A Hölz-design in the generalized hexagon $\mathsf{H}(q)$

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Abstract

In this paper, we give an alternative construction of the Hölz design $\mathsf{D}_{\text{Hölz}}(q)$, for $q \not\equiv 2 \bmod 3$. If $q \equiv 2 \bmod 3$, then our construction yields a $2-(q^3+1, q+1, \frac{q+4}{3})$-subdesign of the Hölz-design. The construction uses two hexagons embedded in the parabolic quadric $\mathsf{Q}(6,q)$.

Mathematics Subject Classification 2000: 51E12.
Key words and phrases: Split Cayley hexagon, Hölz-design, one-point extension, Ahrens-Szekeres generalized quadrangles

1 Introduction

In 1981, Hölz [4] constructed a family of $2-(q^3+1, q+1, q+2)$-designs whose point set coincides with the point set of the Hermitian unital over the field $\mathsf{GF}(q)$, and with an automorphism group containing $\mathsf{PGU}_3(q)$. In fact, the blocks of the design are the blocks of the unital and the Baer-conics lying on the unital (viewed as a Hermitian curve in $\mathsf{PG}(2,q^2)$). We will call the blocks corresponding to the conics *Hölz-blocks*. Here, q is any odd prime power. Two years later, Thas [9] proved that these designs are one-point extensions of the Ahrens-Szekeres generalized quadrangles $\mathsf{AS}(q)$ of order $(q-1, q+1)$ (see [1]). In the present paper, we define, for each odd prime power q, a $2-(q^3+1, q+1, 1+\frac{q+1}{(q+1,3)})$-design by looking at the common point reguli of two split Cayley generalized hexagons represented on the parabolic quadric $\mathsf{Q}(6,q)$. We show that these designs are either isomorphic to the Hölz-designs (for $q \not\equiv 2 \bmod 3$) or subdesigns of the Hölz-designs (for $q \equiv 2 \bmod 3$). The fact that, for $q \equiv 2 \bmod 3$, the Hölz-design has such large subdesigns is apparently unnoticed in the literature. In fact, these subdesigns emerge as a union of orbits under the

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subgroup $\mathsf{PSU}_3(q)$, which acts transitively on the Hölz-blocks of the Hölz-design only if $q \not\equiv 2$ mod 3. Hence we have an alternative, rather unexpected, construction of the Hölz-designs $\mathsf{D}_{\text{Hölz}}(q)$ for $q \not\equiv 2$ mod 3, and we have a geometric explanation of the non-transitivity of the subgroup $\mathsf{PSU}_3(q)$ on the Hölz-blocks if $q \equiv 2$ mod 3.

2 Preliminaries

2.1 Generalized hexagons and the split Cayley hexagon

A *generalized hexagon* Γ (of order (s,t)) is a point-line geometry the incidence graph of which has diameter 6 and girth 12 (and every line is incident with $s+1$ points; every point incident with $t+1$ lines). Note that, if $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set of Γ, then the *incidence graph* is the (bipartite) graph with set of vertices $\mathcal{P} \cup \mathcal{L}$ and adjacency given by incidence. The definition implies that, given any two elements a, b of $\mathcal{P} \cup \mathcal{L}$, either these elements are at distance 6 from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path from a to b. If for two points a, b there exists a unique point collinear with both, then we denote that point by $a \bowtie b$. Finally, the set $a^\perp$ is defined to be the set of all points collinear with a.

In this paper we are mostly interested in the split Cayley hexagons $\mathsf{H}(q)$, for q odd. A model $\mathcal{H}$ of this hexagon, the construction of which is due to Tits [10], can be defined as follows (see [10]; also [11]). Choose coordinates in the projective space $\mathsf{PG}(6,q)$ in such a way that the points of $\mathsf{Q}(6,q)$ satisfy the equation $X_0X_4+X_1X_5+X_2X_6 = X_3^2$, and let the points of $\mathcal{H}$ be all points of $\mathsf{Q}(6,q)$. The lines of $\mathcal{H}$ are the lines on $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$.

To make the points and lines more concrete to calculate with, we will use the coordinatization of $\mathcal{H}$ (see Chapter 3 of [11]; this coordinatization is originally due to De Smet and Van Maldeghem [2]). We thus obtain the labelling of points and lines of $\mathcal{H}$ by i-tuples with entries in the field $\mathsf{GF}(q)$, and two 1-tuples (∞) and $[\infty]$, with $\infty \notin \mathsf{GF}(q)$, as given in Table 1.

In order to complete the description, we need to express the incidence relation with these new coordinates. If we consider the 1-tuples (∞) and $[\infty]$ formally as 0-tuples (because they do not contain an element of $\mathsf{GF}(q)$), then a point, represented by an i-tuple, $0 \le i \le 5$, is incident with a line, represented by a j-tuple, $0 \le j \le 5$, if and only if either $|i-j|=1$ and the tuples coincide in the first $\min\{i,j\}$ coordinates, or $i = j = 5$ and, with notation of Table 1,

$$\begin{cases} k'' = a^3k + l - 3a''a^2 + 3aa', \\ b' = a^2k + a' - 2aa'', \\ k' = a^3k^2 + l' - kl - 3a^2a''k - 3a'a'' + 3aa''^2, \\ b = -ak + a'', \end{cases}$$

POINTS	
Coordinates in $\mathcal{H}$	Coordinates in $\mathsf{PG}(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l,a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,-a'+aa'',$ $1,l+2aa'-a^2a'',-l'+a'a'')$

LINES	
Coordinates in $\mathcal{H}$	Coordinates in $\mathsf{PG}(6,q)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(-l-aa',1,0,-a,0,a^2,-a'),$ $(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 1: Coordinatization of $\mathcal{H}$

or, equivalently,

$$\begin{cases} a''=ak+b, \\ l'=a^3k^2+k'+kk''+3a^2kb+3bb'+3ab^2, \\ a'=a^2k+b'+2ab, \\ l=-a^3k+k''-3ba^2-3ab'. \end{cases}$$

The generalized hexagon $\mathsf{H}(q)$ has the following property (see [11], 1.9.17 and 2.4.15). Let x,y be two opposite points and let L,M be two opposite lines at distance 3 from both x,y. All points at distance 3 from both L,M are at distance 3 from all lines at distance 3 from both x,y. Hence we obtain a set $\mathcal{R}(x,y)$ of $q+1$ points every member of which is at distance 3 from any member of a set $\mathcal{R}(L,M)$ of $q+1$ lines. We call $\mathcal{R}(x,y)$ a *point regulus*, and $\mathcal{R}(L,M)$ a *line regulus*. Any regulus is determined by two of its elements. The two above reguli are said to be *complementary*, i.e. every element of one regulus is at distance 3 from every element of the other regulus. Every regulus has a unique complementary regulus. Now consider again our model $\mathcal{H}$. We will call a line of the quadric $\mathsf{Q}(6,q)$ which does not belong to the hexagon $\mathcal{H}$ an *ideal line*. On the quadric $\mathsf{Q}(6,q)$, every line regulus constitutes a hyperbolic quadric isomorphic to $\mathsf{Q}(3,q)$. Hence there is a unique *opposite regulus*, which is a set of $q+1$ ideal lines that intersect every line of the given regulus in a unique point. Since q is odd, the quadric $\mathsf{Q}(6,q)$ is associated to a unique nondegenerate polarity ρ of $\mathsf{PG}(6,q)$ and the image under the polarity of the 3-space generated by a line regulus is a plane which meets $\mathsf{Q}(6,q)$ exactly in the complementary point regulus. Point reguli of $\mathcal{H}$ thus are simply (some) conics on $\mathsf{Q}(6,q)$.

2.2 Designs and the Hölz-design

A $t-(v,k,\lambda)$-*design* (in this case also briefly designated as a t-*design*), for integers t, v, k and λ with $v>k>1$ and $k\geq t\geq 1$, is an incidence structure $\mathcal{D}$ satisfying following axioms: $\mathcal{D}$ contains v points; each of its blocks is incident with k points; any t points are incident with exactly λ common blocks. For further information on designs we refer to [6].

The following class of 2-designs $\mathsf{D}_{\text{Hölz}}(q)$ is due to G. Hölz [4]. Let $\mathcal{U}$ be a hermitian curve of $\mathsf{PG}(2,q^2)$ (see [3]). A Baer subplane (see [5]) $D\cong\mathsf{PG}(2,q)$ is said to satisfy property (H) if for each point $x\in D\cap\mathcal{U}$ the tangent line L_x to $\mathcal{U}$ at x is a line of D (i.e. $|L_x\cap D|=q+1$). If D satisfies property (H) then one can show that, if $|D\cap\mathcal{U}|\geq 3$, then $|D\cap\mathcal{U}|=q+1$. In this case, the points of $D\cap\mathcal{U}$ are collinear if q is even, and the points of $D\cap\mathcal{U}$ are collinear or form an oval in D, if q is odd. If D_1 and D_2 are Baer subplanes satisfying property (H) and if $|D_1\cap D_2\cap\mathcal{U}|\geq 3$, then $D_1\cap\mathcal{U}=D_2\cap\mathcal{U}$. If moreover $D_i\cap\mathcal{U}$ is an oval of D_i, then $D_1=D_2$.

Let q be odd. If x and y are distinct points of $\mathcal{U}$, then

(1) there are exactly $q+1$ Baer subplanes D in $\mathsf{PG}(2,q^2)$ which satisfy property (H) and for which $D\cap\mathcal{U}=xy\cap\mathcal{U}$, and

(2) there are exactly $q+1$ Baer subplanes D in $\mathsf{PG}(2,q^2)$ which satisfy property (H) and for which $D\cap\mathcal{U}$ is an oval of D through x and y.

Let B_1 be the set of all intersections $L\cap\mathcal{U}$ with L a non tangent line of $\mathcal{U}$, and let B' be the set of all intersections $D\cap\mathcal{U}$ with D a Baer subplane of $\mathsf{PG}(2,q^2)$ satisfying property (H) and containing at least three points of $\mathcal{U}$. Finally, let $B^*=B'-B_1$.

Then $S_1=(\mathcal{U},B_1,\in)$ is a $2-(q^3+1,q+1,1)$ design which we will call the *Hermitian design*; $S'=(\mathcal{U},B',\in)$ is a $2-(q^3+1,q+1,q+2)$ design, the *Hölz*-design denoted by $\mathsf{D}_{\text{Hölz}}(q)$, and $S^*=(\mathcal{U},B^*,\in)$ is a $2-(q^3+1,q+1,q+1)$ design. Moreover any two distinct blocks of these designs have at most two points in common. We will call the elements of B_1 the *Hermitian blocks*, and the members of B^* the *Hölz-blocks*.

2.3 Main Result

In this paper, we will prove the following theorem:

Main Result. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two models of* $\mathsf{H}(q)$ *isomorphic to the model $\mathcal{H}$ as defined above. Define the following incidence structure with point set $\mathcal{P}$ and line set $\mathcal{L}$. The points are the common lines of $\mathcal{H}_1$ and $\mathcal{H}_2$. The blocks are the line reguli entirely contained in $\mathcal{P}$, together with the nonempty sets of elements of $\mathcal{P}$ that are incident with a common point regulus of $\mathcal{H}_1$ and $\mathcal{H}_2$. Then, for each $\mathcal{H}_1$, there exists a suitable choice of $\mathcal{H}_2$ such that, for $q\not\equiv 2 \bmod 3$, this incidence structure is isomorphic to $\mathsf{D}_{\text{Hölz}}(q)$ and, for $q\equiv 2 \bmod 3$, it is a $2-(q^3+1,q+1,1+\frac{q+1}{3})$ subdesign of $\mathsf{H}(q)$, invariant under $\mathsf{PSU}_3(q)$ acting naturally on $\mathsf{D}_{\text{Hölz}}(q)$.*

3 Construction and first properties of some designs derived from the split Cayley hexagons

Consider the model of $\mathsf{H}(q)$, q odd, as described in the section 2.1. We refer to that model as $\mathcal{H}_1$. Consider the hyperplane Π with equation $\nu X_1 + X_5 = 0$. From the equation of $\mathsf{Q}(6,q)$, it is clear that Π meets $\mathsf{Q}(6,q)$ in an elliptic quadric isomorphic to $\mathsf{Q}^-(5,q)$ if and only if $-\nu$ is a nonsquare. In this case, by Thas [8], the set $\mathcal{S}$ of lines of $\mathcal{H}_1$ in Π is a *spread* of $\mathcal{H}_1$, i.e., a set of q^3+1 mutually opposite lines. This spread is called *Hermitian* because endowed with the line reguli entirely contained in it, it is isomorphic to a Hermitian design. One easily calculates that

$$\mathcal{S} = \{[\infty]\} \cup \{[x,y,z,-\nu x,\nu y] : x,y,z \in \mathsf{GF}(q)\}.$$

Now consider the point $p = \Pi^\rho$ with coordinates $(0,1,0,0,0,\nu,0)$ in $\mathsf{PG}(6,q)$. Every line through p and a point x of $\Pi \cap \mathsf{Q}(6,q)$ intersects $\mathsf{Q}(6,q)$ only in x. Any other line through p and a point y on the quadric intersects $\mathsf{Q}(6,q)$ in a second point y'. The involution g mapping y to y' and fixing all points of Π extends to an involutive collineation of $\mathsf{PG}(6,q)$, which we also denote by g. It is actually easy to see that g does not preserve $\mathcal{H}_1$. Indeed, the set of lines of $\mathcal{H}_1$ through a point x of $\Pi \cap \mathsf{Q}(6,q)$ fill up a plane of $\mathsf{Q}(6,q)$, and this plane is fixed under g only if it contains p, clearly a contradiction.

We now define $\mathcal{H}_2$ as $\mathcal{H}_1^g$, and we know $\mathcal{H}_1 \neq \mathcal{H}_2$. Henceforth, we shall use the convention of writing a point regulus of $\mathcal{H}_i$ with a subindex i, $i = 1,2$. So the point regulus determined by two points a,b in $\mathcal{H}_i$ is denoted by $\mathcal{R}_i(a,b)$. Since line reguli are determined by $\mathsf{Q}(6,q)$, such a notation for line reguli is superfluous.

Denote by Ω the set of point reguli of $\mathcal{H}_1$ that are also point reguli of $\mathcal{H}_2$. Clearly, Ω contains the subset Ω_1 of all point reguli complementary to the $\frac{(q^3+1)q^3}{(q+1)q}$ line reguli in $\mathcal{S}$.

We now define the following set Ω_2 of point reguli common to $\mathcal{H}_1$ and $\mathcal{H}_2$. Consider two arbitrary lines L and M of $\mathcal{S}$, let a be any point on L, and denote by Θ the 3-space generated by L and M. Let b be the point on M collinear with a on $\mathsf{Q}(6,q)$. Then b is at distance 4 from a in the incidence graph of both $\mathcal{H}_1$ and $\mathcal{H}_2$. Put $r = a \bowtie b$ and denote r^g by r'. Obviously $r'a$ and $r'b$ are lines of $\mathcal{H}_2$, implying r' belongs to Θ^ρ. Hence both r and r' belong to the point regulus in both $\mathcal{H}_1$ and $\mathcal{H}_2$ complementary to $\mathcal{R}(L,M)$. Therefore r' is collinear in $\mathcal{H}_1$ with two points a' and b' (obviously distinct from a and b, respectively) on L and M respectively. Since g is an involution, the lines ra' and rb' of $\mathsf{PG}(6,q)$ are lines of $\mathcal{H}_2$. The point regulus $\mathcal{R}_1(a,b')$ is complementary to $\mathcal{R}(rb,r'a')$ and the point regulus $\mathcal{R}_2(a,b')$ is complementary to $\mathcal{R}(ra',r'b)$. Since rb and $r'a'$ generate the same 3-space Υ in $\mathsf{PG}(6,q)$ as ra' and $r'b$, we conclude that $\mathcal{R}_1(a,b') = \mathcal{R}_2(a,b')$. Moreover, it is clear that Υ is invariant under g, and hence $p \in \Upsilon$. This implies now that $\mathcal{R}_1(a,b')$, which belongs to $\Upsilon^\rho \subseteq p^\rho$, is entirely contained in Π.

The set Ω_2 consist of all point reguli $\mathcal{R}_1(a,b')$, for all choices of L, M and a (but a and b' determine L and M uniquely, so there is no need to include L and M in the notation).

Lemma 1. *With the above notation, we have $\Omega = \Omega_1 \cup \Omega_2$.*

Proof. From our discussion above we already have $\Omega_1 \cup \Omega_2 \subseteq \Omega$. Now suppose that $\Omega \neq \Omega_1 \cup \Omega_2$. Then there are opposite lines A, B of $\mathcal{H}_1$ spanning a 3-space Υ, which meets $\mathsf{Q}(6,q)$ in a hyperbolic quadric, and such that the regulus R' of that quadric opposite to the regulus R containing A and B is a set of $q+1$ lines belonging to $\mathcal{H}_2$. Let A intersect Π in a and let B intersect Π in b'. Then the line ab' does not belong to $\mathsf{Q}(6,q)$ as otherwise it would belong to $\mathcal{H}_2$ as well, a contradiction. Hence $\Pi \cap \Upsilon$ intersects $\mathsf{Q}(6,q)$ in an irreducible conic $\mathcal{C}$. Hence the set of lines R'^g is a line regulus in $\mathcal{H}_1$ containing $\mathcal{C}$, and so is R.

Suppose $R'^g \neq R$. Then R'^g and R generate a 4-space in $\mathsf{PG}(6,q)$. This 4-space contains all lines of R and in addition two points of the complementary point regulus (collinear to a and b', respectively). Since a line regulus and its complement generate $\mathsf{PG}(6,q)$, a line regulus and two points of the complementary point regulus generate a 5-space, a contradiction. Hence $R^g = R'$. But now defining L and M as the unique lines of Π in $\mathcal{H}_1$ incident with a and b', respectively, we see that A^g meets B in a point r', and B^g meets A in a point r. We also have $r^g = (A \cap B^g)^g = A^g \cap B = r'$. Moreover, L belongs to $r^\rho \cap r'^\rho$ (since L is at distance 3 from r in the incidence graph of $\mathcal{H}_1$ and at the same distance from r' in the incidence graph of $\mathcal{H}_2$), and, likewise, M too. Hence both r and r' belong to the point regulus complementary to the line regulus containing L and M and so r is collinear with a point b on M in $\mathcal{H}_1$ and with a point a' on L in $\mathcal{H}_2$. But now we have, with the very same notation, the situation described above, and we conclude that the point regulus $\mathcal{R}_1(a,b') = \mathcal{R}_2(a,b')$ is complementary to the line regulus $\Upsilon \cap \mathsf{Q}(6,q)$, and belongs to Ω_2.

The lemma is proved. ∎

Lemma 2. *The stabilizer G of Π inside the automorphism group of $\mathcal{H}_1$ stabilizes Ω. Also, G acts doubly transitively on $\mathcal{S}$, and the stabilizer of two elements L, M of $\mathcal{S}$ acts transitively on the set of points incident with L.*

Proof. In order to prove the first assertion, it suffices to show that any element $h \in G$ stabilizes $\mathcal{H}_2$. Since g is the unique involution stabilizing $\mathsf{Q}(6,q)$ and fixing all points of Π, and since h stabilizes both Π and $\mathsf{Q}(6,q)$, we have $g^h = g$. Hence $h = h^g$ stabilizes $\mathcal{H}_1^g = \mathcal{H}_2$.

Now $G \leq \mathsf{SU}_3(q) : 2$, with the natural action on $\mathcal{S}$ as a Hermitian unital. Whence the doubly transitivity. Let $G^* = G \cap \mathsf{SU}_3(q)$. Then, the stabilizer $G^*_{L,M}$ of two lines L, M has order $2(q^2-1)$. Since q is odd, every element σ of $G_{L,M}$ which fixes one point of L has to fix a second point of L, and hence σ is a product of generalized homologies, in the sense of Chapter 4 of [11], see 4.6.6 of [11]. We now consider the explicit form of $\mathcal{S}$ as given in the beginning of the current section. We can take $L = [\infty]$ and $M = [0,0,0,0,0]$. Given the explicit forms of generalized homologies as in 4.5.11 of [11], it is now easy to see that the only generalized homologies fixing $[\infty]$ and $[0,0,0,0,0]$ and the point (∞) are given by the following actions on the coordinates:

$$\begin{array}{rcl} (a,l,a',l',a'') & \mapsto & (\epsilon a, \epsilon K l, K a', \epsilon K^2 l', \epsilon K a''), \\ [k,b,k',b',k''] & \mapsto & [Kk, \epsilon K b, \epsilon K^2 k', K b', \epsilon K k''], \end{array}$$

with $K \in \mathsf{GF}(q)$ arbitrary, and $\epsilon \in \{1, -1\}$. This group fixes all points on $[\infty]$, and hence is normal in $G^*_{L,M}$. By the orbit counting formula, the orbit of (∞) under $G_{L,M}$ has length $q+1$ and hence constitutes all points of L.

The lemma is proved. ■

Now we define $\mathsf{D}_{\mathrm{Hex}}(q)$ as the incidence geometry with point set the elements of $\mathcal{S}$ and block set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, with

$$\mathcal{B}_1 = \{\mathcal{R}(L, M) : L, M \in \mathcal{S}, L \neq M\},$$
$$\mathcal{B}_2 = \{\{L\mathrm{I}p : p \in \omega_2, L \in \mathcal{S}\} : w_2 \in \Omega_2\}.$$

We remark that we consider repeated blocks as one block.

We have the following proposition, which is already a substantial part of our Main Result:

Proposition 1. *The incidence geometry* $\mathsf{D}_{\mathrm{Hex}}(q)$*, q odd, is* $2-(q^3+1, q+1, q+2)$*-design for* $q \not\equiv 2 \bmod 3$*, and it is a* $2-(q^3+1, q+1, \frac{q+4}{3})$*-design otherwise. In any case, two distinct blocks never meet in more than two points.*

Proof. We determine the number of blocks in $\mathsf{D}_{\mathrm{Hex}}(q)$ through two given points of $\mathsf{D}_{\mathrm{Hex}}(q)$. Note that by the 2-transitivity of G on the point set of $\mathsf{D}_{\mathrm{Hex}}(q)$, this number is already a constant. So we have to count the number of "repeated blocks", i.e., the number of times a block is defined by different point reguli. By transitivity of G, we may take the block B_1 defined by the point regulus $R := \mathcal{R}((\infty), (0, 0, 0, 0, 0))$ and the block B_2 defined by the point regulus $R' := \mathcal{R}((a), (A, 0, 0, 0, 0))$, with $(a, 0, 0)^g = (A, 0, 0)$ and $a \in \mathsf{GF}(q) \setminus \{0\}$. Suppose some point x of $B_1 \setminus \{(\infty), (0, 0, 0, 0, 0)\}$ is on the same line of $\mathcal{S}$ as some point $y \in B_2$. Such a point x has coordinates $(0, 0, -\nu a', 0, 0)$, $a' \in \mathsf{GF}(q) \setminus \{0\}$ and is incident with the line $L = [a', 0, 0, -\nu a', 0]$ of $\mathcal{S}$.

Now, the point y has coordinates $(A, 0, ., ., .)$ in $\mathsf{H}(q)$, and from the incidence relation described in Section 2.1, we infer that y is incident with L if and only if $-A^3 + 3A\nu a' = 0$. This is equivalent to $A^2 = 3\nu$. Hence such a point y exists if and only if -3 is not a square in $\mathsf{GF}(q)$, i.e., $q \equiv 2 \bmod 3$. This already shows that, if $q \not\equiv 2 \bmod 3$, then there are no repeated blocks, no two distinct blocks meet in more than two points, and an easy counting argument concludes the proof of the proposition (taking into account that no member of $\mathcal{B}_1$ can meet any member of $\mathcal{B}_2$ in more than two points, which is easy to see).

Now suppose $q \equiv 2 \bmod 3$. Then the equation $A^2 = 3\nu$ has two solutions in A giving rise to two point x'_1, x'_2. This implies that x, x'_1 and x'_2 are three points of distinct point reguli R, R'_1, R'_2 (respectively) in Ω_2 which are on the same spread line $L = [a', 0, 0, -\nu a', 0]$. Since the equation $A^2 = 3\nu$ does not depend on x, we see that the point reguli R, R'_1, R'_2 determine the same element of $\mathcal{B}_2$. By transitivity we thus obtain that any two lines of the spread determine $\frac{q+1}{3}$ elements of $\mathcal{B}_2$. A simple counting argument now concludes the proof of the proposition completely. ■

Remark. The previous proof also implies that G acts transitively on the set $\mathcal{B}_2$. If $q \equiv 2 \bmod 3$, then the center of G^*, which has order 3, fixes all points of

$\mathsf{D}_{\rm Hex}(q)$ and permutes the point reguli belonging to Ω_2 in such a way that point reguli that represent same elements of $\mathcal{B}_2$ are permuted amongst themselves. Hence the center acts trivially on $\mathsf{D}_{\rm Hex}(q)$ and we obtain a faithful action of $\mathsf{PSU}_3(q)$ on $\mathsf{D}_{\rm Hex}(q)$, transitive on the points and with two orbits on the blocks.

4 $\mathsf{D}_{\rm Hex}(q)$ vs. $\mathsf{D}_{\rm Hölz}(q)$

In this section we will show that each block of $\mathsf{D}_{\rm Hex}(q)$ is a block of $\mathsf{D}_{\rm Hölz}(q)$. This will complete the proof of our Main Result.

Proposition 2. $\mathsf{D}_{\rm Hex}(q)$ *is a subdesign of* $\mathsf{D}_{\rm Hölz}(q)$. *In particular, these designs coincide if and only if* $q \not\equiv 2 \bmod 3$.

Proof. It is well known that the line reguli in $\mathcal{S}$ correspond to the Hermitian blocks. Hence we only have to prove that any block of type $\mathcal{B}_2$ is a Hölz-block.

By transitivity on the blocks of type $\mathcal{B}_2$ we may only consider the block defined by the point regulus $\mathcal{R}((\infty),(0,0,0,0,0))$. One easily calculates that this block equals

$$B = \{[\infty]\} \cup \{[k,0,0,-\nu k,0] : k \in \mathsf{GF}(q)\}.$$

We now need to relate the the Hermitian spread $\mathcal{S}$ of $\mathcal{H}_1$ to a Hermitian curve, $\mathcal{U}$, in $\mathsf{PG}(2,q^2)$. Let $\gamma \in \mathsf{GF}(q^2)$ be such that $\gamma^2 = -\nu$ (with ν given previously). We already established that the lines of $\mathcal{H}_1$ in the hyperplane Π with equation $X_5 = -\nu X_1$ in $\mathsf{PG}(6,q)$ form the Hermitian spread

$$\mathcal{S} = \{[\infty]\} \cup \{[k,b,k',-\nu k,\nu b] : k,b,k' \in \mathsf{GF}(q)\}$$

in $\mathsf{H}(q)$.

Now we extend $\mathsf{PG}(6,q)$ to $\mathsf{PG}(6,q^2)$, thereby also extending $\mathsf{Q}(6,q)$ to $\mathsf{Q}(6,q^2)$ (having the same equation) and $\mathsf{H}(q)$ to $\mathsf{H}(q^2)$ (the Grassmannian coordinates of the lines of $\mathsf{H}(q^2)$ satisfy exactly the same six equations above as is the case for $\mathsf{H}(q)$; here we identify $\mathsf{H}(q)$ with the model $\mathcal{H}_1$ for clarity). Let σ be the involution in $\mathsf{H}(q^2)$ defined by applying the map $x \to x^q$ to every coordinate of any element in $\mathsf{H}(q^2)$. It is obvious that σ fixes $\mathsf{H}(q)$ pointwise. By [12], the hyperplane Π, viewed as a hyperplane of $\mathsf{PG}(6,q^2)$, defines in $\mathsf{H}(q^2)$ the subhexagon $\Gamma(p,p')$ of order $(1,q^2)$ of $\mathsf{H}(q^2)$ (with notation of 1.9 of[11]) and $\Gamma(p,p') \cap \mathsf{H}(q) = \mathcal{S}$, where p is a point of $\mathsf{H}(q^2) \setminus \mathsf{H}(q)$ on $[\infty]$ and p' is the point on $[0,0,0,0,0]$ at distance 5 from p^σ. With the terminology of Chapter 1 of [11], we know that $\Gamma(p,p')$ is the double of a Desarguesian projective plane $\Pi_{p,p'}$. Let π^+ (respectively π^-) be the plane of $\mathsf{PG}(6,q^2)$ generated by the points p, p'^σ and $p^\sigma \bowtie p'$ (respectively p^σ, p' and $p'^\sigma \bowtie p$). We know that π^+ and π^- can be thought of as the point set and the line set, respectively, of $\mathsf{PG}(2,q^2)$.

According to [12], the lines of $\mathcal{S}$ meet the plane π^+ in the points of a hermitian curve, which we will call $\mathcal{U}$. We now establish an explicit algebraic correspondence. We may choose p to be the point (γ) on the line $[\infty]$. Hence π^+ is generated by the points $p = (\gamma)$, $p'^\sigma = (\gamma,0,0,0,0)$ and $p' \bowtie p^\sigma = (-\gamma,0,0)$ of $\mathsf{H}(q^2)$. Let $\overline{p}_0$ be the fixed coordinate tuple $(\gamma,0,0,0,0,0,1)$ of p. Likewise let $\overline{p}_1 = (0,0,-\gamma,0,1,0,0)$, be the fixed representative of p'^{σ}, and then we have $\overline{p}_2 = (0, 1, 0, \gamma, 0, \gamma^2, 0)$, representing $p' \bowtie p^{\sigma}$.

We introduce coordinates in π^+ by mapping a point $r_0.\overline{p}_0 + r_1.\overline{p}_1 + r_2.\overline{p}_2$ of $\mathsf{PG}(6, 9)$ to the point (r_0, r_1, r_2) in $\mathsf{PG}(2, q^2)$.

Now we claim that the equation in π^+ of the Hermitian curve $\mathcal{U}$ corresponding to $\mathcal{S}$ is given by

$$\mathcal{U} : -2\gamma X_2 X_2^q = X_0 X_1^q - X_1 X_0^q$$

and the isomorphism $\Phi : \mathcal{S} \to \mathcal{U}$ is given by

$$[\infty]^{\Phi} = (1, 0, 0), \qquad [k, b, k', -\nu k, \nu b]^{\Phi} = (\gamma(b^2 + \nu k^2) - \nu kb + k', -1, \gamma k + b).$$

Indeed, since the line $[\infty]$ meets Π in p it is obvious that we map this line to the point $(1, 0, 0)$. Consider a general line, $[k, b, k', -\nu k, \nu b]$, of the spread $\mathcal{S}$. Using Table 1 and the coordinates of points in π^+, a simple calculation yields $\Phi([k, b, k', -\nu k, \nu b]) = (\gamma(b^2 + \nu k^2) - \nu kb + k', -1, \gamma k + b)$. The point $(1, 0, 0)$ clearly satisfies the given equation of $\mathcal{U}$ and therefore it suffices to check whether a general point $(\gamma(b^2 + \nu k^2) - \nu kb + k', -1, \gamma k + b)$, with $k, b, k' \in \mathsf{GF}(q)$, is a point on $\mathcal{U}$, and that is an easy calculation. The claim follows.

We will now show that B^{Φ} contains the points of an oval on $\mathcal{U}$, which determine a Baer subplane D satisfying property (H).

By our previous claim the lines of $\mathcal{S}$ corresponding to B are mapped onto the points

$$\{(1, 0, 0)\} \cup \{(\gamma\nu k^2, -1, \gamma k) : k \in \mathsf{GF}(q)\},$$

or, since $\gamma^2 = -\nu$,

$$\{(1, 0, 0)\} \cup \{(l^2, \gamma, l) : l \in \mathsf{GF}(q)\}.$$

Now all of these points satisfy the quadratic equation $X_0 X_1 = \gamma {X_2}^2$, which shows that they are contained in a conic of $\mathsf{PG}(2, q^2)$.

We now check Property (H).

Consider the points $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, $p_3 = (1, \gamma, 1)$ and $p_4 = (1, \gamma, -1)$ (all of which are points of B^{Φ}). The Baer subplane through these points contains the following additional points: $p_5 = p_1p_2 \cap p_3p_4 = (1, \gamma, 0)$, $p_6 = p_1p_3 \cap p_2p_4 = (1, -\gamma, -1)$ and $p_7 = p_1p_4 \cap p_5p_6 = (0, -\gamma, 1)$. Now, with these additonal points, one can easily see that the Baer subplane through the $q + 1$ points of $\mathcal{U}$ contains $q + 1$ points on $X_1 = 0$, namely

$$\{(1, 0, x) : x \in \mathsf{GF}(q)\} \cup \{(0, 0, 1)\},$$

and its set of q^2 other points is given by:

$$\{(y, \gamma, x) : y, x \in \mathsf{GF}(q)\}.$$

With these explicit forms of the points of D it is easy to check whether this Baer subplane satisfies property (H). Let us first recall that the tangent line at a point (x_0, x_1, x_2) of $\mathcal{U}$ is given by the equation

$$T_p\mathcal{U} : \left(\frac{\partial \mathcal{U}}{\partial X_0}\right)_p X_0 + \left(\frac{\partial \mathcal{U}}{\partial X_1}\right)_p X_1 + \left(\frac{\partial \mathcal{U}}{\partial X_2}\right)_p X_2 = 0$$

with $\left(\frac{\partial \mathcal{U}}{\partial X_i}\right)_p$ the partial derivative with respect to X_i at the point $p = (x_0, x_1, x_2)$. Given the equation of $\mathcal{U}$ we find

$$x_1^q X_0 - x_0^q X_1 + 2\gamma x_2^q X_2 = 0$$

to be the tangent line of $\mathcal{U}$ at the point (x_0, x_1, x_2) over the field $\mathsf{GF}(q^2)$.

To investigate whether D satisfies property (H) we have to consider all points of D on $\mathcal{U}$, determine the tangent line at these points and check if these are a Baer line of D. In particular, as every line of π^+ intersects D in one or in $q+1$ points it suffices to find two points of such a tangent line which are in D to conclude that it is a Baer line of the Baer subplane. Now, the points of D on $\mathcal{U}$ are in fact the points of $B^\Phi = \{(1,0,0)\} \cup \{(l^2, \gamma, l) : l \in \mathsf{GF}(q)\}$. The tangent line at the first point of this set is the line X_1 which we already know is a Baer line of D. Finally, the tangent line at the point (l^2, γ, l), with $l \in \mathsf{GF}(q)$, is the line

$$\gamma X_0 + l^2 X_1 - 2\gamma l X_2 = 0$$

and this line contains the points (l^2, γ, l) and $(-l^2, \gamma, 0)$ of D.

In conclusion, D satisfies property (H) and consequently $\mathsf{D}_{\mathrm{Hex}}(q)$ is a subdesign of $\mathsf{D}_{\mathrm{Hölz}}(q)$. ■

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