end{array}\right. \\
\{bab^2ab^2, abab^2ab^2\} \neq \{bab^2a, abab^2a\} & \Leftrightarrow & \left\{\begin{array}{lllll} bab^2ab^2 \neq bab^2a & \Leftrightarrow & b^2 \neq e & & \\ bab^2ab^2 \neq abab^2a & \Leftrightarrow & aba \neq bab^2abab & & \end{array}\right.
\end{array}$$

The line containing the points $\{abab^2, bab^2\}$, $\{b^2ab^2, ab^2ab^2\}$ and $\{ababab^2, babab^2\}$ and the line through $\{bab^2, abab^2\}$, $\{ab^2abab^2, b^2abab^2\}$ and $\{bab^2abab^2, abab^2abab^2\}$ are the two other lines in the point $\{abab^2, bab^2\}$. Since the collineation $bab^2 \in G$ maps the three lines through $\{e, a\}$ onto the three lines in $\{abab^2, bab^2\}$, the seven points on the three lines through $\{abab^2, bab^2\}$ are mutually different. To obtain the right local structure we have to require that the four points $\{b^2ab^2, ab^2ab^2\}$, $\{ababab^2, babab^2\}$, $\{ab^2abab^2, b^2abab^2\}$ and $\{bab^2abab^2, abab^2abab^2\}$ are different from the points $\{ab^2a, b^2a\}$ and $\{bab^2a, abab^2a\}$. This results in the following conditions:

$$\{b^2ab^2, ab^2ab^2\} \neq \{ab^2a, b^2a\} \Leftrightarrow \begin{cases} b^2ab^2 \neq ab^2a \Leftrightarrow bab \neq aba \\ b^2ab^2 \neq b^2a \Leftrightarrow b^2 \neq e \end{cases}$$

$$\{b^2ab^2, ab^2ab^2\} \neq \{bab^2a, abab^2a\} \Leftrightarrow \begin{cases} b^2ab^2 \neq bab^2a \Leftrightarrow bab^2 \neq ab^2a \Leftrightarrow bab \neq aba \\ b^2ab^2 \neq abab^2a \Leftrightarrow babab \neq aba \end{cases}$$

$$\{ababab^2, babab^2\} \neq \{ab^2a, b^2a\} \Leftrightarrow \begin{cases} babab^2 \neq ab^2a \Leftrightarrow bab \neq ab^2aba \Leftrightarrow babab \neq aba \\ babab^2 \neq b^2a \Leftrightarrow abab^2 \neq ba \Leftrightarrow aba \neq bab \end{cases}$$

$$\{ababab^2, babab^2\} \neq \{bab^2a, abab^2a\} \Leftrightarrow \begin{cases} babab^2 \neq bab^2a \Leftrightarrow ab^2 \neq ba \Leftrightarrow \mathbf{b^2 \neq aba} \\ babab^2 \neq abab^2a \Leftrightarrow \mathbf{aba \neq bab^2abab^2} \end{cases}$$

$$\{ab^2abab^2, b^2abab^2\} \neq \{ab^2a, b^2a\} \Leftrightarrow \begin{cases} b^2abab^2 \neq ab^2a \Leftrightarrow \mathbf{aba \neq bab^2ab} \\ b^2abab^2 \neq b^2a \Leftrightarrow bab^2 \neq e \Leftrightarrow a \neq e \end{cases}$$

$$\{ab^2abab^2, b^2abab^2\} \neq \{bab^2a, abab^2a\} \Leftrightarrow \begin{cases} b^2abab^2 \neq bab^2a \Leftrightarrow babab^2 \neq ab^2a \Leftrightarrow bab^2ab^2 \neq aba \Leftrightarrow aba \neq babab \\ b^2abab^2 \neq abab^2a \Leftrightarrow bab^2abab \neq aba \end{cases}$$

$$\{bab^2abab^2, abab^2abab^2\} \neq \{ab^2a, b^2a\} \Leftrightarrow \begin{cases} bab^2abab^2 \neq ab^2a \Leftrightarrow bab^2abab^2 \neq aba \\ bab^2abab^2 \neq b^2a \Leftrightarrow ab^2abab^2 \neq ba \Leftrightarrow aba \neq bab^2ab^2 \Leftrightarrow babab \neq aba \end{cases}$$

$$\{bab^2abab^2, abab^2abab^2\} \neq \{bab^2a, abab^2a\} \Leftrightarrow \begin{cases} bab^2abab^2 \neq bab^2a \Leftrightarrow bab^2 \neq e \Leftrightarrow a \neq e \\ bab^2abab^2 \neq abab^2a \Leftrightarrow \mathbf{bab^2abab^2ab \neq aba} \end{cases}$$

We have found the following conditions on the group elements e, a and b to obtain the right local structure in the point e: element a has order 2, element b has order 3, element ab has order bigger than 3 and

$$\begin{aligned} &\mathbf{aba \neq bab} \\ &\mathbf{aba \neq bab^2ab} \\ &\mathbf{aba \neq babab} \\ &\mathbf{aba \neq b^2abab} \\ &\mathbf{aba \neq bab^2abab} \\ &\mathbf{aba \neq bab^2abab^2ab} \end{aligned}$$

Remark that $b^2 \neq a$ follows from $a^2 = e$ and $b^3 = e$. Indeed suppose that $b^2 = a$ then $b = b^4 = a^2 = e$. Also, $aba \neq b$ since $aba = b$ gives $ab = ba = ac$ from which $b = c$, a contradiction. We have that

$$aba \neq bab^2abab \Leftrightarrow b^2abab^2ab^2 \neq ab^2a \Leftrightarrow abab^2ab^2a \neq bab^2 \Leftrightarrow ab^2ab^2a \neq b^2abab^2 \Leftrightarrow ab^2a \neq bab^2abab^2 \Leftrightarrow aba \neq bab^2abab^2$$

Because of transitivity of the group it follows that we have the right local structure in each point.

2. $\{e, a\}$, $\{b, ab\}$ and $\{b^{-1}a, ab^{-1}a\}$ belong to a triangle.

   The second triangle in $\{e, a\}$ is then given by the points $\{e, a\}$, $\{aba, ba\}$ and $\{b^{-1}, ab^{-1}\}$. The collineation $.ab$ takes $\{b^{-1}a, ab^{-1}a\}$ onto $\{e, a\}$ onto $\{ab, b\}$ onto $\{abab, bab\}$, which is equal to $\{b^{-1}a, ab^{-1}a\}$. It follows that bab is either $b^{-1}a$ or either $ab^{-1}a$. In the first case $babab$ is equal to e or $(ab)^3 = a$ which is impossible since $.(ab)^3$ is not an involution. Hence $\mathbf{(ab)^3 = e}$. It is easily seen that also $(ac)^3 = (ba)^3 = e$. The lines through $\{e, a\}$ are then given by:

$$\{e, a\} \; \{b, ab\} \; \{aba, ba\}$$

$$\{e, a\} \; \{ababa, baba\} \; \{ab^2aba, b^2aba\}$$

$$\{e,a\}\ \{bab,abab\}\ \{b^2ab,ab^2ab\}$$

The collineation $.b$ takes the point $\{e,a\}$ onto $\{b,ab\}$. The points $\{b,ab\}$, $\{b^2,ab^2\}$ and $\{abab,bab\}$ are then on one line. Just as the points $\{b,ab\}$, $\{bab^2,abab^2\}$ and $\{b^2ab^2,ab^2ab^2\}$. The seven points on the lines through $\{b,ab\}$ are all different. Requiring the right local structure, results in the following conditions:

$$\{b^2,ab^2\}\neq\{ababa,baba\} \Leftrightarrow \begin{cases} b^2\neq ababa \Leftrightarrow b^2\neq b^{-1} \Leftrightarrow \mathbf{b^3\neq e} \\ b^2\neq baba \Leftrightarrow b^3a\neq e \Leftrightarrow \mathbf{b^3\neq a} \end{cases}$$

$$\{b^2,ab^2\}\neq\{ab^2aba,b^2aba\} \Leftrightarrow \begin{cases} b^2\neq ab^2aba \Leftrightarrow b^3aba\neq ab^2a \Leftrightarrow \mathbf{b^3\neq aba} \\ b^2\neq b^2aba \Leftrightarrow e\neq aba \Leftrightarrow b\neq e \end{cases}$$

$$\{b^2,ab^2\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} b^2\neq b^2ab \Leftrightarrow a\neq b \\ b^2\neq ab^2ab \Leftrightarrow b\neq ab^2a \Leftrightarrow bab\neq b^3a \Leftrightarrow e\neq b^4a \Leftrightarrow \mathbf{b^4\neq a} \end{cases}$$

$$\{bab^2,abab^2\}\neq\{ababa,baba\} \Leftrightarrow \begin{cases} bab^2\neq ababa \Leftrightarrow bab^2\neq b^{-1} \Leftrightarrow b^2ab^2\neq e \Leftrightarrow b^2\neq ab^{-2} \Leftrightarrow b^4\neq a \\ bab^2\neq baba \Leftrightarrow b\neq a \end{cases}$$

$$\{bab^2,abab^2\}\neq\{ab^2aba,b^2aba\} \Leftrightarrow \begin{cases} bab^2\neq ab^2aba \Leftrightarrow \mathbf{bab^3\neq aba} \\ bab^2\neq b^2aba \Leftrightarrow ab^2\neq baba \Leftrightarrow ab^3a\neq e \Leftrightarrow b^3\neq e \end{cases}$$

$$\{bab^2,abab^2\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} bab^2\neq b^2ab \Leftrightarrow ab\neq ba \Leftrightarrow b\neq aba \Leftrightarrow b^2ab\neq e \Leftrightarrow b^2\neq b^{-1}a \Leftrightarrow b^3\neq a \\ bab^2\neq ab^2ab \Leftrightarrow bab\neq ab^2a \Leftrightarrow ababa\neq b^2 \Leftrightarrow e\neq b^3 \end{cases}$$

$$\{b^2ab^2,ab^2ab^2\}\neq\{ababa,baba\} \Leftrightarrow \begin{cases} b^2ab^2\neq ababa \Leftrightarrow \mathbf{b^2ab^3\neq e} \\ b^2ab^2\neq baba \Leftrightarrow \mathbf{bab^2\neq aba} \end{cases}$$

$$\{b^2ab^2,ab^2ab^2\}\neq\{ab^2aba,b^2aba\} \Leftrightarrow \begin{cases} b^2ab^2\neq ab^2aba \Leftrightarrow b^2ab^3a\neq ab \Leftrightarrow \mathbf{b^2ab^3\neq aba} \\ b^2ab^2\neq b^2aba \Leftrightarrow b\neq a \end{cases}$$

$$\{b^2ab^2,ab^2ab^2\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} b^2ab^2\neq b^2ab \Leftrightarrow b\neq e \\ b^2ab^2\neq ab^2ab \Leftrightarrow b^2ab\neq ab^2a \Leftrightarrow b^2ab^2aba\neq ab \Leftrightarrow \mathbf{b^2ab^2ab\neq aba} \end{cases}$$

The two other lines in the point $\{aba,ba\}$ are given by $\{ba,aba\}$, $\{b^2a,ab^2a\}$, $\{ababa,baba\}$ and $\{ba,aba\}$, $\{bab^2a,abab^2a\}$, $\{b^2ab^2a,ab^2ab^2a\}$. To have the right local structure we need the following conditions:

$$\{b^2a,ab^2a\}\neq\{ab^2aba,b^2aba\} \Leftrightarrow \begin{cases} b^2a\neq ab^2aba \Leftrightarrow b^2aba\neq ab \Leftrightarrow b\neq ab^2a \Leftrightarrow b^4\neq a \\ b^2a\neq b^2aba \Leftrightarrow e\neq ba \Leftrightarrow a\neq b \end{cases}$$

$$\{b^2a,ab^2a\}\neq\{bab,abab\} \Leftrightarrow \begin{cases} b^2a\neq bab \Leftrightarrow ba\neq ab \Leftrightarrow b\neq aba \Leftrightarrow b^3\neq a \\ b^2a\neq abab \Leftrightarrow b^3\neq e \end{cases}$$

$$\{b^2a,ab^2a\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} b^2a\neq b^2ab \Leftrightarrow e\neq b \\ b^2a\neq ab^2ab \Leftrightarrow b^3ab\neq ab^2 \Leftrightarrow b^3a\neq ab \Leftrightarrow b^3\neq aba \end{cases}$$

$$\{bab^2a,abab^2a\}\neq\{ab^2aba,b^2aba\} \Leftrightarrow \begin{cases} bab^2a\neq ab^2aba \Leftrightarrow bab\neq ab^2a \Leftrightarrow baba\neq ab^2 \Leftrightarrow e\neq ab^3a \Leftrightarrow e\neq b^3 \\ bab^2a\neq b^2aba \Leftrightarrow ab\neq ba \Leftrightarrow b\neq aba \Leftrightarrow b^3\neq a \end{cases}$$

$$\{bab^2a,abab^2a\}\neq\{bab,abab\} \Leftrightarrow \begin{cases} bab^2a\neq bab \Leftrightarrow ba\neq e \\ bab^2a\neq abab \Leftrightarrow bab^3\neq e \Leftrightarrow b^3\neq ab^{-1} \Leftrightarrow b^4\neq a \end{cases}$$

$$\{bab^2a,abab^2a\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} bab^2a\neq b^2ab \Leftrightarrow ab^2a\neq bab \Leftrightarrow b^3\neq e \\ bab^2a\neq ab^2ab \Leftrightarrow bab^3a\neq ab \Leftrightarrow bab^3\neq aba \end{cases}$$

$$\{b^2ab^2a,ab^2ab^2a\}\neq\{ab^2aba,b^2aba\} \Leftrightarrow \begin{cases} b^2ab^2a\neq ab^2aba \Leftrightarrow b^2ab\neq ab^2a \Leftrightarrow b\neq ab^3a \Leftrightarrow aba\neq b^3 \\ b^2ab^2a\neq b^2aba \Leftrightarrow ba\neq a \Leftrightarrow b\neq e \end{cases}$$

$$\{b^2ab^2a,ab^2ab^2a\}\neq\{bab,abab\} \Leftrightarrow \begin{cases} b^2ab^2a\neq bab \Leftrightarrow bab^2a\neq ab \Leftrightarrow bab^2\neq aba \\ b^2ab^2a\neq abab \Leftrightarrow b^2ab^3\neq e \end{cases}$$

$$\{b^2ab^2a,ab^2ab^2a\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} b^2ab^2a\neq b^2ab \Leftrightarrow ba\neq e \\ b^2ab^2a\neq ab^2ab \Leftrightarrow b^2ab^3a\neq ab \Leftrightarrow b^2ab^3\neq aba \end{cases}$$

The third line through $\{ababa,baba\}$ contains the points $\{bab^2aba,abab^2aba\}$ and $\{b^2ab^2aba,ab^2ab^2aba\}$. Those two points need to be different from the points $\{bab,abab\}$ and $\{b^2ab,ab^2ab\}$:

$$\{bab^2aba,abab^2aba\}\neq\{bab,abab\} \Leftrightarrow \begin{cases} bab^2aba\neq bab \Leftrightarrow baba\neq e \Leftrightarrow e\neq ba \\ bab^2aba\neq abab \Leftrightarrow bab\neq abab^2a \Leftrightarrow baba\neq abab^2 \Leftrightarrow e\neq abab^3a \Leftrightarrow a\neq b^4 \end{cases}$$

$$\{bab^2aba,abab^2aba\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} bab^2aba\neq b^2ab \Leftrightarrow ab^2aba\neq bab \Leftrightarrow ab\neq bab^2a \Leftrightarrow aba\neq bab^2 \\ bab^2aba\neq ab^2ab \Leftrightarrow bab\neq ab^2ab^2a \Leftrightarrow e\neq ab^2ab^3a \Leftrightarrow e\neq b^2ab^3 \end{cases}$$

$$\{b^2ab^2aba,ab^2ab^2aba\}\neq\{bab,abab\} \Leftrightarrow \begin{cases} b^2ab^2aba\neq bab \Leftrightarrow bab^2aba\neq ab \Leftrightarrow bab\neq ab^2a \Leftrightarrow e\neq ab^3a \Leftrightarrow e\neq b^3 \\ b^2ab^2aba\neq abab \Leftrightarrow b^2ab\neq abab^2a \Leftrightarrow b^2aba\neq abab^2 \Leftrightarrow b\neq abab^3a \Leftrightarrow aba\neq bab^3 \end{cases}$$

$$\{b^2ab^2aba,ab^2ab^2aba\}\neq\{b^2ab,ab^2ab\} \Leftrightarrow \begin{cases} b^2ab^2aba\neq b^2ab \Leftrightarrow baba\neq e \Leftrightarrow e\neq ba \\ b^2ab^2aba\neq ab^2ab \Leftrightarrow b^2ab\neq ab^2ab^2a \Leftrightarrow b\neq ab^2ab^3a \Leftrightarrow aba\neq b^2ab^3 \end{cases}$$

The line containing the points $\{ab^2aba, b^2aba\}$, $\{b^3aba, ab^3aba\}$ and $\{abab^2aba, bab^2aba\}$ and the line through $\{ab^2aba, b^2aba\}$, $\{bab^3aba, abab^3aba\}$ and $\{b^2ab^3aba, ab^2ab^3aba\}$ are the two other lines in the point $\{ab^2aba, b^2aba\}$. Since the collineation $b^2aba \in G$ maps the three lines through $\{e, a\}$ onto the three lines in $\{ab^2aba, b^2aba\}$, the seven points on the three lines through $\{ab^2aba, b^2aba\}$ are mutually different. To obtain the right local structure we have to require that the four points $\{b^3aba, ab^3aba\}$, $\{abab^2aba, bab^2aba\}$, $\{bab^3aba, abab^3aba\}$ and $\{b^2ab^3aba, ab^2ab^3aba\}$ are different from the points $\{bab, abab\}$ and $\{b^2ab, ab^2ab\}$. This results in the following conditions:

$$
\begin{array}{lcl}
\{b^3aba, ab^3aba\} \neq \{bab, abab\} & \Leftrightarrow & \begin{cases} b^3aba \neq bab \Leftrightarrow b^2aba \neq ab \Leftrightarrow b \neq ab^2a \Leftrightarrow b^4 \neq a \\ b^3aba \neq abab \Leftrightarrow b^2 \neq abab^2a \Leftrightarrow ab^3 \neq ba \Leftrightarrow b^3 \neq aba \end{cases} \\
\{b^3aba, ab^3aba\} \neq \{b^2ab, ab^2ab\} & \Leftrightarrow & \begin{cases} b^3aba \neq b^2ab \Leftrightarrow baba \neq ab \Leftrightarrow e \neq ab^2a \Leftrightarrow b^2 \neq e \\ b^3aba \neq ab^2ab \Leftrightarrow b^3ab^2a \neq ab \Leftrightarrow \mathbf{b^3ab^2 \neq aba} \end{cases} \\
\{abab^2aba, bab^2aba\} \neq \{bab, abab\} & \Leftrightarrow & \begin{cases} bab^2aba \neq bab \Leftrightarrow baba \neq e \Leftrightarrow e \neq ba \\ bab^2aba \neq abab \Leftrightarrow bab \neq abab^2a \Leftrightarrow baba \neq abab^2 \Leftrightarrow e \neq abab^3a \Leftrightarrow a \neq b^4 \end{cases} \\
\{abab^2aba, bab^2aba\} \neq \{b^2ab, ab^2ab\} & \Leftrightarrow & \begin{cases} bab^2aba \neq b^2ab \Leftrightarrow ab^2aba \neq bab \Leftrightarrow ab \neq bab^2a \Leftrightarrow aba \neq bab^2 \\ bab^2aba \neq ab^2ab \Leftrightarrow bab^2ab^2a \neq ab \Leftrightarrow \mathbf{bab^2ab^2 \neq aba} \end{cases} \\
\{bab^3aba, abab^3aba\} \neq \{bab, abab\} & \Leftrightarrow & \begin{cases} bab^3aba \neq bab \Leftrightarrow baba \neq e \\ bab^3aba \neq abab \Leftrightarrow bab^2 \neq abab^2a \Leftrightarrow b \neq ab^2ab^2a \Leftrightarrow \mathbf{aba \neq b^2ab^2} \end{cases} \\
\{bab^3aba, abab^3aba\} \neq \{b^2ab, ab^2ab\} & \Leftrightarrow & \begin{cases} bab^3aba \neq b^2ab \Leftrightarrow ab^3aba \neq bab \Leftrightarrow ab^2 \neq bab^2a \Leftrightarrow ab^3 \neq ba \Leftrightarrow b^3 \neq aba \\ bab^3aba \neq ab^2ab \Leftrightarrow bab^2 \neq ab^2ab^2a \Leftrightarrow b \neq ab^3ab^2a \Leftrightarrow aba \neq b^3ab^2 \end{cases} \\
\{b^2ab^3aba, ab^2ab^3aba\} \neq \{bab, abab\} & \Leftrightarrow & \begin{cases} b^2ab^3aba \neq bab \Leftrightarrow bab^3aba \neq ab \Leftrightarrow bab^2 \neq ab^2a \Leftrightarrow abab^2ab^2 \neq ba \Leftrightarrow bab^2ab^2 \neq aba \\ b^2ab^3aba \neq abab \Leftrightarrow b^2ab^2 \neq abab^2a \Leftrightarrow ab^3ab^2 \neq ba \Leftrightarrow b^3ab^2 \neq aba \end{cases} \\
\{b^2ab^3aba, ab^2ab^3aba\} \neq \{b^2ab, ab^2ab\} & \Leftrightarrow & \begin{cases} b^2ab^3aba \neq b^2ab \Leftrightarrow baba \neq e \Leftrightarrow e \neq ba \\ b^2ab^3aba \neq ab^2ab \Leftrightarrow b^2ab^3ab^2a \neq ab \Leftrightarrow \mathbf{b^2ab^3ab^2 \neq aba} \end{cases}
\end{array}
$$

Remark the following equivalences:

$$
\begin{array}{lclclclclcl}
b^2 \neq a & \Leftrightarrow & b \neq ab^{-1} & \Leftrightarrow & b \neq baba & \Leftrightarrow & e \neq aba & \Leftrightarrow & e \neq b & & \\
b^2 \neq aba & \Leftrightarrow & b^3ab \neq e & \Leftrightarrow & b^4 \neq a & & & & & & \\
baba \neq a & \Leftrightarrow & e \neq aba & \Leftrightarrow & e \neq b & & & & & & \\
baba \neq ab & \Leftrightarrow & e \neq ab^2a & \Leftrightarrow & e \neq b^2 & & & & & & \\
abab \neq a & \Leftrightarrow & e \neq b & & & & & & & & \\
abab \neq ba & \Leftrightarrow & e \neq b^2 & & & & & & & & \\
(aba)^2 \neq e & \Leftrightarrow & ab^2a \neq e & \Leftrightarrow & b^2 \neq e & & & & & & \\
(aba)^2 \neq a & \Leftrightarrow & ab^2a \neq a & \Leftrightarrow & b^2 \neq a & & & & & & \\
(aba)^2 \neq b & \Leftrightarrow & ab^2a \neq b & \Leftrightarrow & b^3a \neq bab & \Leftrightarrow & b^4a \neq e & \Leftrightarrow & b^4 \neq a & & \\
(aba)^2 \neq ab & \Leftrightarrow & ab^2a \neq ab & \Leftrightarrow & ba \neq e & \Leftrightarrow & b \neq a & & & & \\
bab \neq aba & \Leftrightarrow & e \neq ab^2a & \Leftrightarrow & b^2 \neq e & & & & & & \\
b \neq ababa & \Leftrightarrow & b^2 \neq e & & & & & & & & \\
abab^{-1} \neq bab^{-1}a & \Leftrightarrow & ab^2aba \neq b^2ab & \Leftrightarrow & ab \neq b^2ab^2a & \Leftrightarrow & bab \neq b^3ab^2a & \Leftrightarrow & e \neq b^3ab^3a & \Leftrightarrow & b^3ab^3 \neq a \\
abab^{-1} \neq abab^{-1}a & \Leftrightarrow & ab^2aba \neq ab^2ab & \Leftrightarrow & a \neq e & & & & & & \\
b^3 \neq aba & \Leftrightarrow & bab^4 \neq e & & & & & & & & \\
bab^3 \neq aba & \Leftrightarrow & b^2ab^4a \neq e & \Leftrightarrow & b^2ab^4 \neq a & & & & & & \\
b^2ab^3 \neq e & \Leftrightarrow & bab^3 \neq b^{-1} & \Leftrightarrow & bab^4 \neq e & & & & & & \\
bab^2 \neq aba & \Leftrightarrow & b \neq ab^2a & \Leftrightarrow & bab \neq b^3a & \Leftrightarrow & e \neq b^4a & \Leftrightarrow & b^4 \neq a & & \\
b^2ab^3 \neq aba & \Leftrightarrow & bab^3ab^3 \neq e & \Leftrightarrow & b^3ab^3 \neq baba & \Leftrightarrow & b^3ab^4a \neq e & \Leftrightarrow & b^3ab^4 \neq a & & \\
b^2ab^2ab \neq aba & \Leftrightarrow & b^3ab^2ab^2a \neq e & \Leftrightarrow & b^3ab^2ab^2 \neq a & \Leftrightarrow & b^2ab^2 \neq ab^{-3}a & & & & \\
b^3ab^2 \neq aba & \Leftrightarrow & b^3ab^3ab \neq e & \Leftrightarrow & b^3ab^3 \neq abab & \Leftrightarrow & ab^4ab^3 \neq e & \Leftrightarrow & ab^4a \neq b^{-3} & \Leftrightarrow & b^3ab^4 \neq a \\
bab^2ab^2 \neq aba & \Leftrightarrow & b^2ab^2ab^3a \neq e & \Leftrightarrow & b^2ab^2ab^3 \neq a & \Leftrightarrow & b^2ab^2 \neq ab^{-3}a & & & & \\
aba \neq b^2ab^2 & \Leftrightarrow & e \neq b^2ab^3ab & \Leftrightarrow & a \neq b^3ab^3 & & & & & & \\
b^2ab^3ab^2 \neq aba & \Leftrightarrow & b^3ab^3ab^3a \neq e & \Leftrightarrow & b^3ab^3ab^3 \neq a & & & & & &
\end{array}
$$

We have found the following conditions on the group elements e, a and b to obtain the right local structure in the point e: element a has order 2, element b has order bigger than 3, element ab has order 3 and

$$\mathbf{a \neq b^4}$$
$$\mathbf{a \neq b^5}$$
$$\mathbf{a \neq b^3ab^3}$$
$$\mathbf{a \neq b^2ab^4}$$
$$\mathbf{a \neq b^3ab^4}$$
$$\mathbf{a \neq b^3ab^3ab^3}$$

Remark that if $a = b^3ab^2ab^2$, then $b^3ab^2 = ab^{-2}a = babaabab = bab^2ab$ or $b^2ab = ab^2a$. It follows that $b = ab^{-2}ab^2a = bab^2ab^3a$, consequently $b^2ab^3 = e$ or $a = b^5$ which is a contradiction. Also, suppose that $a = b^3$ then $b^3ab^3 = a^3 = a$ which is in contradiction with the above conditions. Because of transitivity of the group it follows that we have the right local structure in each point.

5 Configuration 13

5.1 A construction of an infinite class $\mathcal{S}_{(r,s)}$

All members of the infinite class we will describe are quotients of the honeycomb geometry. We give an explicit construction based on the incidence graph. Let $\mathcal{G}$ be the incidence graph of the infinite example, which is the (bipartite) graph obtained from the tiling of the real Euclidean plane into regular hexagons. The (incidence graph of the) members of the infinite class will be described as quotients of this graph.

The parameters r and s in $\mathcal{S}_{(r,s)}$ are integers with $r \geq 0$.

We define a coordinate system for the real Euclidean plane as follows. We choose an arbitrary vertex of $\mathcal{G}$ as the origin $(0,0)$. The unit vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are chosen in such a way that they form an angle of sixty degrees and the end points are vertices of $\mathcal{G}$ at graph-theoretical distance 2 from $(0,0)$ contained in a common hexagon through the origin.

The points of $\mathcal{S}_{(r,s)}$ are the ordered pairs (i,j), with i, j integers and with identification of all pairs $(i,j) + k(r,s) = (i+kr, j+ks)$ with k an integer. The lines of the geometry are the 3-sets $\{(i,j),(i+1,j),(i+1,j-1)\}$ consisting of the three points incident with the line, and where for each point the above identification rule holds. The above line can be identified with the vertex with coordinates $(i+2/3, j-1/3)$.

Let m be the Euclidean distance between the origin and the vertex with coordinates (r,s). Consider the strip S of all vertices of the graph between the X-axis and a line $y=s$ parallel to the X-axis through the vertex (r,s). The X-axis itself is contained in the strip, the line $y=s$ not. Every point can be represented by a pair (i,j) with coordinates i, j in the strip S. Now, if there were two representatives for a point in S, then one would be on a line through the other parallel to $(0,0)(r,s)$ at distance m from each other. Since this is impossible, every point of the geometry has a unique representation (i,j) in the strip S which is therefore called a *fundamental strip*.

5.2 A construction of an infinite class $\mathcal{R}_{(r,s)}$

All members of the infinite class we will describe are quotients of the honeycomb geometry. We give an explicit construction based on the incidence graph. Let $\mathcal{G}$ be the incidence graph of the infinite example, which is the (bipartite) graph obtained from the tiling of the real Euclidean plane into regular hexagons. The (incidence graph of the) members of the infinite class will be described as quotients of this graph.

The parameters r and s in $\mathcal{R}_{(r,s)}$ are non negative integers with $r \geq s$ and $r+s \geq 3$.

We define a coordinate system for the real Euclidean plane as follows. We choose an arbitrary vertex of $\mathcal{G}$ as the origin $(0,0)$. The unit vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are chosen in such a way that they form an angle of sixty degrees and the end points are vertices of $\mathcal{G}$ at graph-theoretical distance 2 from $(0,0)$ contained in a common hexagon through the origin.

The points of $\mathcal{R}_{(r,s)}$ are the ordered pairs (i,j), with i, j integers and with identification of all pairs $(i,j)+k(r,s)+l(r,-r-s) = (i+kr+lr, j+ks-lr-ls)$ with k, l integers. The lines of the geometry are the 3-sets $\{(i,j),(i+1,j),(i+1,j-1)\}$ consisting of the three points incident with the line, and where for each point the above identification rule holds. The above line can be identified with the vertex with coordinates $(i+2/3, j-1/3)$.

Let m be the Euclidean distance between the origin and the vertex with coordinates (r,s). By applying the cosine rule in the triangle $(0,0)(r,0)(r,s)$ we find that $m^2 = r^2+rs+s^2$. It is easy to see that the quadrangle formed by the vertices $(0,0)(r,s)(2r,-r)(r,-r-s)$ is a rhombus with length of the sides equal to m. Every point can be represented by a pair (i,j) with coordinates i, j in the rhombus without the line segments $[(r,s)(2r,-r)]$ and $[(2r,-r)(r,-r-s)]$. We will note this domain as $\mathcal{D}$. Now, if there were two representatives for a point in $\mathcal{D}$, then one would be on a line through the other parallel to $(0,0)(r,s)$ or to $(0,0)(r,-r-s)$ at distance m from each other. Since this is impossible, every point of the geometry has a unique representation (i,j) in the domain $\mathcal{D}$ which is therefore called a *fundamental domain*.

In order to count the number of points in the geometry, we have to count the number of vertices corresponding to points in the fundamental domain $\mathcal{D}$. The area of $\mathcal{D}$ is equal to $\frac{\sqrt{3}}{2}r^2 + \sqrt(3)rs$. The area of one hexagon is equal to $\frac{\sqrt{3}}{2}$. The number of hexagons in $\mathcal{D}$ is r^2+2rs. We can assume that every hexagon contributes one vertex representing a point of the geometry and one vertex representing a line. Hence, the geometry $\mathcal{R}_{(r,s)}$ contains r^2+2rs points and also r^2+2rs lines.

Remark that for $s=0$ we obtain the square geometries and for $r=s$ we obtain the triple square geometries.

5.3 Classification of collineation groups acting point and line transitive, but not flag transitive on the honeycomb geometry

A group of collineations of the honeycomb geometry acting point and line transitive but not flag transitive is equivalent to a group of automorphisms of its incidence graph fixing the two partition sets, acting transitive on the vertices in each partition set and for which the stabilizer of any vertex is not transitive onto its neighbors.

We distinguish the case of a sharp point (and hence sharp line) transitive group and the case of a non sharp point (and hence non sharp line) transitive group.

The only isometries for the Euclidean plane are translations, rotations, reflections and glide reflections (where we suppose that the translation vector is in the direction of the reflection axis and is different from the zero vector).

Suppose that a and b are two vertices belonging to the same partition set of the incidence graph of the honeycomb geometry and belonging to a same hexagon. Let c be the vertex incident with both a and b and d the vertex at graph-theoretical distance 2 from both a and the third point f of the hexagon through a and b different from b.

- Sharp point transitive collineation group G

  We consider the following cases:

  - A unique rotation r maps a onto b. Remark that its center is the center of a hexagon. Suppose that $r \in G$. From now on we denote by r^+, resp. r^- the rotation over $+120$, resp. -120 degrees.
    * The rotation r' taking a onto d belongs to G. Remark that the center of this rotation is the center of the hexagon containing a and d. It is easily seen that $G =< r, r' >$ is a group satisfying our conditions.
    * The translation with vector $\overrightarrow{ad}$ belongs to G. Since composition of the rotation r and this translation gives a rotation with center a vertex of a hexagon, we get a contradiction.
    * There are exactly two glide reflections taking a onto d one with axis parallel to edge $\{a, e\}$ and one with axis parallel to edge $\{d, e\}$ where e is the vertex adjacent to a and d. Suppose that a glide reflection st taking a onto d belongs to G. If the glide reflection has axis parallel to edge $\{d, e\}$ then $r^-(st) = r^-(ts)$ is a reflection which is in contradiction with sharp point transitivity. If its axis is parallel to edge $\{a, e\}$ then $r^-(st)$ is a glide reflection $s't'$. Considering st followed by $s't'$ gives a rotation with center the center of the hexagon through a and d. Hence the group G can not be sharp point transitive.
  - A unique translation t maps a onto b. Suppose that $t \in G$.
    * The translation t' taking a onto f belongs to G. It is easily seen that $G =< t, t' >$ is a group satisfying our conditions.
    * The rotation with center the center of the hexagon through a and b belongs to G. But this is in contradiction with sharp point transitivity.
    * There are exactly two glide reflections taking a onto f one with axis parallel to edge $\{a, g\}$ and one with axis parallel to edge $\{f, g\}$ where g is the vertex

adjacent to a and f. Suppose that a glide reflection st' taking a onto f belongs to G. If the glide reflection has axis parallel to edge $\{f, g\}$ then $(st')(st')$ is a translation $2t'$ with vector parallel to the axis of s and with size twice the size of the translation vector of t'. But then also $2t' - t$ is a translation taking a onto f which is in contradiction with sharp point transitivity. If its axis is parallel to edge $\{a, g\}$ then it is easily seen that $G =< t, st' >$ satisfies our conditions.

- There are two glide reflections st taking a onto b, one with axis of s parallel to edge $\{a, c\}$ and one with axis of s parallel to the edge $\{b, c\}$. We consider the case where G contains one glide reflection mapping a onto b.
  * The glide reflection st with axis parallel to edge $\{b, c\}$ belongs to G.
    · The translation t' taking a onto f belongs to G. From the previous it is easily seen that we come to a contradiction.
    · The rotation with center the center of the hexagon through a and b belongs to G. But this is in contradiction with sharp point transitivity.
    · There are exactly two glide reflections taking a onto f one with axis parallel to edge $\{a, g\}$ and one with axis parallel to edge $\{f, g\}$ where g is the vertex adjacent to a and f. Suppose that a glide reflection $s't'$ taking a onto f belongs to G. If the glide reflection has axis parallel to edge $\{f, g\}$ then $G =< st, s't' >$ is a group satisfying our needs. It is easily seen that this group is the same as $< t'', st >$ where $\overrightarrow{t''} = \overrightarrow{(s't')(st)} - \overrightarrow{(st)(st)}$ which we already obtained in a previous case. If its axis is parallel to edge $\{a, g\}$ then the translation $(st)(st)$ and the glide reflection $(st)(s't')(st)(s't')(s't')$ both take vertex a onto the same vertex, which gives a contradiction.
  * The glide reflection st with axis parallel to edge $\{a, c\}$ belongs to G.
    · The translation t' taking a onto f belongs to G. From the previous it is easily seen that the group $G =< t', st >$ is a group satisfying our needs.
    · The rotation with center the center of the hexagon through a and b belongs to G. But this is in contradiction with sharp point transitivity.
    · There are exactly two glide reflections taking a onto f one with axis parallel to edge $\{a, g\}$ and one with axis parallel to edge $\{f, g\}$ where g is the vertex adjacent to a and f. Suppose that a glide reflection $s't'$ taking a onto f belongs to G. If the glide reflection has axis parallel to edge $\{f, g\}$ then the translation $(s't')(s't')$ and the glide reflection $(s't')(st)(s't')(st)(st)$ both take vertex a onto the same vertex, which gives a contradiction. If its axis is parallel to edge $\{a, g\}$ then $(st)(s't')$ is a rotation with center the vertex g. Because G is not flag transitive,

this is impossible.

- Non sharp point transitive collineation group G

  Then G certainly contains reflections. Moreover, every vertex is on exactly one reflection axis. Consequently the order of the stabilizer of a point (or a line) is equal to 2 and there are exactly two collineations taking a point (or a line) onto another point (or another line). We consider two possibilities:

  – All reflections have parallel axes, say parallel to edge $\{b, c\}$. Let s be the reflection with axis through vertex a. It follows that the translation t with vector $\overrightarrow{cg}$ belongs to G. We consider the following possibilities for mapping a onto b:

    * A translation t' takes a onto b. It is easily seen that $G =< s, t, t' >$ is a group which satisfies the conditions.
    * A rotation r with center the center of the hexagon through a and b maps a onto b. Since sr^+ gives a second reflection through vertex b this case cannot occur.
    * There are two glide reflections $s''t''$ taking a onto b, one with axis of reflection parallel to edge $\{a, c\}$ and one with axis parallel to the edge $\{b, c\}$. The second case gives rise to the same group $G =< s, t, t' >$ as in a previous subcase. In the first case the translation $(s''t'')(s''t'')$ with vector two times the translation vector of t'' belongs to G. It follows that the translation t' belongs to G (indeed $2\overrightarrow{t''} - \overrightarrow{t} \in G$). But then st' belongs to G and there are three different collineations mapping the point a onto the point b. This gives a contradiction.

  – Let s be the reflection with axis through a. We suppose that this axis is parallel to the edge $\{b, c\}$. It is easily seen that if the axis of the reflection in b is parallel to the axis of s then we get back to the previous case where all reflections are parallel. Hence the direction of the axis of the reflection s' in b is determined by the vector $\overrightarrow{bg}$. We consider the following possibilities for mapping a onto d:

    * The rotation r' with center the center of the hexagon through a and d belongs to G. Since also $r \in G$ it follows that all rotations with center the center of a hexagon belong to G. It is easy to see that the group $G =< s, s', r' >$ satisfies the conditions.
    * A translation t takes a onto d. Then tss' is a rotation with center the vertex b. Hence this case leads to a contradiction.
    * There are two glide reflections $s''t''$ taking a onto d, one with axis of reflection parallel to edge $\{a, e\}$ and one with axis parallel to the edge $\{e, d\}$. The first case gives rise to the same group $G =< s, s', r' >$ as in a previous

subcase. In the second case the rotation $ss''t''$ with center e belongs to G. This gives a contradiction.

5.4 Proof

- $\mathcal{V}_x$ is a singleton
- $\mathcal{V}_x$ is not a singleton

  We choose a coordinate system for the real Euclidean plane. Let $\tilde{x}$ be a reference vertex in $\mathcal{V}_x$. Then this vertex is chosen as the origin. The unit vector on the X-axis is chosen to be the vector $\overrightarrow{\tilde{x}\tilde{x_1}}$ with $\tilde{x_1}$ a vertex corresponding to the point x_1 at graph-theoretical distance 2 from $\tilde{x}$. We may assume that x_1 and y_1 are collinear in Γ. Then the unit vector on the Y-axis is chosen to be the vector $\overrightarrow{\tilde{x}\tilde{y_1}}$ with $\tilde{y_1}$ a vertex in $\mathcal{V}_{y_1}$ at graph-theoretical distance 2 from both $\tilde{x}$ and $\tilde{x_1}$.

  1. $G = < r_1, r_2 >$

     The group G contains all rotations over $+120$ and -120 degrees with center the center of a hexagon. It is easily seen that this group also contains some translations since the combination of two different rotations over opposite degrees gives a translation.

     Let $\tilde{x}$ be a reference vertex in $\mathcal{V}_x$ and $\tilde{x}'$ be another vertex in $\mathcal{V}_x$ at minimal Euclidean distance m from $\tilde{x}$. Either some rotation r or either some translation t of G takes $\tilde{x}$ onto $\tilde{x}'$.

     - Some rotation r maps $\tilde{x}$ onto $\tilde{x}'$. It follows that $\tilde{x}'^r = \tilde{x}''$ which is also in $\mathcal{V}_x$. The rotation r_1 taking $\tilde{x}$ on $\tilde{y_1}$ and $\tilde{x_1}$ takes $\tilde{x}'$, resp. $\tilde{x}''$ onto $\tilde{y_1}'$ and $\tilde{x_1}'$, resp. $\tilde{y_1}''$ and $\tilde{x_1}''$. Since these four vertices are at graph-theoretical distance 2 from a vertex in $\mathcal{V}_x$ and since the minimal Euclidean distance between vertices representing the same point is equal to m it follows easily that we have four new vertices in $\mathcal{V}_x$ such that these vertices together with $\tilde{x}'$ and $\tilde{x}''$ form a regular hexagon with center $\tilde{x}$. From now on we will suppose that $\tilde{x}'$ is lying between the positive X- and the positive Y-axis. The coordinates of $\tilde{x}'$ are given by the tuple (r, s) with $r, s \geq 0$. Without loss of generality we may assume that $r \geq s$ (interchanging X- and Y-axis if necessary). Applying successively rotations of 60 degrees, a tedious calculation shows that the coordinates of the six vertices in $\mathcal{V}_x$ on the regular hexagon around $\tilde{x}$ are (r, s), $(-s, r+s)$, $(-r-s, r) = -(r, s) + (-s, r+s)$, $-(r, s) = (-r, -s)$, $-(-s, r+s) = (s, -r-s)$ and $(r, s) - (-s, r+s) = (r+s, -r)$. Remark that the first two vectors generate the others by taking sums. In fact, by the minimality of m, all elements of $\mathcal{V}_x$ are generated by (r, s) and $(-s, r+s)$ by taking sums. Hence a generic element of $\mathcal{V}_x$ has coordinates $k(r,s)+l(-s,r+s)=(kr-ls,ks+lr+ls)$ with k and l integers.

Consider an arbitrary vertex $\tilde{z}$ in $V(\mathcal{I}(\Gamma))$ corresponding to a point z of the geometry Γ. Either a rotation or a translation takes vertex $\tilde{x}$ onto $\tilde{z}$ and is the lifting of a collineation mapping the point x onto the point z. Hence we see that the vertices of $\mathcal{V}_z$ are parameterized by $(i,j)+k(r,s)+l(-s,r+s)=(i+kr-ls,j+ks+lr+ls)$ with k and l integers, and with (i,j) the coordinates of $\tilde{z}$.

A line of the geometry can be described by a 3-set of coordinates of the three points incident with that line: a possible representation is given by $\{(i,j),(i+1,j),(i+1,j-1)\}$.

We thus recognize the geometry $\mathcal{G}_{(r,s)}$. Remark that since $\tilde{x}$ can not be on the bisector of the positive X- and positive Y-axis, the triple square geometries don't arise from this case.

– Some translation t of G maps $\tilde{x}$ onto $\tilde{x}'$. Using this translation and the rotation taking $\tilde{y}_1$ onto $\tilde{x}$ it follows that we have six vertices on a regular hexagon around $\tilde{x}$, all at distance m from $\tilde{x}$. A same reasoning as above gives us again the geometry $\mathcal{G}_{(r,s)}$.

2. $G=<t_1,t_2>$

Let $\tilde{x}$ be a reference vertex in $\mathcal{V}_x$ and $\tilde{x}'$ be another vertex in $\mathcal{V}_x$ at minimal Euclidean distance m from $\tilde{x}$. We distinguish two possible cases: there are exactly two vertices $\tilde{x}'$ and $\tilde{x}''$ in $\mathcal{V}_x$ at distance m from $\tilde{x}$ or there are more than two vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$. The coordinates of $\tilde{x}'$ are given by the tuple (r,s).

In the first case we may assume that $r\geq 0$. It is easily seen that a generic element of $\mathcal{V}_x$ has coordinates $k(r,s)$ with k an integer. Consider an arbitrary vertex $\tilde{z}$ in $V(\mathcal{I}(\Gamma))$ corresponding to a point z of the geometry Γ. The translation with vector $\overrightarrow{\tilde{x}\tilde{z}}$ is the lifting of a collineation in the collineation group G mapping the point x onto the point z. Hence we see that the vertices of $\mathcal{V}_z$ are parameterized by $(i,j)+k(r,s)=(i+kr,j+ks)$ with k an integer and with (i,j) the coordinates of $\tilde{z}$. A line of the geometry can be described by a 3-set of coordinates of the three points incident with that line: a possible representation is given by $\{(i,j),(i+1,j),(i+1,j-1)\}$. We thus recognize the geometry $\mathcal{S}_{(r,s)}$.

In the second case it follows that a third vertex $\tilde{x}'''$ in $\mathcal{V}_x$ at distance m from $\tilde{x}$ is also at distance m from $\tilde{x}'$ or $\tilde{x}''$. It is then easy to see that there are six vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$ lying on a regular hexagon. Analogously to a previous case we deduce that we get the geometry $\mathcal{G}_{(r,s)}$.

3. $G=<t',st>$

Let $\tilde{x}$ be a reference vertex in $\mathcal{V}_x$ and $\tilde{x}'$ be another vertex in $\mathcal{V}_x$ at minimal

Euclidean distance m from $\tilde{x}$. Either some glide reflection or either some translation of G takes $\tilde{x}$ onto $\tilde{x}'$.
Suppose first that $\tilde{x}'$ belongs to the region strictly between the X-axis and the line $s = -r$ containing the line $2s = -r$ but that $\tilde{x}'$ is not on the line $2s = -r$. Remark that $\tilde{x_1}'^{st^{-1}} = \tilde{x}''$ gives a vertex in $\mathcal{V}_x$ which is the image of $\tilde{x}'$ under reflection through the line $2s = -r$. The triangle $(\tilde{x}\tilde{x}'\tilde{x}'')$ is an isosceles triangle with $\widehat{\tilde{x}'\tilde{x}\tilde{x}''}$ less than sixty degrees. It follows easily that the length of $[\tilde{x}'\tilde{x}'']$ is less than m which gives a contradiction.
Secondly, suppose that $\tilde{x}'$ is strictly between the Y-axis and the line $s = -2r$ forming an angle of 30 degrees. Then $\tilde{x_1}'st^{-1} = \tilde{x}''$ belongs to $\mathcal{V}_x$ and $\widehat{\tilde{x}'\tilde{x}\tilde{x}''}$ is strictly between 120 and 180 degrees. If some translation t'' takes $\tilde{x}$ onto $\tilde{x}'$ then $\tilde{x}^{-t''} = \tilde{x}'''$ belongs to $\mathcal{V}_x$. But then $\widehat{\tilde{x}''\tilde{x}\tilde{x}'''}$ is strictly between zero and 60 degrees inducing that the length of $[\tilde{x}''\tilde{x}''']$ is less than m, which is a contradiction. If some glide reflection $s''t''$ takes $\tilde{x}$ onto $\tilde{x}'$ then $\tilde{x}^{s''t''^{-1}} = \tilde{x}''$ belongs to $\mathcal{V}_x$ and is the image of $\tilde{x}'$ under reflection through the Y-axis. Since $\widehat{\tilde{x}'\tilde{x}\tilde{x}''}$ is less than 60 degrees it follows that the length of $[\tilde{x}'\tilde{x}'']$ is less than m, a contradiction.
Next, suppose that $\tilde{x}'$ belongs to the region strictly between the Y-axis and the line $r = s$ forming an angle of 30 degrees. Based on the previous observation it follows that we again come to a contradiction.
There remain four possible regions for $\tilde{x}'$: strictly between the line $s = -2r$ and $s = -r$ making an angle of 30 degrees or strictly between the X-axis and the line $r = s$ also making an angle of 30 degrees. Call region 1, the region between the positive X-axis and the line $r = s$. The other regions are then called region 2 up till region 4 in anti-clockwise order starting from region 1. It is easily seen that there are exactly four vertices in $\mathcal{V}_x$ at distance m from the reference vertex $\tilde{x}$ each of them lying in a different region. Also, without loss of generality, we can say that $\tilde{x}'$ belongs to region 1 and hence its coordinates (r, s) are non negative integers with $r \geq s$.
In the case of a translation t'' mapping $\tilde{x}$ onto $\tilde{x}'$ the vertex $\tilde{x}'' \in \mathcal{V}_x$ in region 4 is taken onto $\tilde{x}''^{t''}$ in $\mathcal{V}_x$ which is at distance bigger than m from the reference vertex. It is easily seen that the coordinates of the three vertices $\tilde{x}'$, $\tilde{x}''$ and $\tilde{x}''^{t''}$ are given by (r, s), $(r, -r-s)$ and $(2r, -r)$ respectively. A generic element of $\mathcal{V}_x$ has coordinates $k(r, s) + l(r, -r-s) = (kr + lr, ks - lr - ls)$ with k and l integers.
Consider an arbitrary vertex $\tilde{z}$ in $V(\mathcal{I}(\Gamma))$ corresponding to a point z of the geometry Γ. Either the translation with vector $\overrightarrow{\tilde{x}\tilde{z}}$ or either a glide reflection with axis parallel to the axis of s taking $\tilde{x}$ onto $\tilde{z}$ is the lifting of a collineation in the collineation group G mapping the point x onto the point z. In both cases we see that the vertices of $\mathcal{V}_z$ are parameterized by $(i, j) + k(r, s) + l(r, -r-s) =$

$(i+kr+lr, j+ks-lr-ls)$ with k and l integers, and with (i,j) the coordinates of $\tilde{z}$.

A line of the geometry can be described by a 3-set of coordinates of the three points incident with that line: a possible representation is given by $\{(i,j),(i+1,j),(i+1,j-1)\}$.

We thus recognize the geometry $\mathcal{R}_{(r,s)}$.

In the case of a glide reflection $s''t''$ taking $\tilde{x}$ onto $\tilde{x}'$. Let $\tilde{x}''$ be the vertex in $\mathcal{V}_x$ at distance m from $\tilde{x}$ lying in region 4. A same reasoning as above, with domain $\mathcal{D}$ the rhombus $\tilde{x}\tilde{x}'\tilde{x}'^{s''t''}\tilde{x}''$ gives us again the geometry $\mathcal{R}_{(r,s)}$.

Finally we have a look at the cases where $\tilde{x}'$ belongs to the X-axis, the line $r=s$, the Y-axis, the line $s=-2r$, the line $s=-r$ or the line $2s=-r$.

First suppose that $\tilde{x}'$ is on the X-axis. It follows easily that in the case of a translation $\tilde{x}^{t''}=\tilde{x}'$, there are six vertices in $\mathcal{V}_x$ at distance m from the reference vertex and that we become the square geometries. If a glide reflection $s''t''$ takes $\tilde{x}$ onto $\tilde{x}'$ then there are certainly four vertices in $\mathcal{V}_x$ at distance m from the reference vertex: $\tilde{x}^{s''t''^{-1}}=\tilde{x}''$, $\tilde{x}_1'^{st^{-1}}=\tilde{x}'''$ and $\tilde{x}_1''^{st^{-1}}$. The automorphism taking $\tilde{x}'$ onto $\tilde{x}'''$ is a translation with length of the translation vector equal to m. It follows that there are two more vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$, lying on the Y-axis. We again recognize the square geometries.

Next suppose that $\tilde{x}'$ belongs to the line $r=s$. In the case of a translation taking $\tilde{x}$ onto $\tilde{x}'$ we get six vertices of $\mathcal{V}_x$ on a regular hexagon around $\tilde{x}$, at distance m from $\tilde{x}$. We hence recognize the triple square geometries. If a glide reflection $s''t''$ takes $\tilde{x}$ onto $\tilde{x}'$ then there are certainly four vertices in $\mathcal{V}_x$ at distance m from the reference vertex: $\tilde{x}^{s''t''^{-1}}=\tilde{x}''$, $\tilde{x}_1'^{st^{-1}}=\tilde{x}'''$ and $\tilde{x}_1''^{st^{-1}}$. The automorphism taking $\tilde{x}'$ onto $\tilde{x}''$ is a translation with length of the translation vector equal to m. It follows that there are two more vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$, lying on the line $2s=-r$. We again recognize the triple square geometries.

If $\tilde{x}'$ belongs to the Y-axis then some translation t'' maps $\tilde{x}$ onto $\tilde{x}'$. If there are only two vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$ then we recognize the strip geometries. If there are more than two vertices in $\mathcal{V}_x$ at distance m from the reference vertex then there is a vertex $\tilde{x}''\in\mathcal{V}_x$ lying on the line $2s=-r$ or on the line $s=-r$ or on the X-axis for which $d(\tilde{x},\tilde{x}'')=m$. Suppose that $\tilde{x}''$ belongs to the line $2s=-r$. The length of the translation vector mapping $\tilde{x}$ onto $\tilde{x}''$ is equal to m and it is easily seen that m is equal to n times the length of the translation vector $2t$, with n a positive integer. Since the length of $2t$ is equal to $\sqrt{3}$ and since m is a positive integer in the case that $\tilde{x}'$ is on the Y-axis, it follows that this case can not occur. If $\tilde{x}''$ is on the line $s=-r$ then it follows easily that we obtain the square geometries. Analogously for $\tilde{x}''$ lying on the X-axis.

Next we look at the possibility where $\tilde{x}'$ belongs to the line $s = -2r$. In the case of a translation t'' mapping $\tilde{x}$ onto $\tilde{x}'$ it is easily seen that we obtain the triple square geometries. If a glide reflection $s''t''$ takes $\tilde{x}$ onto $\tilde{x}'$ then $\tilde{x}'$, $\tilde{x}^{s''t''^{-1}} = \tilde{x}''$, $\tilde{x}'^{s''t''}$, $\tilde{x}''^{s''t''^{-1}}$, $\tilde{x_1}''^{st^{-1}}$ and $\tilde{x_1}'^{st^{-1}}$ are six vertices of $\mathcal{V}_x$ at distance m from $\tilde{x}$. We again recognize the triple square geometries.

If $\tilde{x}'$ belongs to the line $s = -r$, then the case of a translation mapping $\tilde{x}$ onto $\tilde{x}'$ gives rise to the square geometries. If a glide reflection $s''t''$ takes $\tilde{x}$ onto $\tilde{x}'$ then $\tilde{x_1}'^{st^{-1}} = \tilde{x}''$ belongs to $\mathcal{V}_x$ and lies on the X-axis at distance m from $\tilde{x}$. From the previous it follows that we obtain the square geometries.

Finally we consider the case where $\tilde{x}'$ is lying on the line $2s = -r$. Then some translation t'' maps $\tilde{x}$ onto $\tilde{x}'$. If there are only two vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$ then we recognize the strip geometries. If there are more than two vertices in $\mathcal{V}_x$ at distance m from the reference vertex then there is a vertex $\tilde{x}'' \in \mathcal{V}_x$ lying on the line $s = -2r$ or on the line $s = r$ or on the Y-axis for which $d(\tilde{x}, \tilde{x}'') = m$. Suppose that $\tilde{x}''$ belongs to the Y-axis. The length of the translation vector mapping $\tilde{x}$ onto $\tilde{x}''$ is equal to m and it is easily seen that m is a positive integer. But since $\tilde{x}'$ belongs to the line $2s = -r$ it also follows that m is equal to n times the length of the translation vector $2t$, with n a positive integer. Since the length of $2t$ is equal to $\sqrt{3}$ it follows that this case can not occur. If $\tilde{x}''$ is on the line $s = r$ then it follows easily that we obtain the triple square geometries. Analogously for $\tilde{x}''$ lying on the line $s = -2r$.

4. $G = < s, t, t' >$

   Let $\tilde{x}$ be a reference vertex in $\mathcal{V}_x$ and $\tilde{x}'$ be another vertex in $\mathcal{V}_x$ at minimal Euclidean distance m from $\tilde{x}$. We suppose that s has axis $r = s$, $\overrightarrow{t} = \overrightarrow{\tilde{x}\tilde{y_1}}$ and $\overrightarrow{t'} = \overrightarrow{\tilde{x}\tilde{x_1}}$. It is easily seen that $\tilde{x}'$ can not be strictly between the X-axis and the line $r = s$ (angle of 30 degrees) and not strictly between the line $r = s$ and the Y-axis (angle of 30 degrees). Suppose that $\tilde{x}'$ is strictly between the lines $s = -2r$ and $2s = -r$ (angle of 60 degrees) but not on the line $s = -r$. Then a translation t'' takes $\tilde{x}$ onto $\tilde{x}'$. But then it follows that $\tilde{x}^{t''^{-1}}$ and $\tilde{x}'^{s}$ are in $\mathcal{V}_x$ at distance m from $\tilde{x}$ and that $d(\tilde{x}^{t''^{-1}}, \tilde{x}'^{s}$ is less than m, a contradiction.

   If $tildex'$ belongs to the region strictly between the Y-axis and the line $s = -2r$ (angle of 30 degrees) or the region strictly between the X-axis and the line $2s = -r$ (angle of 30 degrees) then it is easily seen that there is a vertex in $mathcalV_x$ at distance m from the reference vertex with coordinates (r, s) for which $r < 0$, $s > 0$ and $s > -2r$. The vertex $\tilde{x}'' = \tilde{x}'^{s} \in \mathcal{V}_x$ has then coordinates (s, r) and is also at distance m from $\tilde{x}$. The vertex $\tilde{x}''' = \tilde{x}''^{t''}$ with t'' a translation with vector $\overrightarrow{\tilde{x}\tilde{x}'}$ belongs to $\mathcal{V}_x$ and has coordinates $(r + s, r + s)$. Let us consider a rotation over $+60$ degrees of the coordinate system. The new coordinates for $\tilde{x}'$, $\tilde{x}''$ and $\tilde{x}'''$ are respectively $(r + s, -r)$, $(r + s, -s)$ and $(2s + 2r, -s - r)$. The new coordinates of $\tilde{x}'$ are positive integers with X-coordinate bigger than the Y-coordinate. A generic element of $\mathcal{V}_x$ has coordinates $k(r+s,-r)+l(r+s,-s) = (kr+ks+lr+ls,-kr-ls)$ with k and l integers. Consider an arbitrary vertex $\tilde{z}$ in $V(\mathcal{I}(\Gamma))$ corresponding to a point z of the geometry Γ. A translation with vector $\overrightarrow{\tilde{x}\tilde{z}}$ is the lifting of a collineation in the collineation group G mapping the point x onto the point z. The vertices of $\mathcal{V}_z$ are parameterized by $(i,j)+k(r+s,-r)+l(r+s,-s) = (i+kr+ks+lr+ls, j-kr-ls)$ with k and l integers, and with (i,j) the coordinates of $\tilde{z}$. We thus recognize the geometry $\mathcal{R}_{(r+s,-r)}$ with $(r+s)^2+2(r+s)(-r) = s^2-r^2$ points and lines.

Suppose that $\tilde{x}'$ is on the X-axis or on the Y-axis. It is then easy to see that we obtain the square geometries.

If $\tilde{x}'$ is on the line $r=s$ then we get the stripe geometries in the case that there are exactly two vertices in $\mathcal{V}_x$ at distance m from the reference vertex. Suppose that there are more than two vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$. Then we easily recognize the triple square geometries.

In the case that $\tilde{x}'$ is on the line $s=-2r$ or on the line $2s=-r$ we obtain the triple square geometries.

Finally we consider the case where $\tilde{x}'$ is on the line $s=-r$. If there are only two vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$ then we recognize the stripe geometries. If more than two, then there are exactly six vertices in $\mathcal{V}_x$ at distance m from the reference vertex and we recognize the square geometries.

5. $G = < s, s', r >$

Let $\tilde{x}$ be a reference vertex in $\mathcal{V}_x$ and $\tilde{x}'$ be another vertex in $\mathcal{V}_x$ at minimal Euclidean distance m from $\tilde{x}$. We suppose that s has axis $r=s$, that s' has axis $s=-2r+1$ and r has center the center of the hexagon through $\tilde{x}$ and $\tilde{x_2}$ with $\tilde{x_2}$ in $\mathcal{V}_{x_2}$.

Remark that the group G contains glide reflections with axes the axes of the reflections and also axes lying symmetrically between to parallel reflection axes and parallel to those two axes. Given two different vertices v and w of the honeycomb geometry belonging to the same partition set. Then either a rotation or either a translation takes v onto w. Indeed, suppose G contains both a rotation r' and a translation t mapping v onto w. Then it follows that $r'^{-1}t$ is a rotation with center w, which is impossible. The second automorphism in G taking v onto w is then a reflection or a glide reflection.

It is easily seen that $\tilde{x}'$ can not be strictly between the X-axis and the line $r=s$ (angle of 30 degrees) and not strictly between the line $r=s$ and the Y-axis (angle of 30 degrees). Suppose that $\tilde{x}'$ is strictly between the Y-axis and the line $s=-r$ (angle of 60 degrees) but not on $s=-2r$. Let $\tilde{a}$ be in $\mathcal{V}_a$ at graph-theoretical distance 2 from both $\tilde{y_1}$ and $\tilde{x_1}$ and with a the third point on the line through y_1 and x_1. Let $\tilde{z_1}$ be in $\mathcal{V}_{z_1}$ at graph-theoretical distance 2 from the reference vertex. Let $\tilde{b}$ be in $\mathcal{V}_b$ belonging to the same hexagon as

$\tilde{y_1}$ and $\tilde{z_1}$. The translation $(r^+s')^2$ has translation vector $\tilde{x}\tilde{a}$. The translation $((r^-s)^2)^{-1}$ has translation vector $\tilde{x}\tilde{b}$. Now, since $\tilde{y_1}'$ is at graph-theoretical distance 2 from $\tilde{x}'$, $\tilde{a}'$ and $\tilde{b}'$ and since the reflection s' fixes the point y_1 it follows that the distance between $\tilde{y_1}'$ and $\tilde{y_1}''$, with $\tilde{y_1}'' = \tilde{y_1}'^{s'}$ is smaller than m, a contradiction. Next, if $\tilde{x}'$ is strictly between the line $s = -r$ and the X-axis (angle of 60 degrees) but not on the line $2s = -r$ then $\tilde{x}'^s$ is strictly between $s = -r$ and the Y-axis and not on $s = -2r$ which gives a contradiction as shown in the previous.

Suppose that $\tilde{x}'$ is on the X-axis. Then $\tilde{x}'' = \tilde{x}'^s$ belongs to $\mathcal{V}_x$ and is on the Y-axis at distance m from the reference vertex. The vertices $\tilde{y_1}'$ and $\tilde{y_1}''$ are in $\mathcal{V}_{y_1}$ at graph-theoretical distance 2 from $\tilde{x}'$, resp. $\tilde{x}''$. Reflecting those two vertices through the reflection axis in $\tilde{x_1}$ gives two vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$, one on the Y-axis and one on the line $s = -r$. The images of those two vertices under s give two new vertices in $\mathcal{V}_x$ at distance m from $\tilde{x}$, one on the X-axis and one on the line $s = -r$. It is then easy to see that we obtain the square geometries. If $\tilde{x}'$ is on the Y-axis, then there is also a vertex $\tilde{x}''$ in $\mathcal{V}_x$ on the X-axis at distance m from $\tilde{x}$ which brings us back to the previous. If $\tilde{x}'$ is on the line $s = -r$ then $\tilde{y_1}'^{s''}$, with $\tilde{y_1}' \in \mathcal{V}_{y_1}$ and at graph-theoretical distance 2 from $\tilde{x}'$ and s'' the reflection with axis through $\tilde{x_1}$, gives a vertex $\tilde{x}''$ in $\mathcal{V}_x$ at distance m from $\tilde{x}$ lying on the X-axis. Hence also this case gives rise to the square geometries.

Next we consider the case where $\tilde{x}'$ is on the line $r = s$. Then some translation t maps $\tilde{x}$ onto $\tilde{x}'$ and $\tilde{x}^{t^{-1}} = \tilde{x}''$ belongs to $\mathcal{V}_x$ at distance m from $\tilde{x}$. The vertices $\tilde{y_1}'$ and $\tilde{y_1}''$ of $\mathcal{V}_{y_1}$ which are at graph-theoretical distance 2 from $\tilde{x}'$ and $\tilde{x}''$ respectively are on the line $s = r + 1$. The image of those two vertices under s' are hence on the line $2s = -r+2$. Reflection through the axis $2s = -r+1$ gives two new vertices $\tilde{x}'''$ and $\tilde{x}^{iv}$ in $\mathcal{V}_x$ at distance m from $\tilde{x}$ on the line $2s = -r$. Considering the images of those two vertices under the reflection s it is easily seen that we get exactly six vertices in $\mathcal{V}_x$ at distance m from the reference vertex, two on the line $r = s$, two on the line $2s = -r$ and two on the line $s = -2r$. Hence we recognize the triple square geometries. In the case that $\tilde{x}'$ is on the line $s = -2r$ then there is some translation t taking $\tilde{x}$ onto $\tilde{x}'$ and the vertex $\tilde{x}'^{st}$ belongs to $\mathcal{V}_x$ at distance m from $\tilde{x}$ and is on the X-axis. This brings us back to the previous case. Analogously for $\tilde{x}'$ on the line $2s = -r$.

6 Configuration 24

For this configuration, the lines in Γ_x^l are x_1y_1, x_1z_2, y_1y_2 and x_2z_1. Looking at the local structure in the point x_1 gives two possibilities: x_2 is incident either with the third point on x_1z_2 or either with the third point on x_1y_1. First, let's consider the case where x_2 is incident with the third point on the line x_1z_2, say a. We distinguish two possible cases: the case where x_2z_1a is a line of Γ and the case where x_2z_1 and x_2a are two different lines of Γ. For the first case we look at the dual configuration which appears to be configuration 30. Since this configuration cannot occur, also its dual cannot occur. The dual configuration in the second case is configuration 21 which cannot occur. Hence, x_2 is incident with the third point on the line x_1y_1, say a. If x_2z_1a is a line, the dual configuration is configuration 29 which cannot occur, hence x_2z_1 and x_2a are two different lines of Γ. We will now give a full description of geometries with this local structure.

We can find a unique triangle xx_1y_1 in the point x with the property that the third line through x is incident with two other points, one on the third line of each other point of the triangle. Because of point transitivity, every point has a unique triangle with that property. Let's call this property ($\square$). Remark that if $t = uvw$ is the unique triangle in u with property ($\square$), then t is also the unique triangle with property ($\square$) in the points v and w. In the above configuration, $t_0 = y_2z_2b$, $t_1 = xx_1y_1$ and $t_2 = x_2z_1a$, with b the point incident with the lines x_1z_2 and y_1y_2, are triangles with the above mentioned property. We see that all points of triangle t_2 are the third points on the edges of triangle t_1 and the points of t_1 are the third points on the edges of triangle t_0. Because of point transitivity every triangle with property ($\square$) has points which are the third points on the edges of a unique other triangle with the same property and has a unique triangle with property ($\square$) with points the third points on its sides. Based on this observation, we construct a directed graph $\mathcal{G}(V, E)$ with vertex set the set of triangles with property ($\square$) in the geometry Γ. There is a directed edge from vertex t to vertex t' if the three points of the corresponding triangle t' are the third points on the lines of triangle t. It's clear that $\mathcal{G}$ is a union of directed cycles or an infinite directed 2-valent graph. Since the segment $t \to t' \to t''$ defines three lines each containing three points through each point of t', a directed cycle defines a connected bislim component of Γ. Since we assume that Γ is connected, the graph $\mathcal{G}$ is either a directed cycle if Γ is finite or either an infinite directed 2-valent graph if Γ is infinite. Let's denote $\mathcal{G}$ by $t_0 \to t_1 \to t_2 \to \cdots \to t_{n-1} \to t_0$ or by $\cdots \to t_{-n} \to t_{-n+1} \to \cdots \to t_0 \to t_1 \to \cdots \to t_n \to \cdots$. It's clear that in the first case Γ has $3n$ points and lines.

There is a collineation g in G mapping x onto x_1. The line x_2z_1 has image x_2a and hence x_2 is mapped onto itself or onto a. Let's consider the first possibility. In triangle $t_0 = y_2z_2b$ the point z_2 is fixed by g and the points y_2 and b are mapped onto each other. For triangle $t_1 = xx_1y_1$ the point y_1 is fixed, while the other two points are interchanged. In triangle $t_2 = x_2z_1a$ the fix point is given by x_2 and z_1 and a are mapped onto each other. Inductively, we see that every triangle with property ($\Box$) has one point fixed by g and two points which are mapped onto each other by g. Secondly, if x_2 is taken onto a, then g maps y_2 onto z_2 onto b onto y_2, x onto x_1 onto y_1 onto x, x_2 onto a onto z_1 onto x_2. By induction we can conclude that g acts cyclically onto the points of each triangle with property ($\Box$). Remark that if the geometry Γ contains a collineation of the first type then also one of the second type. Indeed, there is a collineation h in G mapping x_2 onto a. If h is of type two, mapping x_2 onto a onto z_1 onto x_2 then it's proved. If h is of type one mapping x_2 and a onto each other and fixing z_1 then gh maps x_2 onto a onto z_1 onto x_2, hence of type two.

Let C be the cycle of n triangles $t_0 \to t_1 \to t_2 \to \cdots \to t_{n-1} \to t_0$. The question is: which point of t_0 is incident with which line of t_{n-1}? If we choose one point of t_0 and one line of t_{n-1} to be incident, then the other incidences are determined by a collineation of type 2. We use the following representation for the n triangles: all triangles are equilateral and t_{i+1} is represented inside t_i, for all $i \in \{0, \ldots, n-2\}$. Say $t_i = u_i^1 u_i^2 u_i^3$ for $0 \leq i \leq n-1$ and u_i^j is incident with $u_{i-1}^{j-1} u_{i-1}^{j+1}$ where j is taken modulo 3 and where $1 \leq i \leq n-1$. Call Γ_1 the geometry where u_0^1 is incident with $u_{n-1}^1 u_{n-1}^2$, Γ_2 the geometry where $\overline{u}_0^1$ is incident with $\overline{u}_{n-1}^1 \overline{u}_{n-1}^3$ and Γ_3 the geometry where $\tilde{u}_0^1$ is incident with $\tilde{u}_{n-1}^2 \tilde{u}_{n-1}^3$. We define ϕ as mapping u_i^1 onto $\overline{u}_i^1$, u_i^2 onto $\overline{u}_i^3$ and u_i^3 onto $\overline{u}_i^2$ for $0 \leq i \leq n-1$. It's easy to see that ϕ is an isomorphism between Γ_1 and Γ_2. Suppose now that Γ_1 is isomorphic to Γ_3. Let ψ be the reflection through the axis $u_0^1 u_1^1$. It's easy to see that ψ is a collineation for Γ_3 but not for Γ_1, hence Γ_1 and Γ_3 cannot be isomorphic. Hence, a cycle of triangles C defines two non-isomorphic geometries on $3n$ points and $3n$ lines. For n equal to 3, we obtain two isomorphic geometries on 9 points and 9 lines and the Pappus geometry. Both non-isomorphic geometries don't have local structure 24, hence $n \geq 4$. We now give a description of the two geometries with local structure 24 for each $n \geq 4$.

First, consider the cyclic group $\mathbb{Z}_{3n}$ of order $3n$. We define a geometry Γ with point set the elements of $\mathbb{Z}_{3n}$ and line set the triples $\{x, x+n, x+i\}$ where $x \in \mathbb{Z}_{3n}$ and $1 \leq i \leq 3n-1$, a fixed number different from $\frac{n}{2}, n, \frac{3n}{2}, 2n$ and $\frac{5n}{2}$. Note this geometry with $\Gamma(n,i)$. It's easy to see that Γ is a bislim geometry admitting a point and line transitive collineation group. Indeed, consider the action of the group $\mathbb{Z}_{3n}$ onto itself. It's also easy to see that Γ has the above mentioned local configuration. Clearly, Γ contains triangles $x(x+n)(x+2n)$ with property ($\Box$) but no digons. We call $d_j = j(n+j)(2n+j)$ the n triangles with property ($\Box$), for $0 \leq j \leq n-1$. Let's look at a directed cycle associated with Γ: $d_0 \to d_{i \bmod n} \to d_{2i \bmod n} \to \cdots \to d_0$. There exists some $k \in \mathbb{N}_0$ such that $ki \bmod n = 0$. Let $m \in \mathbb{N}_0$ be the smallest number for which $mi \bmod n = 0$ or equivalently $\gcd(n,i) = \frac{n}{m}$. If m is equal to n and hence $\gcd(n,i) = 1$, then the cycle associated to Γ is given by $d_0 \to d_{i \bmod n} \to d_{2i \bmod n} \to \cdots \to d_{(n-1)i \bmod n} \to d_0$. It follows that Γ is a connected geometry. If m is different from n then $d_0 \to d_{i \bmod n} \to d_{2i \bmod n} \to \cdots \to d_{(m-1)i \bmod n} \to d_0$ is a cycle containing m triangles and representing a connected component with $3m$ points and $3m$ lines of Γ. Let j be a point of Γ belonging to triangle $d_{j \bmod n}$ not contained in the above mentioned cycle, then $d_{j \bmod n} \to d_{j+i \bmod n} \to d_{j+2i \bmod n} \to \cdots \to d_{j+(m-1)i \bmod n} \to d_{j \bmod n}$ is another directed cycle representing another connected component of Γ. Since Γ contains n triangles, Γ consists of $\frac{n}{m} = \gcd(n,i)$ connected components. Since we assume that Γ is connected, we demand that $\gcd(n,i) = 1$. It's easy to see that every point of Γ can be written as $kn + li$ with $k \in \{0,1,2\}$ and $0 \leq l \leq n-1$. For our convenience, we rename the triangles in the following way: d_j is the triangle $ji, ji+n, ji+2n$ with $0 \leq j \leq n-1$.

How many different geometries are obtained for fixed n and varying i? Since $ni \bmod n = 0$ we have that $ni = 0 \bmod 3n$ or $ni = n \bmod 3n$ or $ni = 2n \bmod 3n$ and hence $i = 0 \bmod 3$ or $i = 1 \bmod 3$ or $i = 2 \bmod 3$. Let $\Gamma_1(\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$, respectively $\Gamma_2(\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be the geometry with point set $\mathbb{Z}_{3n}$ and line set $\{\{x, x+n, x+i_1\} |\ x \in \mathbb{Z}_{3n}\}$, respectively $\{\{x, x+n, x+i_2\} |\ x \in \mathbb{Z}_{3n}\}$. Suppose that ϕ is an isomorphism between Γ_1 and Γ_2 and that the point 0 is mapped onto the point $x \in \mathbb{Z}_{3n}$. It's easily seen that then n is mapped onto either $x+n$ or either $x+2n$. If n is taken onto $x+n$ then the point $kn+li_1$ is mapped onto $x+kn+li_2$. We prove this by induction on the triangles of the geometry. For d_0 we have that $0^\phi = x$, $n^\phi = x+n$ and $(2n)^\phi = x+2n$. Suppose that the assertion is valid for the points ri_1, ri_1+n and ri_1+2n of triangle d_r with $0 \leq r \leq n-2$. The points of triangle d_{r+1} are the third points on the sides of triangle d_r. Hence, $(r+1)i_1$ is mapped onto $x+(r+1)i_2$, $(r+1)i_1+n$ onto $x+n+(r+1)i_2$ and $(r+1)i_1+2n$ onto $x+2n+(r+1)i_2$. Now, let's look at triangle d_{n-1}. We know that $(n-1)i_1$ is taken onto $(n-1)i_2+x$, $(n-1)i_1+n$ onto $x+n+(n-1)i_2$ and $(n-1)i_1+2n$ onto $x+2n+(n-1)i_2$. If $i_1 = 0 \bmod 3$ then $ni_1 = 0 \bmod 3n$ is taken onto $x+ni_2$. Hence, i_2 should be equal to $0 \bmod 3$ to avoid contradictions. If $i_1 = 1 \bmod 3$ then $ni_1 = n \bmod 3n$ is taken onto $x+ni_2$. Hence, i_2 should be equal to $1 \bmod 3$ to avoid contradictions. Finally, if $i_1 = 2 \bmod 3$ then $ni_1 = 2n \bmod 3n$ is taken onto $x+ni_2$. Hence, i_2 should be equal to $2 \bmod 3$ to avoid contradictions. On the other hand, if n is taken onto $x+2n$ then we assert that the point $kn+li_1$ has image $x+k'n+li_2$ with $k' = 2k \bmod 3$ if $l = 0 \bmod 3$, $k' = 2(k+1) \bmod 3$ if $l = 1 \bmod 3$ and $k' = 2(k+2) \bmod 3$ if $l = 2 \bmod 3$. Indeed, for triangle d_0 the assertion holds. Suppose that it's also valid for triangle d_r with $0 \leq r \leq n-2$. If $r = 0 \bmod 3$ then $(r+1)i_1$ is mapped onto $x+2n+(r+1)i_2$, $(r+1)i_1+n$ onto $x+n+(r+1)i_2$ and $(r+1)i_1+2n$ onto $x+(r+1)i_2$. If $r = 1 \bmod 3$ then $(r+1)i_1$ is mapped onto $x+n+(r+1)i_2$, $(r+1)i_1+n$ onto $x+(r+1)i_2$ and $(r+1)i_1+2n$ onto $x+2n+(r+1)i_2$. If $r = 2 \bmod 3$ then $(r+1)i_1$ is mapped onto $x+(r+1)i_2$, $(r+1)i_1+n$ onto $x+2n+(r+1)i_2$ and $(r+1)i_1+2n$ onto $x+n+(r+1)i_2$. Now, let's look at triangle d_{n-1}. If $i_1 = 0 \bmod 3$ then $ni_1 = 0 \bmod 3n$ is taken onto $x+ni_2+2n$ if $n-1 = 0 \bmod 3$, onto $x+ni_2+n$ if $n-1 = 1 \bmod 3$, onto $x+ni_2$ if $n-1 = 2 \bmod 3$. Hence, i_2 should be equal to $1 \bmod 3$ if $n = 1 \bmod 3$, to $2 \bmod 3$ if $n = 2 \bmod 3$ and to $0 \bmod 3$ if $n = 0 \bmod 3$ to avoid contradictions. If $i_1 = 1 \bmod 3$ then $ni_1 = n \bmod 3n$ is taken onto $x+ni_2+2n$ if $n-1 = 0 \bmod 3$, onto $x+ni_2+n$ if $n-1 = 1 \bmod 3$, onto $x+ni_2$ if $n-1 = 2 \bmod 3$. Hence, i_2 should be equal to $0 \bmod 3$ if $n = 1 \bmod 3$, to $1 \bmod 3$ if $n = 2 \bmod 3$ and to

2 mod 3 if $n = 0 \bmod 3$ to avoid contradictions. If $i_1 = 2 \bmod 3$ then $ni_1 = 2n \bmod 3n$ is taken onto $x+ni_2+2n$ if $n-1 = 0 \bmod 3$, onto $x+ni_2+n$ if $n-1 = 1 \bmod 3$, onto $x+ni_2$ if $n-1 = 2 \bmod 3$. Hence, i_2 should be equal to 2 mod 3 if $n = 1 \bmod 3$, to 0 mod 3 if $n = 2 \bmod 3$ and to 1 mod 3 if $n = 0 \bmod 3$ to avoid contradictions. We can conclude that if $n = 0 \bmod 3$ then all geometries $\Gamma(n,i)$ are isomorphic. If $n = 1 \bmod 3$ then there are two classes of mutual non-isomorphic geometries. All geometries contained in one class are isomorphic. The first class consists of geometries $\Gamma(n,i)$ for which $i = 0 \bmod 3$ or $i = 1 \bmod 3$, the second class contains the geometries $\Gamma(n,i)$ with $i = 2 \bmod 3$. If $n = 2 \bmod 3$, an analogous result occurs: the first class consists of the geometries $\Gamma(n,i)$ with $i = 0 \bmod 3$ or $i = 2 \bmod 3$, the second class contains the geometries $\Gamma(n,i)$ for which $i = 1 \bmod 3$.

If $n = 0 \bmod 3$, consider the group $\mathbb{Z}_3 \times \mathbb{Z}_n$. Let Γ be the geometry with point set the elements of the group and line set the set of triples $\{(i,j),(i+1,j),(i,j+1)\}$ where $i \in \mathbb{Z}_3$ and $j \in \mathbb{Z}_n$, incidence is natural. It's clear that Γ is a connected bislim geometry with local structure 24. The geometry admits a point and line transitive collineation group which is not flag transitive. Indeed, consider the action of the group onto itself. The triangles with property ($\Box$) are given by $(0,j)(1,j)(2,j)$ with j an integer modulo n. We still have to prove that this geometry, say Γ_2 is not isomorphic to the geometry Γ_1 with point set $\mathbb{Z}_{3n}$ and line set $\{\{x, x+1, x+n\} \mid x \in \mathbb{Z}_{3n}\}$. We try to construct an isomorphism ϕ. The point 0 of Γ_1 is mapped onto the point (i,j) of Γ_2 with $i \in \mathbb{Z}_3$ and $j \in \mathbb{Z}_n$. The point n is then taken onto $(i+1,j)$ or $(i+2,j)$. First we consider the case where $n^\phi = (i+1,j)$. Then $(2n)^\phi = (i+2,j)$. Consider the directed cycle of triangles of Γ_1: $d_0 = (0,n,2n) \to d_1 = (1,n+1,2n+1) \to d_2 = (2,n+2,2n+2) \to \cdots \to d_m = (m,n+m,2n+m) \to d_{m+1} = (m+1,n+m+1,2n+m+1) \to \cdots \to d_{n-1} = (n-1,n+n-1,2n+n-1) \to d_0$. It's easy to see that ϕ is fully determined. Indeed, consider the points of triangle d_1: 1 is taken onto $(i,j+1)$, $n+1$ onto $(i+1,j+1)$ and $2n+1$ onto $(i+2,j+1)$. Based on the above mentioned cycle we can describe each point of Γ_1 as $kn+l$ with $k \in \{0,1,2\}$ and $0 \leq l \leq n-1$. We assert that $kn+l$ has image $(i+k,j+l)$ under ϕ. Suppose that the assertion holds for triangle d_m with $0 \leq m < n-1$ then $(m+1)^\phi = (i,j+m+1)$, $(n+m+1)^\phi = (i+1,j+m+1)$ and $(2n+m+1)^\phi = (i+2,j+m+1)$. The triple $\{n-1,n,n+n-1\}$ is a line of Γ_1. Its image under ϕ is then given by $\{(i,j+n-1),(i+1,j),(i+1,j+n-1)\}$ which is not a line of Γ_2. Hence ϕ cannot be an isomorphism between both geometries. Next, we consider the second possibility where n is taken onto $(i+2,j)$ and hence $2n$ onto $(i+1,j)$. The points 1, $n+1$ and $2n+1$ of triangle d_1 are then mapped onto $(i+2,j+1)$, $(i+1,j+1)$ and $(i,j+1)$ respectively. Similarly we have that $2^\phi = (i+1,j+2)$, $(n+2)^\phi = (i,j+2)$ and $(2n+2)^\phi = (i+2,j+2)$. We assert that the point $kn+l$ with $k \in \{0,1,2\}$ and $0 \leq l \leq n-1$ has image $(i+2(k+1),j+l)$ if $l = 1 \bmod 3$, $(i+2k,j+l)$ if $l = 0 \bmod 3$ and $(i+2(k+2),j+l)$ if $l = 2 \bmod 3$. Suppose that the assertion holds for d_m with $0 \leq m < n-1$. Then the points $m+1$, $n+m+1$ and $2n+m+1$ of triangle d_{m+1}

are taken onto $(i+2, j+m+1)$, $(i+1, j+m+1)$ and $(i, j+m+1)$ respectively if $m = 0 \bmod 3$, onto $(i+1, j+m+1)$, $(i, j+m+1)$ and $(i+2, j+m+1)$ respectively if $m = 1 \bmod 3$ and onto $(i, j+m+1)$, $(i+2, j+m+1)$ and $i+1, j+m+1)$ respectively if $m = 2 \bmod 3$. Now, $\{n-1, n, n+n-1\}$ is a line of the geometry Γ_1 which has image $\{(i+1, j+n-1), (i+2, j), (i, j+n-1)\}$ under ϕ. Since this is not a line of Γ_2, we can conclude that we cannot find an isomorphism between Γ_1 and Γ_2.

Hence, all geometries with local structure 24 are fully described.

7 Configuration 51

In this case, the lines $x_1y_1y_2$, x_1z_1, x_2y_2 and x_2z_2 are the only lines between points of $\Gamma_2(x)$. Every point in Γ_x has a special characteristic in that configuration. The point y_2 for example is the unique point in Γ_x on the unique line of three points in Γ_x^l which is incident with a point not on that line and not on the line through x and y_2, which itself is incident with the third point on the line xy_2. Let's call this characteristic $(*)$. It's easy to see that geometries with this local structure have a sharply point (and hence sharply line) transitive collineation group. Hence, there is a unique collineation g in G taking x_1 onto x. Obviously, x has characteristic $(*)$ in Γ_{x_1}. Let v be any point of the geometry Γ, then it's easy to see that there is a unique point w for which v has characteristic $(*)$ in the configuration of w. The collineation g hence maps x onto y_2 onto x_2 onto z_2 onto $\ldots$. It's clear that we obtain a cycle of different points of Γ if Γ is finite. Look at a segment $a_1 \to a_2 \to a_3 \to a_4$. We easily see that $a_1a_2a_4$ and a_3a_4 are lines of the geometry Γ. Consequently, the cycle of points defines a connected bislim component of the geometry Γ and since Γ is connected, the cycle contains all points of the geometry Γ. If Γ is infinite we obtain a path of all (different) points of the geometry.

On the other hand, consider the geometry Γ with point set $\mathcal{P}$ the elements of $\mathbb{Z}_n$ with $n \in \mathbb{N} \cup \{\infty\}$ and line set $\mathcal{L} = \{\{x, x+1, x+3\} \mid x \in \mathbb{Z}_n\}$. This is a bislim geometry with local structure the above mentioned configuration 53 and admitting a point and line transitive collineation group, the action of $\mathbb{Z}_n$ onto itself. Remark that for $n = 7, 8$ and 9 we get respectively the Fano geometry, the Möbius-Kantor geometry and a geometry on 9 points, which have a local structure different than configuration 51. Hence, n is bigger than or equal to 10. For n equal to 10 we obtain a geometry which is not isomorphic to the Desargues geometry.

References

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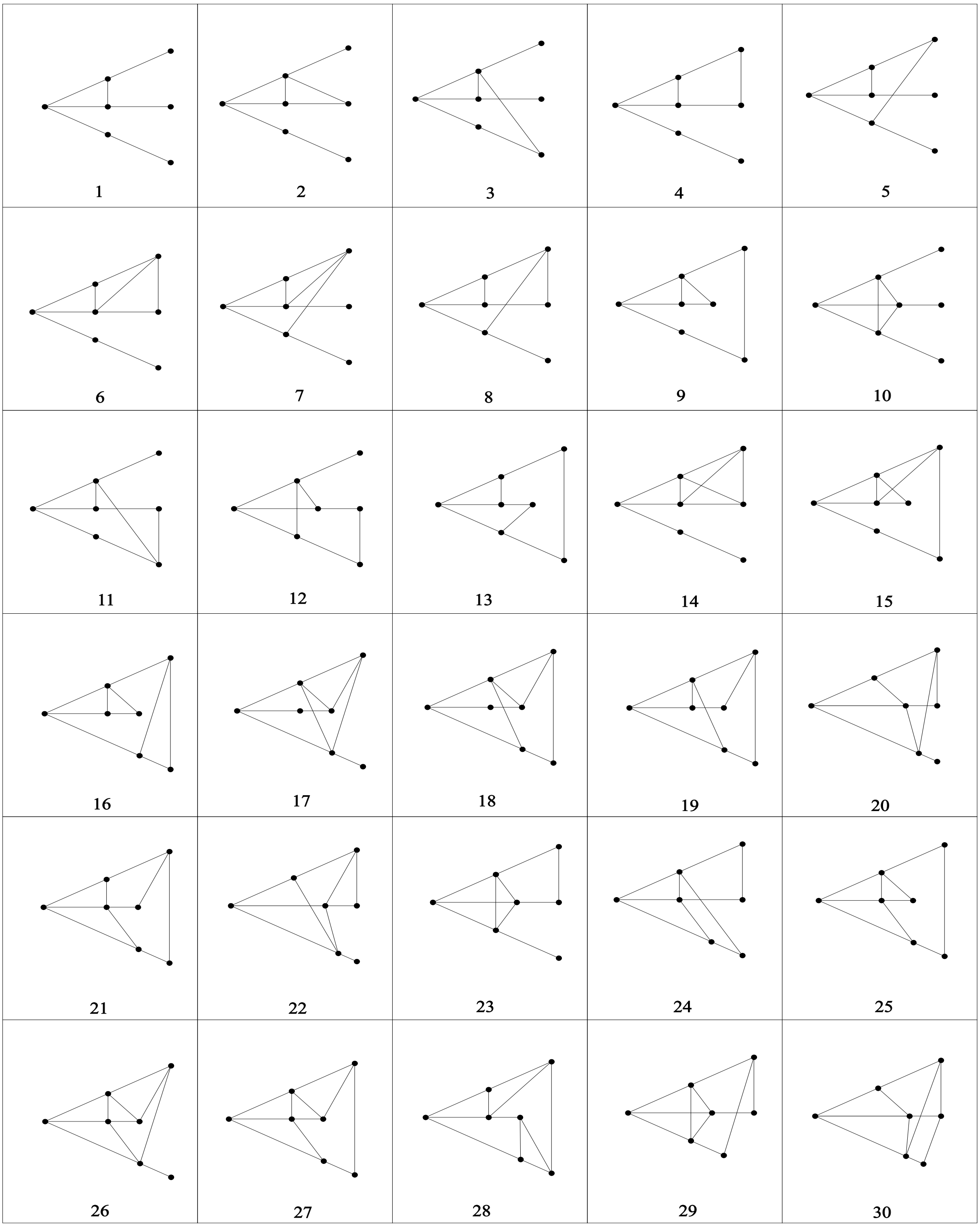
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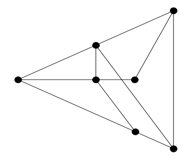

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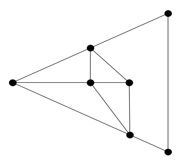

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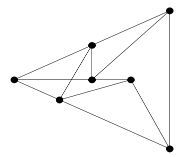

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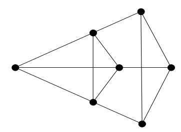

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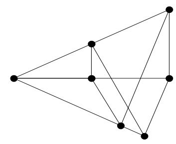

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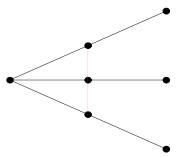

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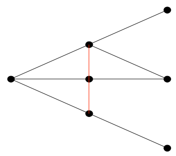

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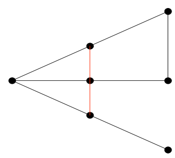

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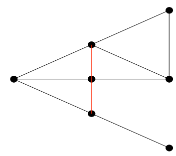

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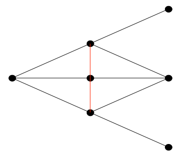

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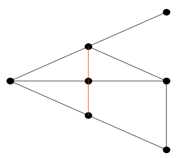

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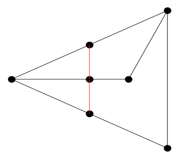

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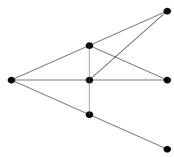

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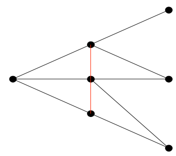

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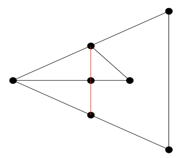

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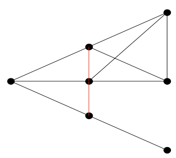

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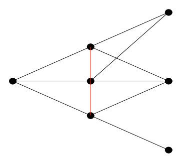

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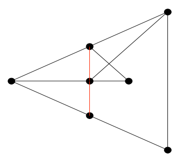

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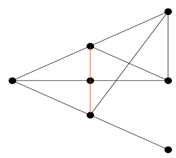

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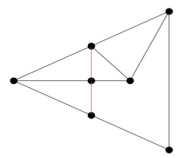

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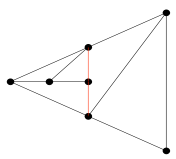

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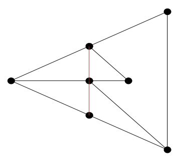

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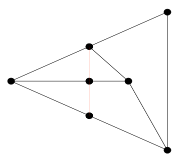

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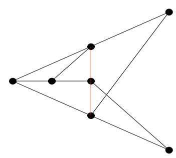

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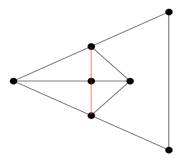

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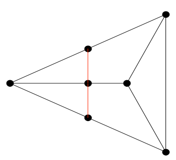

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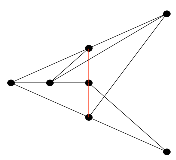

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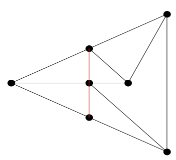

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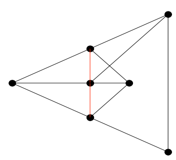

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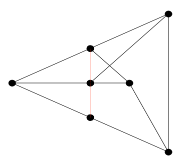

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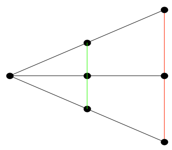
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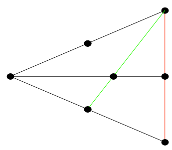
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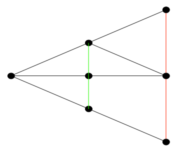
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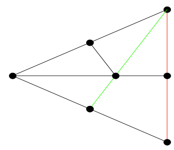
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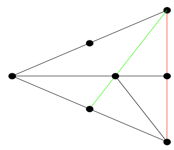
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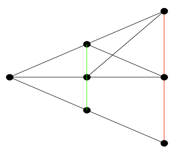
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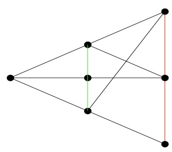
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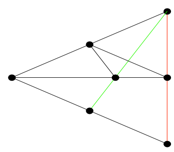
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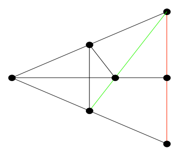
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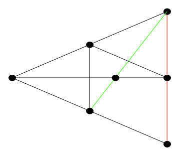
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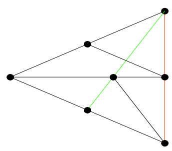
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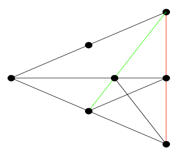
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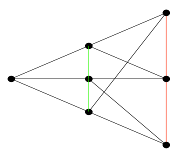
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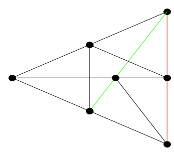
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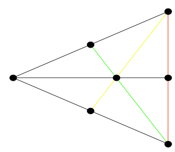
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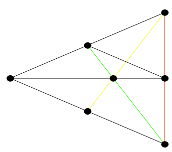
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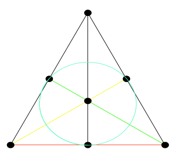
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