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Combinatorial characterizations of convexity and apartments in buildings

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Abstract

We present a new criterion for a set of chambers to be convex in a building. We apply this criterion to apartments, which are known to be the thin convex chamber subcomplexes of buildings. We also prove an additional characterization of apartments in terms of certain Weyl distances between chambers, generalizing our results in the spherical case, obtained by the present authors in *Annals Combin.* 4 (2000), 125–137, using opposition of chambers.

1 Introduction

Originally, buildings were defined as certain simplicial chamber complexes containing "a lot" of isomorphic apartments (made precise with two axioms; apartments are thin chamber sub-complexes). At a certain moment, people started to look at buildings as structures where the chambers are no longer simplices, i.e. sets of vertices, but where they are the "atoms" themselves. In the definitions that emerge from this viewpoint the apartments are no longer needed. Still, one wants to use them, and therefore one wants to recognize them. For instance, in [1], some properties of the opposition map are proved using apartments, which are first recognized as sets of chambers satisfying some conditions stated in terms of opposition. Namely, a set of chambers in a spherical building is an apartment if and only if each chamber in the set is opposite exactly one other chamber of the set, and each chamber outside the set is opposite an even number of chambers of the set. Noting that opposite chambers are merely chambers at a certain fixed Weyl distance from each other, one could wonder if in this characterization of apartments one can replace "opposition" with "being at Weyl distance w", for a fixed element w of the Weyl group. In [2], we showed the "only if" part of such a statement. In the present paper, we generalize the "if" part.

Our motivation is partly purely esthetic and curiosity, but partly also that investigations like this bring other beautiful and useful properties to the surface. In this case, we establish along the way a simple characterization of convexity, which in its turn implies a considerable generalization of Brown's characterization of apartments in [3]. In fact, our results show that, for buildings, the notion of convexity is equivalent to the much weaker global notion of connectivity along with a local notion of convexity. Hence buildings are peculiar metric spaces where a particular intuition is needed. To illustrate this, we just mention the following consequence of our results below. In an arbitrary spherical building of irreducible type and rank at least 3, a connected set $\mathcal{M}$ of chambers with the property that the convex closure of any two chambers of $\mathcal{M}$ at numerical distance at most 4 is contained in $\mathcal{M}$, is itself convex. Noting that the diameter of spherical buildings is unbounded, we see that this is rather surprising and clashes with our intuition from ordinary (discrete) metric spaces.

We state our main results in a more precise way in the next section, after introducing the necessary notation.

2 Notation and Statement of the Main Results

We spend some time on the definition and some basic properties of a building in order to turn this paper into a more introductory one, for the sake of the casual reader of these proceedings. The interested reader is recommended to consult [3], [4], [5] and [8] for more information, background and results.

2.1 Background and History

Buildings are certain kinds of incidence geometries — created by Jacques Tits in the early sixties as the natural geometries associated to groups of Lie type — and as such in fact just a class of multipartite graphs. The classical and original definition uses chamber simplicial complexes. It is a purely geometric definition, but rather involved in that already for the simplest class of examples, namely the projective spaces, it is a nontrivial exercise to show that they satisfy the axioms of a building.

In this classical definition, the concept of a *chamber* plays a central role. When viewed as a multipartite graph, a chamber of a building is just a maximal clique, which then contains exactly one vertex of each multipartition class. If these classes were seen as political parties, then a chamber is just a committee where every party is represented by exactly one member. Interpreting the word "chamber" as a stylish version of "room", the real estate terminology is a logical consequence. Indeed, a "building" consists of a set of chambers. Certain subcomplexes will be called apartments, etc.

Since the concept of a chamber plays such a prominent role in the definition of a building — and not only in the definition, but also in many properties and applications — one is tempted to look for a definition where the chambers are the atoms. It was again Jacques Tits who first developed this idea by introducing the concept of a *chamber system*. This definition was optimized after the introduction of *twin buildings*, which emerge as the geometries naturally associated to Kac-Moody groups. At present, buildings are viewed as metric spaces where the distance has values in a *Coxeter group*. That is exactly how we will define them. So it is convenient to first look at the value set of our metric spaces.

2.2 Coxeter Systems

Let n be a positive nonzero integer, and let M be a symmetric $n \times n$ matrix with entries m_{ij} in the set of natural numbers union $\{\infty\}$ satisfying $m_{ii} = 1$, and $m_{ij} \geq 2$, for $i \neq j$, with $i, j \in \{1, 2, \ldots, n\}$. The matrix M is called a *Coxeter matrix*. The *Coxeter group of rank* n associated with the Coxeter matrix M is the group W with presentation

$$\langle s_1, s_2, \ldots, s_n \,|\, (s_i s_j)^{m_{ij}} = \mathrm{id}, i, j \in \{1, 2, \ldots, n\}\rangle,$$

where we use id to denote the identity element in W, and $(s_i s_j)^\infty = \mathrm{id}$ by definition means that there is no relation for $s_i s_j$ if $m_{ij} = \infty$. Coxeter groups have nice properties. In particular, the order of the element $s_i s_j$ is precisely equal to m_{ij}. In the sequel we will often denote m_{ij} as $m(s_i, s_j)$. Hence, putting $i = j$, we see that $S = \{s_1, s_2, \ldots, s_n\}$ is a generating set of involutions. Moreover, S is a minimal generating set (in the set-theoretic sense). One can show that, for each subset $S' \subseteq S$, the subgroup $W' = \langle S' \rangle$ is a Coxeter group (see (P1) below). It is, however, not true that W determines S, not even up to conjugacy or isomorphism. In fact, not even the Coxeter matrix is, up to conjugation with a permutation matrix, determined by W. For instance consider the group $W = \langle (1\ 2), (3\ 4), (4\ 5), (5\ 6)\rangle \leq \mathbf{S}_6$. Then $W \cong 2 \times \mathbf{S}_4$ is a Coxeter group of rank 4 with matrix $(m_{ij})_{1 \leq i,j \leq 4}$ where $m_{ii} = 1$, $m_{1j} = 2 = m_{24}$ and $m_{23} = m_{34} = 3$. But W can also be presented as $W = \langle (1\ 2)(3\ 4)(5\ 6), (3\ 4), (3\ 5)\rangle$. We thus obtain a Coxeter group of rank 3 with a Coxeter matrix having 1 on the diagonal and $2, 3, 4$ off the diagonal.

Hence we take S as part of the definition and talk about a *Coxeter system* (W, S). The cardinality of S is referred to as the *rank* of the Coxeter system. If $w \in W$, $w \neq \mathrm{id}$, then we can write w as a product of elements of S, and it will be convenient to consider the sequence of elements of S that defines w. Therefore, we distinguish between the element $w = s_{i_1}s_{i_2}\dots s_{i_k}$ and the word $f = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$, $i_j \in \{1, 2, \dots, n\}$, $1 \le j \le k$, $k \in \mathbb{N}$. In general, a word in elements of S defines a unique element of W (by multiplying the elements of the word), but one element of W of course defines a lot of words. We call the word f above *reduced* if the corresponding element w of W cannot be written with less than the number of elements of f (the identity can be written by definition with zero elements of S — it corresponds to the empty word). If f above is reduced, then the number k is called the *length* of w and denoted by $\ell(w)$. Hence the length of the identity id is equal to 0. We note that $\ell(ws) = \ell(w)+1$ or $\ell(w)-1$ for all $w \in W$ and $s \in S$. ($\ell(w)-1 \le \ell(ws) \le \ell(w)+1$ is clear, and $\ell(ws) \ne \ell(w)$ because each of the defining relations for W involves an even number of elements of S.)

In this and the next subsection, we will collect some well-known basic properties of Coxeter systems and of buildings. In the following, (W, S) always denotes a Coxeter system.

(P1) For any subset S' of S, the pair $(W' = \langle S' \rangle, S')$ is a Coxeter system. If we denote by ℓ' its length function, then the length function ℓ of (W, S) satisfies $\ell_{|W'} = \ell'$. Furthermore, for any $w \in W'$ and any word $f = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$ of length $k = \ell(w)$ with letters $s_{i_j} \in S$ which represents w, necessarily $s_{i_j} \in S'$ for all $1 \le j \le k$.

(P2) W is finite if and only if there exists an element w_0 such that $\ell(w_0 s) < \ell(w_0)$ for all $s \in S$. If W is finite, then $w_0 \in W$ satisfies $\ell(w_0 s) < \ell(w_0)$ for all $s \in S$ if and only if $\ell(w_0) = \max\{\ell(w) \mid w \in W\} =: n_0$. There is precisely one $w_0 \in W$ with $\ell(w_0) = n_0$. This in particular implies $w_0 = w_0^{-1}$, because $\ell(w^{-1}) = \ell(w)$ for all $w \in W$. The element w_0 also satisfies $\ell(w_0) = \ell(w) + \ell(w^{-1}w_0)$ for all $w \in W$.

(P3) If (W, S) has rank 2, then W is the dihedral group $\mathbf{D}_{2m}$ of order $2m$, where $m = m(s_1, s_2)$. If $m < \infty$, then $m = n_0$ (with n_0 as in (P2)), and w_0 admits precisely two reduced representations (of length m), namely $w_0 = s_1 s_2 s_1 \dots = s_2 s_1 s_2 \dots$

Combining (P1) and (P3), we obtain

(P4) If $s, t \in S$ satisfy $m(s, t) < \infty$, then there are precisely two reduced words of length $m(s, t)$ with letters in $\{s, t\}$, namely $p(s, t) := (s, t, s, \dots)$ and $p(t, s) = (t, s, t, \dots)$.

2.3 Axioms for and Basic Properties of Buildings

Now we are ready to define the notion of a building Δ. Let $\mathcal{C}(\Delta)$ be a set whose elements we call *chambers*. Let (W, S) be a Coxeter system. Let $\delta : \mathcal{C}(\Delta) \times \mathcal{C}(\Delta) \longrightarrow W$ be a map, which we call the *(Weyl) distance map*; in particular $\delta(C, D)$ is the distance from the chamber C to the chamber D. Then $\Delta = (\mathcal{C}(\Delta), \delta)$ is a *building of type* (W, S) (and W is called the *Weyl group* of Δ) if the conditions (B1), (B2), (B3), (B4) and (B5) below are satisfied. Let us introduce these conditions now.

The first axiom is the analogue of the axiom in an ordinary metric space that states that the distance between two elements is zero if and only if the elements are the same.

(B1) For all $C, D \in \mathcal{C}(\Delta)$ we have $\delta(C, D) = \mathrm{id}$ if and only if $C = D$.

The analogue of the symmetry of an ordinary metric would be here that the distance from C to D is the inverse of the distance from D to C (thinking about distance as the group element that is needed to go from one chamber to the other). It will turn out that we only have to require this for distances in S. Actually, since all elements in S are involutions, we have symmetry in these cases.

(B2) For all $C, D \in \mathcal{C}(\Delta)$ we have that $\delta(C, D) \in S$ implies $\delta(C, D) = \delta(D, C)$.

The analogue of the triangle inequality will likewise only be required for elements of S.

(B3) For any three chambers $C, D, E \in \mathcal{C}(\Delta)$ we have that $\delta(C, D) = \delta(D, E) \in S$ implies $\delta(C, E) \in \{\mathrm{id}, \delta(C, D)\}$.

It may be clear by now that distances with values in S play a privileged role. This comes with some special terminology, that we introduce now.

Two chambers $C, D \in \mathcal{C}(\Delta)$ are called *s-adjacent* if $\delta(C, D) = s \in S$. *Adjacent* chambers are chambers which are s-adjacent for some $s \in S$. A *gallery* γ is a sequence of chambers $\gamma = (C_0, C_1, \ldots, C_k)$ such that C_{i-1} and C_i are adjacent, for all $i \in \{1, 2, \ldots, k\}$. The word $(s_1, \ldots, s_k)$, where $s_i = \delta(C_{i-1}, C_i)$, for all i, is called the *type* of γ. We also say that the gallery γ *connects* C_0 with C_k, and that its *length* is equal to k. If every gallery connecting C_0 with C_k is of length at least k, then we say that γ is *minimal*, and that it is *stretched* between C_0 and C_k. Also, we call k the *numerical distance* between C_0 and C_k and set $d(C_0, C_k) = k$.

The next axiom is the central one, and it is in fact very strong. It actually determines the distance from any chamber to any other when all distances between chambers with values in S are given.

(B4) For any two chambers $C, D \in \mathcal{C}(\Delta)$ and any reduced word $(s_1, \ldots, s_k)$, $k \in \mathbb{N} \cup \{0\}$, we have $\delta(C, D) = s_1 \ldots s_k$ if and only if there exists a gallery of type $(s_1, \ldots, s_k)$ connecting C with D.

The last axiom is to exclude degenerate cases, which we do not want. It is more or less equivalent to requiring in a geometry that every element is incident with at least two elements of each type.

(B5) For every chamber C and every element $s \in S$, there exists at least one chamber s-adjacent to C.

An example of a building of type (W, S) is given by $\Sigma(W, S) := (W, \delta)$, with distance map $\delta : W \times W \longrightarrow W : (w, w') \mapsto w^{-1}w'$. We call this the *standard apartment of type* (W, S). For any building $\Delta = (\mathcal{C}(\Delta), \delta)$ of type (W, S), an isometric image of the standard apartment of type (W, S) in Δ will be called an *apartment* of Δ.

Further important sub-buildings of $\Delta = (\mathcal{C}(\Delta), \delta)$ are obtained as follows. For every Coxeter subsystem $(W' = \langle S' \rangle, S')$, $S' \subseteq S$, and every chamber C, the metric $\delta_{S'}$ induced by δ on $\mathcal{C}(\Delta_{S'}(C)) := \{D \in \Delta \,|\, \delta(C, D) \in W'\}$ defines a building $\Delta_{S'}(C) = (\mathcal{C}(\Delta_{S'}(C)), \delta_{S'})$. (The building axioms for $\Delta_{S'}(C)$ follow from those for Δ and from (P1).) $\Delta_{S'}(C)$ is called the *residue* of C of type S' in Δ and $|S'|$ is the *rank* of this residue. Residues of rank 1 are called *panels*. The building Δ is called *thin* (respectively *weak*, *thick*) if each panel contains precisely two (respectively, at least two, at least three) chambers. Note that any building is weak by Axiom (B5), and that we only use this notion when we want to stress that Δ is not required to be thick. We will also use the notions "thin" and "weak" for sets of chambers; a set $\mathcal{M}$ of chambers is thin (weak) if for each chamber C of $\mathcal{M}$ and for each $s \in S$, there is a unique (at least one) chamber of $\mathcal{M}$ which is s-adjacent to C.

Rank 2 residues with finite Weyl group $\langle S' \rangle$, $S' = \{s_i, s_j\}$, have some nice special properties. One of them is the fact that two chambers of a rank 2 residue at maximal numerical distance $m(s_i, s_j)$ from each other can be connected by exactly two minimal galleries (of length $m(s_i, s_j)$), see (P4) and (P7) below.

Now we turn to some basic properties of buildings, which we partly already indicated while introducing the axioms. All of these facts are standard consequences of the building axioms and can be found in the references we gave at the beginning of this section.

Let a building $\Delta = (\mathcal{C}(\Delta), \delta)$ of type (W, S) be given. The antisymmetry of δ reads as follows.

(P5) $\delta(D, C) = \delta(C, D)^{-1}$ for all $C, D \in \mathcal{C}(\Delta)$

Axiom (B3) generalizes to the following "triangle inequality".

(P6) If $\delta(C, D) = w$ and $\delta(D, E) = s \in S$, then $\delta(C, E) \in \{w, ws\}$, and $\delta(C, E) = ws$ whenever $\ell(ws) > \ell(w)$.

The next property links minimal galleries to reduced words.

(P7) A gallery $\gamma = (C_0, C_1, \ldots, C_k)$ of type $f = (s_1, \ldots, s_k)$ is minimal in Δ if and only if the word f is reduced. Combining this with Axiom (B4), we deduce $d(C, D) = \ell(\delta(C, D))$ for all $C, D \in \mathcal{C}(\Delta)$.

Because of (P7), the numerical distance $d(C, D)$ is often also called the *gallery distance* between C and D. We mention another property of minimal galleries.

(P8) If $\gamma = (C_0, C_1, \ldots, C_k)$ is a gallery of reduced type f and if $\gamma' = (C'_0, C'_1, \ldots, C'_k)$ is another gallery of the same type f with the same extremities $C'_0 = C_0$ and $C'_k = C_k$, then $\gamma' = \gamma$.

2.4 Spherical Buildings

In the spherical case, some interesting things happen. Remember that a building is spherical if its associated Weyl group W is finite. Given a finite Coxeter system (W, S), then obviously the length of a reduced word is bounded. Let n_0 be the maximal length of a reduced word occurring in (W, S). By (P2), every reduced word of length n_0 defines *the same element* w_0 *of* W, which is an involution. We call w_0 the *longest element* of (W, S). Chambers at distance w_0 are usually called *opposite.*

For instance, in the dihedral group of order $2n$ generated by two "adjacent" reflections s, t, the longest element is equal to $stst\ldots$ (n factors), which is also equal to $tsts\ldots$ (n factors), and these are the only two expressions of the longest element arising from a reduced word.

In a building of arbitrary spherical type, it easy to see that there are more chambers opposite a given chamber C than there are chambers at any other fixed distance $w \in W \setminus \{w_0\}$ from C. If the building is "sufficiently thick", then there are even "much more" chambers opposite C than there are chambers not opposite C. (In the case of finite buildings, one can make these slightly vague remarks more precise by counting the number of chambers in question.) Hence we may conclude that opposition is the "general position" of two arbitrary chambers. This translates (after some non-trivial work!) into properties such as the following (see [1]).

A thick spherical building of type (W, S) is, up to a permutation of the set S, completely determined by the pairs of opposite chambers. Also, a surjective mapping from the chamber set of one thick spherical building of type (W, S) to the chamber set of another thick spherical building of type (W, S) preserving opposition and non-opposition of pairs of chambers is induced by an isometry, possibly after renaming the elements of S for one of the buildings.

In showing these properties, the following proposition was established.

Proposition 1 *A set $\mathcal{M}$ of chambers of a thick spherical building Δ is the set of chambers of an apartment if and only if for every chamber $C \in \mathcal{M}$, the number of chambers in $\mathcal{M}$ opposite C is equal to one, and for every chamber $D \notin \mathcal{M}$, the number of chambers of $\mathcal{M}$ opposite D is an even number.*

While the above mentioned properties were generalized to other distances than opposition, and to non-spherical buildings, too, only the "only-if" part of Proposition 1 was proved to have an analogue for arbitrary elements $w \in W$, with W not necessarily finite, see Proposition 1.6 of [2]. In the present paper, we complement our results by establishing an analogue of Proposition 1 for arbitrary buildings. As a special case, the "if" part of the above proposition will be reproved.

2.5 Main Results

We now introduce some specific notation in order to be able to state our main results.

Let (W, S) be a Coxeter system, and let $s, t \in S$. If $m(s,t) < \infty$, then $p(s,t) = (s, t, s \ldots)$, of length $m(s,t)$, is a reduced word in s and t (see (P4)).

Let $\Delta = (\mathcal{C}(\Delta), \delta)$ be a building of type (W, S), for some Coxeter system (W, S), and let $\mathcal{M} \subseteq \mathcal{C}(\Delta)$ be a set of chambers. Then $\mathcal{M}$ is *convex* if, for every two chambers C, D of $\mathcal{M}$, all galleries stretched between C and D belong to $\mathcal{M}$. The *convex closure* $\mathrm{cl}(\mathcal{M})$ of a set $\mathcal{M}$ of chambers is the smallest convex set of chambers containing $\mathcal{M}$. The set $\mathcal{M}$ is called 2-*convex* if, for every gallery $\gamma = (C_0, C_1, \ldots, C_m)$ of type $p(s,t)$ contained in $\mathcal{M}$ (where $s, t \in S$ satisfy $m(s,t) < \infty$), the (by (P8) uniquely determined) gallery $\gamma' = (C'_0, C'_1, \ldots, C'_m)$ of type $p(t,s)$ with $C'_0 = C_0$ and $C'_m = C_m$ is also contained in $\mathcal{M}$.

The notion of 2-convexity is a priori weaker than the notion of 2-local convexity. We call a set of chambers $\mathcal{M}$ 2-*locally convex* if, whenever two chambers C, D of $\mathcal{M}$ belong to the same rank 2 residue, all chambers of the convex closure $\mathrm{cl}(\{C, D\})$ belong to $\mathcal{M}$.

A set of chambers is *connected* if any two chambers in the set can be joined by a gallery which is completely contained in the set. A *connected component* of a set of chambers is a subset which is maximal with respect to being connected. We denote the diameter of a building Δ by $\mathrm{diam}(\Delta)$. If finite, it is the largest numerical distance between chambers (equivalently, it is the largest length of a reduced gallery).

The first proposition we will prove in this paper reads as follows:

Proposition 2 *Let $\Delta = (\mathcal{C}(\Delta), \delta)$ be a building of type (W, S), and let $\mathcal{M} \subseteq \mathcal{C}(\Delta)$ be a set of chambers. Then $\mathcal{M}$ is convex whenever $\mathcal{M}$ is connected and 2-convex.*

Let $\Delta = (\mathcal{C}(\Delta), \delta)$ be a building of type (W, S), let $\mathcal{M} \subseteq \mathcal{C}(\Delta)$ be a nonempty set of chambers, and let $w \in W$ be arbitrary. For any chamber $C \in \mathcal{C}(\Delta)$, we denote by $n_{\mathcal{M},w}(C)$ the number of chambers $X \in \mathcal{M}$ such that $\delta(C, X) = w$. Then we define the following Condition (E_w) for $\mathcal{M}$:

(E_w) For each $C \in \mathcal{M}$, we have $n_{\mathcal{M},w}(C) = 1$;
for each $C \in \mathcal{C}(\Delta) \setminus \mathcal{M}$, we have $n_{\mathcal{M},w}(C) \equiv 0 \bmod 2$.

For $\ell \in \mathbb{N}$, we say that $\mathcal{M}$ satisfies Condition (E_ℓ) if it satisfies Condition (E_w) for all $w \in W$ with $\ell(w) = \ell$. One easily checks that $\mathcal{M}$ satisfies (E_1) if and only if $\mathcal{M}$ is thin.

We shall prove:

Proposition 3 *Let $\Delta = (\mathcal{C}(\Delta), \delta)$ be a thick building of type (W, S), and let N be a positive integer not exceeding the diameter of Δ, i.e. such that $N \leq \sup\{\ell(w) \,|\, w \in W\}$. Let $\mathcal{M}$ be a nonempty set of chambers. Then $\mathcal{M}$ is the set of chambers of an apartment of Δ if and only if $\mathcal{M}$ satisfies* (E_ℓ) *for all $\ell \geq N$.*

If we want to use Condition (E_ℓ) for a set of chambers for only one value of ℓ, then we only have to consider connectivity.

Proposition 4 *Let $\Delta = (\mathcal{C}(\Delta), \delta)$ be a thick building of type (W, S), and let m be defined as*

$$m = \max(\{m(s,t) \,|\, s,t \in S \text{ and } m(s,t) < \infty\} \cup \{1\}).$$

Let $\mathcal{M}$ be a nonempty set of chambers. If (E_ℓ) *holds for $\mathcal{M}$ for some $\ell \geq m$, with $\ell \leq \mathrm{diam}(\Delta)$, then every connected component of $\mathcal{M}$ is the set of chambers of an apartment of Δ.*

Combined with the "only if" part of Proposition 3, we immediately obtain the following

Corollary 5 *Let Δ and m be as in Proposition 4. Then a nonempty set $\mathcal{M}$ of chambers of Δ is the set of chambers of an apartment of Δ if and only if $\mathcal{M}$ is connected and satisfies (E_ℓ) for some ℓ with $m \leq \ell \leq \mathrm{diam}(\Delta)$.*

We will mention some other corollaries later on, and also present some counterexamples to show that our assumptions are optimal (see Section 5). In particular, we shall show that an $\mathcal{M}$ satisfying the assumptions of Proposition 4 need not be connected, even if Δ is spherical and $\ell = \mathrm{diam}(\Delta) - 1$.

3 2-Convexity

Before we prove Proposition 2 we need the notion of (elementary) homotopy.

Let (W, S) be a Coxeter system, and let $s, t \in S$ with $s \neq t$ and $m(s,t) < \infty$. Then a word $f_1 p(s,t) f_2$ is called *elementary homotopic to* the word $f_1 p(t,s) f_2$. The transitive closure of the relation "…is elementary homotopic to…" is the relation "*…is homotopic to…*".

It is well known that, if a word f is not reduced, then it is homotopic to a word containing (s, s) for some $s \in S$. It is also well known that two reduced words representing the same element in W are homotopic to each other.

We can now prove Proposition 2.

Suppose $\mathcal{M}$ is a connected and 2-convex set of chambers in the building $\Delta = (\mathcal{C}(\Delta), \delta)$ of type (W, S). First we note that the definition of (elementary) homotopy and 2-convexity immediately implies the following:

(i) *If a gallery* γ *of type* f *is contained in* $\mathcal{M}$ *and* f' *is homotopic to* f*, then* $\mathcal{M}$ *also contains a gallery* γ' *of type* f' *with the same extremities as* γ*.*

Now let C, D be arbitrary chambers of $\mathcal{M}$. By connectivity, there is some gallery $\gamma = (C_0, C_1, \ldots, C_n)$, with $C_0 = C$ and $C_n = D$, contained in $\mathcal{M}$. Assume that γ is of minimal length in $\mathcal{M}$ with these properties. We want to show that then γ is also a minimal gallery in Δ. Let f be the type of γ, and suppose that f is not reduced. Let f' be a word homotopic to f and containing the word (s, s), for some $s \in S$. Choose a gallery γ' in $\mathcal{M}$ of type f' having the same extremities as γ (which is possible by (i)). Then γ' contains a sub-gallery (D_1, D_2, D_3), with D_1, D_2, D_3 mutually s-adjacent. We can now delete either D_2 (if $D_1 \neq D_3$) or D_2 and D_3 (if $D_1 = D_3$) in γ' to obtain a shorter gallery connecting C and D in $\mathcal{M}$, which contradicts our minimality assumption concerning γ. This shows

(ii) *For any* $C, D \in \mathcal{M}$*, there is a gallery* γ *connecting* C *and* D*, which is minimal in* Δ *and which is contained in* $\mathcal{M}$*.*

In order to prove Proposition 2, we have to show that *every* gallery of reduced type connecting two chambers $C, D \in \mathcal{M}$ is contained in $\mathcal{M}$. But the types of two galleries with the same extremities C and D and of reduced type are automatically homotopic to each other since they both represent the element $\delta(C, D) \in W$. And if f (and hence also any f' homotopic to f) is *reduced*, then there is by (P8) only one gallery of type f in Δ which connects C and D. Combining (i) and (ii), the proof of Proposition 2 is now complete.

We mention two immediate corollaries.

Corollary 6 *Any thin connected and* 2*-convex set* $\mathcal{M} \subseteq \mathcal{C}(\Delta)$ *is (the set of chambers of) an apartment of* Δ*.*

Indeed, by Proposition 2, $\mathcal{M}$ is convex, and the result follows from Section IV,4 in [3].

Corollary 7 *If* $\mathcal{M} \subseteq \mathcal{C}(\Delta)$ *is weak, connected and* 2*-convex, then* $(\mathcal{M}, \delta/\mathcal{M} \times \mathcal{M})$ *is a sub-building of* $(\mathcal{C}(\Delta, \delta)$*.*

This follows immediately from the definition of buildings as given in Section 2. One can rephrase this statement as follows.

Corollary 8 *If* $\mathcal{M} \subseteq \mathcal{C}(\Delta)$ *is connected, and if the intersection of* $\mathcal{M}$ *with the chamber set of any rank* 2 *residue of* Δ *is either empty or (the chamber set of a) building, then* $\mathcal{M}$ *itself is the chamber set of a building of the same type as* Δ*.*

4 Characterizations of Apartments

4.1 Proposition 3

We now prove Proposition 3. We shall freely use the properties (P1) to (P8) collected in Section 2.

Henceforth we let $\Delta = (\mathcal{C}(\Delta), \delta)$ be a (not necessarily thick, for the time being) building of type (W, S). We assume that $\mathcal{M}$ is a nonempty set of chambers of Δ. The "only if" part of Proposition 3 is proved in [2], Proposition 1.6. In order to prove the "if" part, we show some lemmas.

Lemma 9 *Suppose Δ is thick. Let $C, C' \in \mathcal{M}$, let $\delta(C, C') = w$, and let $\gamma = (C_0, C_1, \ldots, C_k)$ be a gallery stretched between $C_0 = C$ and $C_k = C'$. If (E_w) and $(\mathrm{E}_{w^{-1}})$ are satisfied for $\mathcal{M}$, then $C_1 \in \mathcal{M}$.*

Set $D := C_1$ and $s := \delta(C, D)$. Since Δ is thick, there exists a chamber E of Δ which is s-adjacent with both C and D. Since γ is minimal, $\ell(sw) < \ell(w)$; set $w' = sw$. Now $\delta(C', C) = w^{-1}$ and $\delta(C', D) = w'^{-1}$. Hence $\delta(C', E) = w'^{-1}s = w^{-1}$, and $\delta(E, C') = w$. Condition $(\mathrm{E}_{w^{-1}})$ for $\mathcal{M}$ implies that $E \notin \mathcal{M}$.

Now we claim that, for arbitrary $X \in \mathcal{M} \setminus \{C'\}$, the condition $\delta(D, X) = w$ is equivalent to $\delta(E, X) = w$. Indeed, assume for instance $\delta(D, X) = w$. Then $\delta(C, X), \delta(E, X) \in \{w, w'\}$. But if $\delta(C, X) = w$, then C has distance w to two chambers of $\mathcal{M}$, namely C' and X. Hence $\delta(C, X) = w'$. Since $\ell(w') < \ell(w)$, property (P6) (or rather its "inverse"; see also (P5)) implies that $\delta(E, X) = sw' = w$. Similarly $\delta(E, X) = w$ implies $\delta(D, X) = w$. The claim is proved.

Consequently, since $\delta(E, C') = w \neq \delta(D, C')$, we see that $n_{\mathcal{M},w}(E) = n_{\mathcal{M},w}(D) + 1$. Since $E \notin \mathcal{M}$, Condition (E_w) for $\mathcal{M}$ implies that $n_{\mathcal{M},w}(E)$ is even; hence $n_{\mathcal{M},w}(D)$ is odd, implying by (E_w) that $D \in \mathcal{M}$. □

Lemma 10 *Let $0 < l \leq \sup\{\ell(w') \,|\, w' \in W\}$, and let $s \in S$. Then there exists $w \in W$ with $\ell(sw) < \ell(w) = l$.*

By assumption there exists $u \in W$ with $\ell(u) = l$. If $\ell(su) < \ell(u)$, then we are done (because we can then put $u = w$). So assume that $\ell(su) > \ell(u)$. Since $u \neq \mathrm{id}$, we can write $u = u't$, with $\ell(u') < \ell(u)$ and $t \in S$. Hence $su't$ is reduced of length $l + 1$, and we put $w = su'$. □

Lemma 11 *If Δ is thick, and if for some $0 < \ell \leq \mathrm{diam}(\Delta)$, the condition (E_ℓ) holds for $\mathcal{M}$, then $\mathcal{M}$ is thin.*

First we show that $\mathcal{M}$ is *weak*, i.e. any chamber in $\mathcal{M}$ is, for any $s \in S$, s-adjacent to at least one other chamber in $\mathcal{M}$. Let $C \in \mathcal{M}$ and $s \in S$ be given. By Lemma 10, there exists $w \in W$ such that $\ell(sw) < \ell(w) = \ell$. Hence we may write $w = st_2t_3 \ldots t_\ell$, with $t_i \in S$, $i \in \{2, 3, \ldots, \ell\}$. Let $C' \in \mathcal{M}$ be the unique chamber in $\mathcal{M}$ with

$\delta(C, C') = w$. Let $\gamma = (C, C_1, C_2, \ldots, C_\ell)$, $C_\ell = C'$, be a minimal gallery of type $(s, t_2, t_3, \ldots, s_\ell)$. Lemma 9 now shows that $C_1 \in \mathcal{M}$.

Secondly, we show that $\mathcal{M}$ is thin. Indeed, suppose by way of contradiction that C, D, E are three mutually s-adjacent chambers contained in $\mathcal{M}$. Let $C' \in \mathcal{M}$ be such that $\delta(C, C') = w$, with w as in the first part of the proof. Also, we may take $D = C_1$, with C_1 as above. Then $\ell(w^{-1}ss) > \ell(w^{-1}s)$ and so (P6) implies that $\delta(C', E) = w^{-1}ss = w^{-1} = \delta(C', C)$, contradicting Condition ($\mathrm{E}_{w^{-1}}$). □

We now introduce a condition on $\mathcal{M}$ that we denote by (C_ℓ), with ℓ a natural number.

(C_ℓ) For every pair of chambers C, D of $\mathcal{M}$ with $d(C, D) = \ell$, the set $\mathrm{cl}(\{C, D\})$ is contained in $\mathcal{M}$.

Lemma 12 *If $\mathcal{M}$ is a weak set of chambers of Δ, and if ℓ is a natural number with $\ell \leq \mathrm{diam}(\Delta)$, then Condition* ($\mathrm{C}_\ell$) *for $\mathcal{M}$ implies Condition* (C_n)*, for all natural $n \leq \ell$.*

Given $C, D \in \mathcal{M}$ such that $d(C, D) < \ell$, we have to show that $\mathrm{cl}(\{C, D\}) \subseteq \mathcal{M}$.

Set $u = \delta(C, D)$. Since $\ell(u) < \ell \leq \mathrm{diam}(\Delta)$, there exists (in view of (P2)) some $t_1 \in S$ with $\ell(ut_1) = \ell(u) + 1$. Proceeding inductively, we find $t_1, t_2, \ldots, t_m$, $m = \ell - \ell(u)$, such that $\ell(ut_1t_2\ldots t_m) = \ell$. Set $w := ut_1t_2\ldots t_m$.

Since $D \in \mathcal{M}$ and $\mathcal{M}$ is weak, there exists $D_1 \in \mathcal{M}$ with $\delta(D, D_1) = t_1$. Inductively we find $D_i \in \mathcal{M}$ with $\delta(D_{i-1}, D_i) = t_i$, for $i \in \{1, 2, \ldots, m\}$ (where $D_0 := D$). By Axiom (B4) we conclude that $\delta(C, D_m) = w$ and in particular $d(C, D_m) = \ell(w) = \ell$. Condition ($\mathrm{C}_\ell$) implies that $\mathrm{cl}(\{C, D\}) \subseteq \mathrm{cl}(\{C, D_m\}) \subseteq \mathcal{M}$. □

Remark. *Suppose that Δ is spherical and that $d = \mathrm{diam}(\Delta)$. Then, if* ($\mathrm{C}_d$) *is satisfied for $\mathcal{M} \subseteq \mathcal{C}(\Delta)$, and $\mathcal{M}$ contains at least two opposite chambers, $\mathcal{M}$ is convex.*

Indeed, $\mathcal{M}$ contains an apartment Σ (as the convex closure of two opposite chambers in $\mathcal{M}$). For any $C \in \mathcal{M}$, we find a chamber $D \in \Sigma$ opposite C as follows. First choose an arbitrary chamber $X \in \Sigma$ and set $w := \delta(C, X) \in W$. Then the longest element w_0 of W admits, by (P2), a reduced expression $w_0 = wv$ with $\ell(w_0) = \ell(w) + \ell(v)$, where $v := w^{-1}w_0$. Choosing D as the unique chamber in Σ with $\delta(X, D) = v$, an easy induction using (P6) now shows $\delta(C, D) = \delta(C, X)\delta(X, D) = wv = w_0$. By assumption the unique apartment containing C, D is contained in $\mathcal{M}$. Since any $C \in \mathcal{M}$ is an element of an apartment contained in $\mathcal{M}$, the set of chambers $\mathcal{M}$ is weak. Lemma 12 completes the proof of the remark.

We now arrive at the crux of the proof of Proposition 3.

Lemma 13 *If Δ is thick and $\ell \leq \mathrm{diam}(\Delta)$, then Condition* ($\mathrm{E}_\ell$) *for $\mathcal{M}$ implies Condition* (C_ℓ) *for $\mathcal{M}$.*

Given $C, D \in \mathcal{M}$ with $d(C, D) = \ell$, we must show that $\mathrm{cl}(\{C, D\}) \subseteq \mathcal{M}$. Equivalently, we have to show that, given any minimal gallery $\gamma = (C_0, C_1, \ldots, C_\ell)$ stretched between $C = C_0$ and $D = C_\ell$, every chamber of γ belongs to $\mathcal{M}$.

Set $w := \delta(C,D)$, $s := \delta(C,C_1)$ and $w' := sw$. Then $\ell(w')+1 = \ell(w) = \ell$ and $\delta(C_1,D) = w'$. By Lemma 9, we also have $C_1 \in \mathcal{M}$. Since $\ell(w') < \ell(w)$, the element w' cannot be the longest element of W, hence there exists $t \in S$ such that $\ell(w't) = \ell(w')+1 = \ell$ (see (P2)). By Lemma 11, $\mathcal{M}$ is thin. Hence there exists $D' \in \mathcal{M}$ such that $\delta(D,D') = t$, and so, using (P6) again, we obtain $\delta(C_1,D') = w't$. But then $(C_1, C_2, \ldots, C_{\ell-1}, D, D')$ is a gallery of length ℓ stretched between C_1 and D'. Lemma 9 shows us that $C_2 \in \mathcal{M}$. Proceeding inductively, we obtain $C_i \in \mathcal{M}$, for all $i \in \{1, 2, \ldots, \ell\}$. □

We can now prove Proposition 3. Indeed, under the given conditions, the set $\mathcal{M}$ is thin by Lemma 11. But then Lemma 12 and 13 imply that $\mathcal{M}$ is also convex. Hence $\mathcal{M}$ is the set of chambers of an apartment of Δ, see for instance [3], Section IV,4. on page 88.

Remark. If W is finite, and w_0 denotes the longest element of (W,S), with length n_0, then as a special case of Proposition 3, we can characterize the set of chambers of an apartment by the Condition (E_{n_0}), which coincides with Condition (E_{w_0}). This is Proposition 1 above.

4.2 Proposition 4

We now prove Proposition 4.

We are given a nonempty set $\mathcal{M}$ of chambers of a thick building $\Delta = (\mathcal{C}(\Delta), \delta)$ satisfying (E_ℓ), for some $\ell \geq m$, where m is defined as

$$m = \max(\{m(s,t) \mid s,t \in S \text{ and } m(s,t) < \infty\} \cup \{1\}).$$

We have to show that every connected component of $\mathcal{M}$ is an apartment. As before, it suffices to prove that $\mathcal{M}$ is thin and that every connected component is convex.

Now, from Lemma 11, we deduce that $\mathcal{M}$ is *thin*. The result for $m = 1$ already follows, since $\mathcal{M}$ is trivially 2-convex, and so every connected component is convex by Proposition 2.

Lemma 13 implies that $\mathcal{M}$ satisfies (C_ℓ), and with Lemma 12 we conclude that $\mathcal{M}$ satisfies (C_n) for all $n \leq m$. But then it is clear that $\mathcal{M}$ is 2-*convex* ($\mathcal{M}$ contains $\mathrm{cl}\{C,D\}$ for any two chambers C,D which are contained in some spherical rank 2 residue of Δ and thus satisfy $d(C,D) \leq m$).

Hence every connected component is convex by Proposition 2, and so Proposition 4 is proved.

5 Some Counterexamples

In this section, we assume that the reader is somewhat familiar with the construction of buildings of type A_n, the spherical ones of rank 2, and affine buildings of type $\tilde{A}_2$.

5.1 About connectivity in Proposition 4

We show that the assumptions in Proposition 4 are, in general, never strong enough to conclude that $\mathcal{M}$ is connected. This is not surprising if the building is not spherical. For instance, in an affine building of type (W, S), and for given natural number ℓ, it is often possible to construct two apartments which are at (numerical) distance $> \ell$ from each other (take for instance a tree). In the spherical case, taking $\ell = \text{diam}(\Delta)$, we are reduced to Proposition 1, and $\mathcal{M}$ is automatically connected. Hence there remains to investigate the values $\ell < \text{diam}(\Delta)$, for a spherical building Δ. We will, however, show that for type A_n, the assumptions for $\ell = \text{diam}(\Delta) - 1$ are not strong enough to guarantee the connectivity of $\mathcal{M}$.

So we consider a building Δ of type A_n, i.e. a building arising from a projective space of dimension n over the skew field $\mathbb{K}$ by taking as set of chambers the maximal flags of the geometry. In this building, an apartment is the barycentric subdivision of an n-simplex, i.e. the set of chambers arising from all subspaces spanned by the proper nonempty subsets of a set of $n+1$ independent projective points. Such a set corresponds to some basis of the underlying vector space $\mathbb{K}^{n+1}$. So an apartment of Δ is completely and uniquely determined by a basis of $\mathbb{K}^{n+1}$.

We first show the following technical lemma.

Lemma 14 *Suppose that $|\mathbb{K}| \geq \binom{2n+1}{n}$, then we can find a set S of $2n+2$ vectors in $\mathbb{K}^{n+1}$ such that every subset of S of cardinality $n+1$ is a basis of $\mathbb{K}^{n+1}$.*

We start with an arbitrary basis $\{e_0, e_1, \ldots, e_n\}$ of $\mathbb{K}^{n+1}$ (for instance, the standard basis). Let there be given i vectors $e_{n+1}, e_{n+2}, \ldots, e_{n+i}$, $i \in \{0, 1, \ldots, n\}$, so that every subset of size $n+1$ of $\{e_0, e_1, \ldots, e_{n+i}\}$ is a basis of $\mathbb{K}^{n+1}$. We show that we can find a vector e_{n+i+1} such that every subset of size $n+1$ of $\{e_0, e_1, \ldots, e_{n+i+1}\}$ is a basis of $\mathbb{K}^{n+1}$.

Indeed, it is sufficient for e_{n+i+1} to be not contained in any hyperplane spanned by n vectors of the set $\{e_0, e_1, \ldots, e_{n+i}\}$. Since there are $\binom{n+i+1}{n} \leq \binom{2n+1}{n} \leq |\mathbb{K}|$ such hyperplanes, these hyperplanes cannot fill the entire vector space.

Applying the foregoing argument subsequently for $i = 1, 2, \ldots, n$, the lemma follows. □

Now we remark that two chambers in a spherical building of type A_n are opposite if and only if for each $i \in \{1, 2, \ldots, n\}$ the subspace of (vector) dimension i of one chamber meets the subspace of dimension $n+1-i$ of the other chamber in the zero vector. So clearly, every chamber of the apartment Σ_1 determined by the basis $\{e_0, e_1, \ldots, e_n\}$ is opposite *every* chamber of the apartment Σ_2 determined by $\{e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ (using the above notation and the set guaranteed by Lemma 14).

Hence by the "only if" part of Proposition 3, the set of chambers of $\Sigma_1 \cup \Sigma_2$ satisfies (E_ℓ) for every $\ell < \text{diam}(\Delta)$, and is nevertheless disconnected.

5.2 About the bound on ℓ in Proposition 4

We show with a counterexample that the bound $\ell \geq m = \max(\{m(s,t) \mid s,t \in S$ and $m(s,t) < \infty\} \cup \{1\})$ is sharp. Therefore, let Δ be a thick rank two building associated to a generalized polygon Γ, hence with Weyl group $\mathbf{D}_{2m}$, for some $m \geq 2$. We denote the longest element of $\mathbf{D}_{2m}$ (with respect to two suitable generators) by w_0. The reader not familiar with generalized polygons can put $m = 3$ in which case the generalized polygon is nothing other than an ordinary projective plane. We remark that the chambers of Δ are the incident point-line pairs of Γ, hence edges in the incidence graph of Γ (which is the graph of points and lines of Γ with adjacency being incidence). Also, two chambers consisting of the point-line pairs $\{p, L\}$ and $\{p', L'\}$ are opposite if and only if the distances from the vertices p and L to the vertices p' and L' measured in the incidence graph are contained in $\{m, m-1\}$.

Now let Σ be a cycle of length $2m+2$ in the incidence graph of Γ. Such a cycle exists since we assume that Δ is thick (see [7], Lemma 1.3.2). The cycle Σ defines a *closed* gallery of $2m+2$ chambers $C_0, C_1, \ldots, C_{2m+1}, C_{2m+2} = C_0$, with C_{i-1} adjacent to C_i, $i \in \{1, 2, \ldots, 2m+2\}$. Clearly $d(C_0, C_i) = d(C_0, C_{2m+2-i}) = i$ for $i \in \{1, 2, \ldots, m\}$ and $\delta(C_0, C_i) \neq \delta(C_0, C_{2m+2-i})$ for $i < m$. Hence both C_m and C_{m+2} are opposite C_0. So the distances from the vertices defined by C_m and C_{m+2} to the vertices defined by C_0 are contained in $\{m, m-1\}$, implying easily that also C_0 and C_{m+1} are opposite. We conclude that for each chamber $C \in \Sigma$ and for each $w \in \mathbf{D}_{2m} \setminus \{w_0\}$, there is a unique chamber D in Σ such that $\delta(C, D) = w$.

Consider the apartment Σ_1 containing C_0 and C_m, and the apartment Σ_2 containing the chambers C_{m+1} and C_{2m+1}. One can check easily that every chamber of $\Sigma_1 \cup \Sigma_2$ that does not belong to Σ, belongs to $\Sigma_1 \cap \Sigma_2$. Now let C be any chamber outside $\Sigma_1 \cup \Sigma_2$, and let $w \in \mathbf{D}_{2m}$. By Proposition 1.6 of [2], there are an even number n_i of chambers D in Σ_i with $\delta(C, D) = w$. Adding n_1 and n_2, we count the chambers not in Σ twice (as they belong to $\Sigma_1 \cap \Sigma_2$). We conclude that there are an even number of chambers D in Σ with $\delta(C, D) = w$. Similarly, if $C \in \Sigma_1 \setminus \Sigma = \Sigma_2 \setminus \Sigma$, then there are 0 or 2 chambers D with $\delta(C, D) = w$.

We have shown that (E_w) is satisfied for Σ, for all $w \in \mathbf{D}_{2m} \setminus \{w_0\}$. Yet, Σ is not an apartment. This shows that the bound on ℓ in Proposition 4 is sharp.

Remark. We considered $(2m+2)$-cycles here for the sake of convenience. In fact, if Δ is "thick enough", one can construct $(2m+2k)$-cycles Σ in Δ for any given natural number k such that Condition (E_{m-1}) is satisfied for Σ. Even more: If Γ is such that there are infinitely many points on each line and infinitely many lines through each point, then one can construct a Coxeter complex Σ of type $\tilde{A}_1$ (corresponding to the infinite dihedral group $\mathbf{D}_\infty$) in Δ satisfying (E_{m-1}).

5.3 About Proposition 2

One could wonder if the assumption in Proposition 2 of 2-convexity is optimal. In particular, would it hurt to drop the assumption of 2-convexity in one single rank 2 residue? In other words, if a set $\mathcal{M}$ of chambers is connected, and if in all rank 2 residues but one, $\mathcal{M}$ induces a convex set, is $\mathcal{M}$ necessarily convex?

The answer is no, and we briefly describe an example, without introducing the notions. We refer to e.g. [6] for the definitions, more information and background.

Let Δ be a building of type $\tilde{A}_2$, and let Δ^∞ be the building at infinity (which is a building of type A_2; hence associated to a projective plane). Let Q be a quadrangle in that projective plane. Then Q defines four apartments of Δ^∞, which in their turn define four apartments $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ in Δ. Now the union

$$\mathcal{M} := (\Sigma_1 \cap \Sigma_2) \cup (\Sigma_2 \cap \Sigma_3) \cup (\Sigma_3 \cap \Sigma_4) \cup (\Sigma_4 \cap \Sigma_1)$$

is a set of chambers which is thin and connected. If Δ is viewed as a simplicial complex, then the intersection of the four apartments is a vertex v, and it can be shown that the intersection of $\mathcal{M}$ with the rank 2 residue defined by v is a set of 8 chambers arranged in a quadrangle; hence not convex. All other rank 2 residues intersect $\mathcal{M}$ in a convex set. Our example is complete.

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