Generalized Polygons in Finite Projective Spaces

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Abstract

We survey the theory of embedded finite generalized polygons in projective spaces. In a first part, we give an overview of the main results concerning generalized quadrangles, and in a second part, we deal with generalized hexagons. We conclude by mentioning some results on homogeneous and grumbling embeddings of quadrangles and hexagons.

1 Introduction

In 1974 Buekenhout and Lefèvre [1] published their beautiful theorem classifying all finite generalized quadrangles fully embedded in $\mathrm{PG}(d, q)$. In 1976 and 1977 Buekenhout and Lefèvre [2] and Lefèvre-Percsy [9] extended this classification to all finite polar spaces fully embedded in $\mathrm{PG}(d, q)$. Deeply influenced by this pioneering work, De Clerck and Thas [4] determined in 1978 all partial geometries fully embedded in finite projective space, and in the same year Thas [15] classified all partial geometries fully embedded in $\mathrm{AG}(d, q)$. In 1980 Dienst [5, 6] proved the analogue of the Buekenhout-Lefèvre theorem for infinite generalized quadrangles. During the last 25 years many papers were written on embeddings of generalized quadrangles, generalized hexagons, polar spaces, partial geometries, semipartial geometries, dual semipartial geometries, $(0, \alpha)$-geometries, (α, β)-geometries in projective and affine spaces, also considering weaker kinds of embeddings than the full embeddings. Here the most difficult cases are the embeddings of the generalized polygons, and we will survey here what is known about their possible embeddings in $\mathrm{PG}(d, q)$.

Weak (or polarized) and lax embeddings of generalized quadrangles in finite projective spaces were considered from 1998 on by Thas and Van Maldeghem [18, 25, 27]; a small, but interesting, case was overlooked and this was solved very recently. As to the generalized hexagons, there we find a good and very useful embedding theorem already in 1979, in a paper by Cameron and Kantor [3] on "2-Transitive and antiflag transitive collineation groups of finite projective spaces". From 1996 on Thas and Van Maldeghem [16, 19, 20, 21, 22, 23, 24, 26, 27] wrote nine papers on that difficult problem, but still many questions have to be solved. Finally, most of the problems on embeddings of generalized octagons are still widely open.

2 Embeddings and generalized polygons

2.1 Embeddings

A *lax embedding* of a connected point-line geometry $\mathcal{S}$ with point set P in a finite projective space $\mathrm{PG}(d, K), d \geq 2$ and K a field, is a monomorphism θ of $\mathcal{S}$ into the geometry of points and lines of $\mathrm{PG}(d, K)$ satisfying

(1) the set P^θ generates $\mathrm{PG}(d, K)$.

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *laxly embedded* in $\mathrm{PG}(d, K)$.

A *weak* or *polarized embedding* in $\mathrm{PG}(d, K)$ is a lax embedding which also satisfies

(2) for any point x of $\mathcal{S}$, the subspace generated by the set

$$X = \{y^\theta \parallel y \in P \text{ is not at maximum distance from } x\}$$

meets P^θ precisely in X.

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *weakly* or *polarizedly embedded* in $\mathrm{PG}(d, K)$.

A *full embedding* in $\mathrm{PG}(d, K)$ is a lax embedding with the additional property that for every line L of $\mathcal{S}$, all points of $\mathrm{PG}(d, K)$ on the line L^θ have an inverse image under θ. In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *fully embedded* in $\mathrm{PG}(d, K)$.

Usually, we simply say that $\mathcal{S}$ is laxly, or weakly, or fully embedded in $\mathrm{PG}(d, K)$ without referring to θ, that is, we identify the points and lines of $\mathcal{S}$ with their images in $\mathrm{PG}(d, K)$.

2.2 Generalized polygons

A *generalized n-gon*, $n \geq 2$, or a *generalized polygon*, is a non-empty point-line geometry the incidence graph of which has diameter n (i.e. two elements are at most at distance n) and girth $2n$ (i.e. the length of any shortest circuit is $2n$).

A *thick generalized polygon* is a generalized polygon for which each element is incident with at least three elements. In this case, the number of points on a line is a constant, say $s+1$, and the number of lines through a point is also a constant, say $t+1$. The pair (s,t) is called the *order* of the polygon; if $s=t$ we say the polygon has *order* s.

If $\mathcal{S}$ is a finite thick generalized n-gon, then by the theorem of Feit and Higman [7] we have $n \in \{2,3,4,6,8\}$. The digons ($n=2$) are trivial incidence structures, the thick generalized 3-gons are the projective planes (here $s=t$), and the generalized 4-gons, 6-gons, 8-gons are also called *generalized quadrangles, generalized hexagons* and *generalized octagons*, respectively.

It should be mentioned that generalized n-gons were introduced by Tits [28] in 1959 in order to study classical, exceptional and twisted Chevalley groups (of relative rank 2).

3 Fully embedded finite generalized quadrangles

3.1 Finite generalized quadrangles

A finite *generalized quadrangle* (GQ) $\mathcal{S}=(P,B,\mathrm{I})$ of order (s,t) is a non-empty point-line incidence geometry satisfying the following axioms.

(i) Every line is incident with $s+1$ points for some integer $s \geq 1$ and two lines are incident with at most one point.

(ii) Every point is incident with $t+1$ lines for some integer $t \geq 1$ and two points are incident with at most one common line.

(iii) Given any point x and any line L not incident with x, that is, $x \not\mathrm{I}\, L$, there exist a unique point y and a unique line M with $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$.

3.2 The finite classical generalized quadrangles

The geometry of points and lines of a non-singular quadric of projective index 1, that is, of Witt index 2, in $\mathrm{PG}(d,q)$ is a GQ denoted by $Q(d,q)$. Here only the cases $d=3,4,5$ occur and $Q(d,q)$ has order (q,q^{d-3}). The geometry of all points of $\mathrm{PG}(3,q)$, together with all totally isotropic lines of a symplectic polarity in $\mathrm{PG}(3,q)$, is a GQ of order (q,q) denoted by $W(q)$. The geometry of points and lines of a non-singular hermitian variety of projective index 1 in $\mathrm{PG}(d,q^2)$ is a GQ $H(d,q^2)$ of order (q^2,q^{2d-5}). Here $d=3$ or $d=4$. Any GQ isomorphic to one of these examples is called *classical*; the examples themselves are called the *natural embeddings* of the classical GQ. We notice that $W(q)$ is isomorphic to the dual of $Q(4,q)$, and that $H(3,q^2)$ is isomorphic to the dual of $Q(5,q)$; $W(q)$ (respectively, $Q(4,q)$) is isomorphic to its dual if and only if q is even. See Payne and Thas [11] for more details.

3.3 Fully embedded finite generalized quadrangles

Theorem of Buekenhout and Lefèvre [1]. *If $\mathcal{S}$ is a generalized quadrangle fully embedded in* $\mathrm{PG}(d,q)$, *then $\mathcal{S}$ is one of the natural embeddings of the classical generalized quadrangles.*

4 Weakly embedded finite generalized quadrangles

4.1 The universal weak embedding of $W(2)$

For finite polar spaces of rank at least 3, it follows from Thas and Van Maldeghem [17] that any weak embedding in $\mathrm{PG}(d,q)$ is a full embedding in some subspace $\mathrm{PG}(d,q')$ over the subfield $\mathrm{GF}(q')$ of $\mathrm{GF}(q)$. This is certainly not true for GQ as the following counterexample shows. Let x_1,x_2,x_3,x_4,x_5 be the consecutive vertices of a proper pentagon in $W(2)$. Let K be any field and identify x_i, $i\in\{1,2,3,4,5\}$, with the point $(0,\cdots,0,1,0,\cdots,0)$ of $\mathrm{PG}(4,K)$, where the 1 is in the i-th position. Identify the unique point y_{i+3} of $W(2)$ on the line x_ix_{i+1} and different from both x_i and x_{i+1}, with the point $(0,\cdots,0,1,1,0,\cdots,0)$ of $\mathrm{PG}(4,K)$, where the 1's are in the i-th and the $(i+1)$-th position (subscripts are taken modulo 5). Finally, identify the unique point z_i of the line x_iy_i of $W(2)$ different from both x_i and y_i, with the point whose coordinates are all 0 except in the i-th position, where the coordinate is -1, and in the positions $i-2$ and $i+2$, where the coordinate takes the value 1 (again subscripts are taken modulo 5). It is an elementary exercise to check that this defines a weak embedding of $W(2)$ in $\mathrm{PG}(4,K)$. We call this the *universal weak embedding* of $W(2)$ in $\mathrm{PG}(4,K)$.

4.2 Main result

Theorem (Thas and Van Maldeghem [18]). *Let $\mathcal{S}$ be a finite thick generalized quadrangle of order (s,t) weakly embedded in* $\mathrm{PG}(d,q)$. *Then either s is a prime power*, $\mathrm{GF}(s)$ *is a subfield of* $\mathrm{GF}(q)$ *and $\mathcal{S}$ is fully embedded in some subspace* $\mathrm{PG}(d,s)$ *of* $\mathrm{PG}(d,q)$, *or $\mathcal{S}$ is isomorphic to $W(2)$ and the weak embedding is the universal one in a projective 4-space over an odd characteristic finite field.*

Remarks. (1) All weak embeddings in $\mathrm{PG}(3,q)$ of finite thick GQ are classified by Lefèvre-Percsy [10], although she used a stronger definition for "weak embedding".

(2) All polarized lax embeddings of arbitrary generalized quadrangles in arbitrary projective spaces were classified by Steinbach and Van Maldeghem in [12, 13].

5 Laxly embedded finite generalized quadrangles

5.1 Introduction

For thick GQ "being fully or weakly embedded" characterizes the finite classical GQ amongst the others. This is no longer true for laxly embedded GQ. To handle laxly embedded GQ, completely different combinatorial and geometric methods are needed than in the full and the weak case. Also, these methods do not work in the case of laxly embedded GQ in the plane. By extension (of the ground field) and projection, every GQ which admits an embedding in some projective space admits a lax embedding in a plane. This makes the classification problem for $d = 2$ very hard and probably impossible. Hence we will restrict our attention to the case $d \geq 3$. Notice also that by substituting $\mathrm{AG}(d,q)$ for $\mathrm{PG}(d,q)$ in the definition of laxly, weakly and fully embedded GQ, every such embedding in affine space gives rise to a lax embedding in the corresponding projective space.

In 2001 a paper on laxly embedded GQ in $\mathrm{PG}(d,q)$ appeared in Proc. of the London Math. Soc. [25]. There are two theorems on these embeddings with in total a proof of about 35 pages. As the statement of the second theorem is long and technical we will not give the complete formulation here.

5.2 Characterization theorems

Theorem 1 (Thas and Van Maldeghem [25]). *If the generalized quadrangle $\mathcal{S}$ of order (s,t), with $s > 1$, is laxly embedded in $\mathrm{PG}(d,q)$, then $d \leq 5$. Furthermore we have the following isomorphisms.*

(i) If $d = 5$, then $\mathcal{S} \cong Q(5,s)$.

(ii) If $d = 4$, then $s \leq t$.

(a) If $s = t$, then $\mathcal{S} \cong Q(4, s)$.

(b) If $t = s + 2$, then $s = 2$ and $\mathcal{S} \cong Q(5, 2)$.

(c) If $t^2 = s^3$, then $\mathcal{S} \cong H(4, s)$.

(iii) If $d = 3$ and $s = t^2$, then $\mathcal{S} \cong H(3, s)$.

Theorem 2 (Thas and Van Maldeghem [25]). *Suppose that the generalized quadrangle $\mathcal{S}$ of order (s, t) is laxly embedded in* $\mathrm{PG}(d, q)$*, where $d \geq 3$ and $\mathcal{S} \cong Q(5, s), Q(4, s), H(4, s), H(3, s)$ or the dual of $H(4, t)$. Then $\mathcal{S}$ arises by extensions (of the ground field) and projections either from a full embedding, or from a certain unique $\mathcal{S} \cong Q(5, 2)$ in* $\mathrm{PG}(5, p)$ *with p an odd prime, or from a certain unique $\mathcal{S} \cong Q(4, 2)$ in* $\mathrm{PG}(4, p)$ *with p an odd prime, or from a certain unique $\mathcal{S} \cong Q(4, 3)$ in* $\mathrm{PG}(4, q)$, $q \equiv 1 \bmod 3$, *with q either an odd prime or the square of a prime p with* $p \equiv 2 \bmod 3$, *or $\mathcal{S} \cong H(3, 4)$ and is laxly embedded in* $\mathrm{PG}(3, q)$ *with* $q \notin \{2, 3, 5\}$.

Remarks
(a) Note that the dual of $H(4, t)$ does not occur.
(b) With our techniques $W(s)$, with s odd, could not be handled.

5.3 The forgotten case

In our paper of 2001 the last case of Theorem 2 was overlooked. As this lax embedding of $H(3, 4)$ is quite elegant, we will say a few words about it. This is again joint work of Thas and Van Maldeghem [27].

Consider a non-singular cubic surface $\mathcal{F}$ in $\mathrm{PG}(3, K)$, K any commutative field, and assume that $\mathcal{S}$ has 27 lines. Then necessarily $K \neq \mathrm{GF}(q)$ with $q \in \{2, 3, 5\}$; see Chapter 20 of Hirschfeld [8]. Let $\mathcal{S}' = (P', B', \mathrm{I}')$ be the following incidence structure : the elements of P' are the 45 tritangent planes of $\mathcal{F}$ (that are the planes which intersect $\mathcal{F}$ in 3 lines), the elements of B' are the 27 lines of $\mathcal{F}$, and a point $\pi \in P'$ is incident with a line $L \in B'$ if and only if $L \subset \pi$. It is well-known that $\mathcal{S}'$ is the unique generalized quadrangle of order $(4, 2)$. Let β be an anti-isomorphism of $\mathrm{PG}(3, K)$, let $(P')^\beta = P$ and let $(B')^\beta = B$. If I is containment, then $\mathcal{S} = (P, B, \mathrm{I})$ is again isomorphic to $H(3, 4)$, and is contained in the dual surface $\hat{\mathcal{F}}$ of $\mathcal{F}$ which again contains exactly 27 lines. Clearly $\mathcal{S}$ is laxly embedded in $\mathrm{PG}(3, K)$. An *Eckardt point* y of $\mathcal{F}$ is a point contained in 3 lines of $\mathcal{F}$, which are then contained in the tangent plane of $\mathcal{F}$ at y. If $x \in P$, then the 3 lines of $\mathcal{S}$

incident with x are contained in a plane π if and only if $\pi^{\beta^{-1}}$ is an Eckhardt point of $\mathcal{F}$. Thas and Van Maldeghem [27] show that every lax embedding in $\mathrm{PG}(3,K)$ of the unique generalized quadrangle of order $(4,2)$ is of the type described above. Such a lax embedding is uniquely defined by 5 mutually skew lines $A_1, A_2, \cdots, A_5$ with a transversal B_6 such that each five of the six lines are linearly independent (in the sense that their Plücker (or line) coordinates define 5 independent points in $\mathrm{PG}(5,K)$). Such a configuration exists for every commutative field K except for $K = \mathrm{GF}(q)$ with $q = 2, 3$ or 5; see Chapter 20 of Hirschfeld [8]. The embedding is weak if and only if $\mathcal{F}$ has 45 Eckhardt points; in such a case $\mathrm{GF}(4)$ is a subfield of K, see Hirschfeld [8]. Finally, by Thas and Van Maldeghem [18], in that case $\mathcal{S}$ is a full embedding of that generalized quadrangle in a subspace $\mathrm{PG}(3,4)$ of $\mathrm{PG}(3,K)$, so by Buekenhout and Lefèvre [1] is the natural embedding of $H(3,4)$ in that subspace $\mathrm{PG}(3,4)$. Hence we have the following theorem.

Theorem (Thas and Van Maldeghem [27]) *Let K be any commutative field and let $\mathcal{S}$ be a lax embedding of the unique generalized quadrangle of order $(4,2)$ in $\mathrm{PG}(3,K)$. Then $|K| \neq 2, 3, 5$ and $\mathcal{S}$ arises from a unique non-singular cubic surface $\mathcal{F}$ as explained above. Also, the embedding is polarized if and only if $\mathcal{F}$ admits 45 Eckhardt points. In that case $\mathrm{GF}(4)$ is a subfield of K and $\mathcal{S}$ is a natural embedding of $H(3,4)$ in a subspace $\mathrm{PG}(3,4)$ of $\mathrm{PG}(3,K)$.*

6 Fully embedded generalized hexagons

6.1 Finite generalized hexagons

A *finite generalized hexagon* (GH) $\mathcal{H} = (P, B, \mathrm{I})$ of order (s,t), $s,t \geq 1$, is a non-empty point-line incidence geometry satisfying the following axioms.

(i) Every line contains $s+1$ points and two lines are incident with at most one point.

(ii) Every point is on $t+1$ lines and two points are incident with at most one line.

(iii) Given two distinct elements v, w (points and/or lines), there always exists a minimal path $v = v_0 \mathrm{~I~} v_1 \mathrm{~I~} \cdots \mathrm{~I~} v_k = w$ with $k \leq 6$, and if $k < 6$, then the minimal path is unique.

6.2 The finite classical generalized hexagons

Tits [28] defines two classes of thick GH arising from trialities on the non-singular hyperbolic quadric in the projective 7-dimensional space over a commutatieve field. In the finite case the two classes are related to Dickson's simple group $G_2(q)$ respectively the triality group ${}^3D_4(q)$; every GH isomorphic to one of these will be called a *classical generalized hexagon*. In the first case one obtains a generalized hexagon of order (q,q) which lies in a hyperplane of $\mathrm{PG}(7,q)$, and denoted by $H(q)$, in the second case one obtains a generalized hexagon of order $(q,\sqrt[3]{q})$, here denoted by $T(q,\sqrt[3]{q})$. So the former is represented on a non-singular quadric $Q(6,q)$ in $\mathrm{PG}(6,q)$ (its points are all the points of $Q(6,q)$ while its lines are some lines of $Q(6,q)$; see Tits [28] for more details). If q is even, the polar space $Q(6,q)$ is isomorphic to the non-singular symplectic polar space $W(5,q)$ and hence in this case one obtains a representation of $H(q)$ in 5-dimensional space. We call these three representations of the classical GH the *natural embeddings*.

A comprehensive introduction to these classical generalized hexagons, their construction and their properties are contained in Van Maldeghem [29].

6.3 Regularly fully embedded finite generalized hexagons

Any embedding of any thick finite GH $\mathcal{H}$ will be called a *regular* or *ideal embedding* in $\mathrm{PG}(d,q)$ if the following conditions are satisfied.

(i) The points collinear (in $\mathcal{H}$) with any given point in $\mathcal{H}$ are coplanar in $\mathrm{PG}(d,q)$.

(ii) The points not opposite (that is, not at maximum distance from) any given point in $\mathcal{H}$ are contained in a hyperplane of $\mathrm{PG}(d,q)$.

Theorem (Thas and Van Maldeghem [16]). *A finite thick generalized hexagon $\mathcal{H}$ is regularly fully embedded in some* $\mathrm{PG}(d,q)$ *if and only if it is a natural embedding of a classical generalized hexagon.*

Remarks. (a) In the proof we rely on a result on embeddings of GH by Cameron and Kantor [3]. Their assumptions are stronger and they obtain the natural embeddings in $\mathrm{PG}(5,q)$ and $\mathrm{PG}(6,q)$.

(b) In the same paper Thas and Van Maldeghem show that a thick finite generalized octagon does not admit a regular full embedding.

6.4 Flatly fully embedded finite generalized hexagons

Any embedding of any thick finite GH $\mathcal{H}$ will be called a *flat embedding* in $\mathrm{PG}(d,q)$ if condition (i) of 6.3 is satisfied.

Theorem (Thas and Van Maldeghem [19]). *If a finite thick generalized hexagon* $\mathcal{H}$ *of order* (q,t) *is flatly fully embedded in* $\mathrm{PG}(d,q)$*, then* $4 \leq d \leq 7$ *and* $t \leq q$*. Also, if* $d = 7$*, then* $\mathcal{H}$ *is classical and the embedding is natural. If* $d = 6$ *and* $t^5 > q^3$*, then* $\mathcal{H}$ *is classical and the embedding is natural. If* $d = 5$ *and* $q = t$*, then* $\mathcal{H}$ *is classical,* q *is even, and the embedding is the natural one.*

6.5 Weakly (or polarizedly) fully embedded finite generalized hexagons

Any embedding of any thick finite GH $\mathcal{H}$ will be called a *weak embedding* in $\mathrm{PG}(d,q)$ if condition (ii) of 6.3 is satisfied (this is equivalent to the definition given in 2.1).

Theorem (Thas and Van Maldeghem [19]). *If a finite thick generalized hexagon* $\mathcal{H}$ *of order* (q,t) *is weakly fully embedded in* $\mathrm{PG}(d,q)$*, then* $d \geq 5$*. If* $d = 5$*, then* $q = t$*,* q *is even, and* $\mathcal{H}$ *is the natural embedding. If* $d = 6$ *and* q *is odd, then* $q = t$ *and* $\mathcal{H}$ *is the natural embedding.*

6.6 Embeddings of the flag geometries of projective planes in finite projective spaces

The *flag geometry* $\mathcal{H} = (P, B, \mathrm{I})$ of a finite projective plane π of order s is the GH of order $(s,1)$ obtained from π by putting P equal to the set of all flags (that is, the incident point-line pairs) of π, by putting B equal to the set of all lines and points of π, and where I is the natural incidence relation (inverse containment). The following theorem took four papers to be proved, but this classification is essential to handle the embeddings of the GH which are isomorphic to the point-line duals of the classical GH $H(q)$.

Theorem (Thas and Van Maldeghem [20, 21, 22, 23]). *If the flag geometry of the finite projective plane* π *is weakly fully embedded in* $\mathrm{PG}(d,s)$*, then* π *is Desarguesian and* $d \in \{6,7,8\}$*. Also, these embeddings are completely classified.*

Remark. In a fifth paper [24] the authors even weaken the hypotheses (there are cases where the assumption "weakly" can be deleted).

6.7 Full embeddings of the finite dual split Cayley hexagons

Full embeddings of GH which are isomorphic to the point-line duals of the classical GH $H(q)$ were also handled and very strong results were obtained.

Theorem (Thas and Van Maldeghem [26]). *If the generalized hexagon $\mathcal{H}$ is isomorphic to the point-line dual of $H(q)$ and is fully embedded in* $\mathrm{PG}(d,q)$*, then $d \geq 13$. Moreover, if $d = 13$, then the embedding is unique.*

6.8 Remarks

Thas and Van Maldeghem [19, 27] also obtained many results on lax embeddings of GH in finite projective spaces. Finally, all regular embeddings of generalized hexagons in arbitrary projective spaces were classified by Steinbach and Van Maldeghem [14]. Their proof provides an alternative proof for the finite case.

7 Grumbling and homogeneous embeddings

A *grumbling embedding* of a classical generalized polygon defined over a field with characteristic c is an embedding in a projective space defined over a field with characteristic $c' \neq c$. For example, the universal embedding of $W(2)$ in a projective space $\mathrm{PG}(4,q)$, with q odd, is a grumbling embedding.

A *homogeneous embedding* of a classical generalized polygon $\mathcal{S}$ is a lax embedding with the additional property that every collineation of $\mathcal{S}$ is induced by a collineation of the projective space. The natural embeddings are all homogeneous.

It seems that homogeneous embeddings that are not natural, and also grumbling embeddings, are phenomena that occur only for small orders.

7.1 Homogeneous and grumbling embeddings of generalized quadrangles

We already met grumbling embeddings for $W(2)$ and $H(3,4)$. The quadrangle $Q(5,2)$, which is the dual of $H(3,4)$, has a unique embedding in $\mathrm{PG}(5,K)$, for every field K, as follows from Theorems 6.1 and 6.2 of [25] (proved for finite K, but the proof is easily seen to be also valid for infinite (not necessarily commutative) fields), which is moreover homogeneous. We call this the *universal embedding of $Q(5,2)$ over K*.

In [27], the following theorem is proved.

Theorem (Thas and Van Maldeghem [27]). *All non-grumbling homogeneous lax embeddings of $W(2)$, $H(3,4)$ and $Q(5,2)$ arise from their natural embeddings ($W(2)$ also viewed as $Q(4,2)$) by extending the ground field. Apart from the unique universal lax embedding of $W(2)$ in* $\mathrm{PG}(4,K)$*, and the unique universal embedding of $Q(5,2)$ in* $\mathrm{PG}(5,K)$*, for any field K with characteristic unequal to 2, there does not exist any grumbling homogeneous embedding of either $W(2)$, $H(3,4)$ or $Q(5,2)$.*

7.2 Homogeneous and grumbling embeddings of generalized hexagons

Here we restrict ourselves to $H(2)$ and its dual, denoted $H(2)^*$.

We start with the classification of homogeneous full embeddings of $H(2)$ and $H(2)^*$. We will not give an explicit description of the so-called universal embedding of $H(2)^*$ (we refer the interested reader to [27]), but below we explicitly describe an embedding of $H(2)$ in 13-dimensional projective space over any field K. Putting K equal to $\mathrm{GF}(2)$, one obtains the universal full embedding of $H(2)$ in $\mathrm{PG}(13,2)$.

Theorem (Thas and Van Maldeghem [27]). *The hexagon $H(2)$ admits exactly four homogeneous full embeddings: one in* $\mathrm{PG}(13,2)$*, which is the universal embedding, one in* $\mathrm{PG}(12,2)$*, one in* $\mathrm{PG}(6,2)$*, which is the natural embedding in a parabolic quadric, and one in* $\mathrm{PG}(5,2)$*, which is the natural embedding into a symplectic space.*
The hexagon $H(2)^*$ admits exactly one homogeneous full embedding, namely, the universal one in $\mathrm{PG}(13,2)$.

We now give an explicit description of an embedding in $\mathrm{PG}(13,K)$ of $H(2)$, for any field K.

First we define $H(2)$ in an alternative way, see Van Maldeghem [30].

Construction of $H(2)$. We consider the projective plane $\mathrm{PG}(2,2)$. The points of $H(2)$ are the seven points, seven lines, twenty-one flags and twenty-eight antiflags of $\mathrm{PG}(2,2)$ (an antiflag is a non-incident point-line pair). The lines are of two types. For a given flag $\{x,L\}$ of $\mathrm{PG}(2,2)$ (where x is a point of $\mathrm{PG}(2,2)$ and L a line of $\mathrm{PG}(2,2)$ incident with x), the points x, L and $\{x,L\}$ of $H(2)$ form a line of $H(2)$. Also, if x_1, x_2 are the other two points incident with L in $\mathrm{PG}(2,2)$, and if L_1, L_2 are the other two lines incident with x in $\mathrm{PG}(2,2)$, then the set $\{\{x,L\},\{x_1,L_1\},\{x_2,L_2\}\}$ forms a line of $H(2)$.

An embedding of $H(2)$**.** Let V be a 14-dimensional vector space over the arbitrary field K. Let a basis of V be indexed by the points and lines of $\mathrm{PG}(2,2)$. For every point or line

x of $\mathrm{PG}(2,2)$, we denote by $\overline{x}$ the corresponding basis vector of V, which we also identify with a unique point of $\mathrm{PG}(13,K)$. The point of $H(2)$ defined by the point x of $\mathrm{PG}(2,2)$ is represented in $\mathrm{PG}(13,K)$ as the projective point corresponding with the vector of V obtained as the sum of the nine basis vectors of V indexed by the points of $\mathrm{PG}(2,2)$ different from x and the lines of $\mathrm{PG}(2,2)$ incident with x. The point of $H(2)$ defined by the line L of $\mathrm{PG}(2,2)$ is represented in $\mathrm{PG}(13,K)$ as the projective point corresponding with the vector of V obtained as the sum of the five vectors of V indexed by the points of $\mathrm{PG}(2,2)$ not incident with L and the line L of $\mathrm{PG}(2,2)$. The point of $H(2)$ defined by the flag $\{x,L\}$ of $\mathrm{PG}(2,2)$ is represented in $\mathrm{PG}(13,K)$ as the projective point corresponding with the vector of V obtained as the sum of the four vectors of V indexed by the points on L different from x, and the lines through x different from L. Finally, the point of $H(2)$ defined by the antiflag $\{x,L\}$ of $\mathrm{PG}(2,2)$ is represented in $\mathrm{PG}(13,K)$ as the projective point corresponding with the vector $\overline{x}+\overline{L}$.

One can check for his own that this indeed defines a lax embedding of $H(2)$ in $\mathrm{PG}(13,K)$. For $K=\mathrm{GF}(2)$, we obtain the universal embedding. In the general case, the embedding is homogeneous.

The embedding of $H(2)$ in $\mathrm{PG}(12,2)$ as referred to in the previous theorem can be obtained from the above description of the universal embedding of $H(2)$ by projecting from the point of $\mathrm{PG}(13,2)$ corresponding with the vector of V obtained by adding all basis vectors defined by lines of $\mathrm{PG}(2,2)$, onto a hyperplane of $\mathrm{PG}(13,2)$ not containing that point.

Remarks. (1) One can show, see [27], that the hexagons $H(2)$ and $H(2)^*$ both admit a unique homogeneous embedding over the real numbers. For $H(2)$, it is the embedding described above, when putting $K=\mathbb{R}$. In fact, the embeddings of both $H(2)$ and $H(2)^*$ in $\mathrm{PG}(13,K)$, for an arbitrary field K, are projectively unique.

(2) One would like to call the above embedding of $H(2)$ in $\mathrm{PG}(13,K)$ also *universal*. But we hesitate to do so for the simple reason that we cannot prove (yet) that every embedding of $H(2)$ in some projective space over K arises as a projection of the above embedding. If this were true, this would justify the name *universal* (as is the case for $W(2)$). A similar remark applies to $H(2)^*$ and its embedding in $\mathrm{PG}(13,K)$.

(3) Are similar results true for the dual of $T(8,2)$, and for the classical generalized octagon of order $(2,4)$? This are open questions.

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