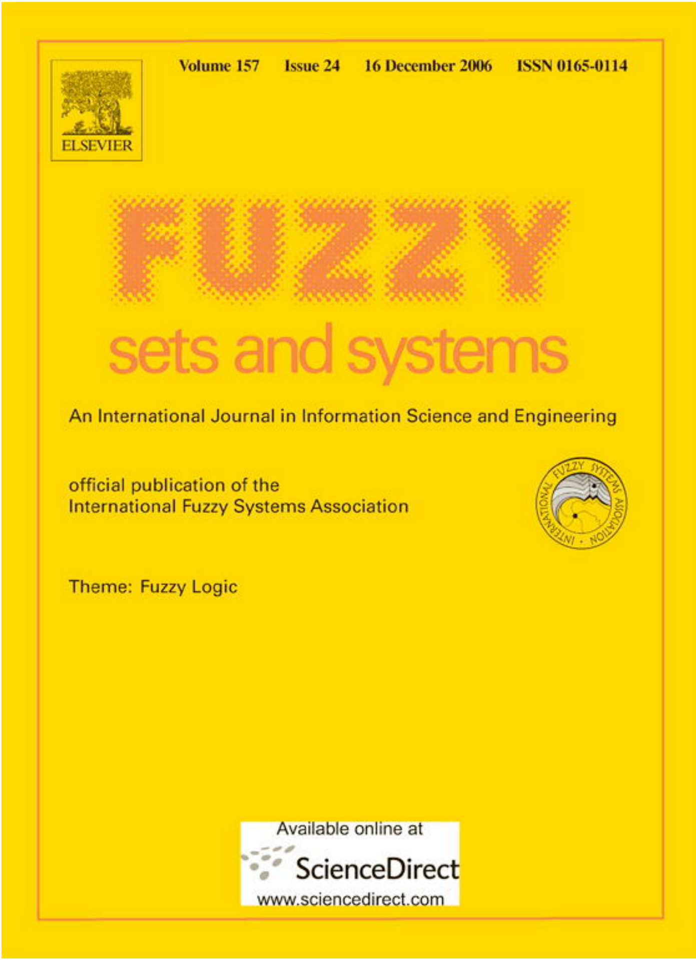
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Fuzzy projective spreads of fuzzy projective spaces

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Abstract

In this paper, we define the notion of a fuzzy spread of a fuzzy projective space. We classify the fuzzy line spreads of the smallest finite projective space, the Fano plane, and prove a general existence theorem for line spreads of arbitrary finite projective planes. We then extend the classical relation between designs and statistics to an application of the fuzzy spreads in statistics, by considering projective spaces as 2-designs. A fuzzy spread then gives a blueprint of how to make groups for test procedures where certain kinds of objects are mixed together and in each group one kind is over-represented.

Keywords: Fuzzy projective space; Fuzzy point; Fuzzy projective spread

1. Introduction

In this paper, we join the worlds of projective geometry and fuzzy mathematics to come to an application in statistics. We start this Introduction with explaining some basic notions on geometry that will be useful for this paper, and proceed with the classical link between geometry and statistics. It is precisely that classical link that we will generalize using fuzzification.

Projective geometry provides a geometric way to study vector spaces. Indeed, a projective space over a skew field K is nothing else than the lattice of proper non-trivial subspaces of a vector space over K. This is the origin of projective geometry. Axiomatizing this situation, one arrives at what could be called *abstract projective spaces*. It turns out that such an abstract projective space (for a formal definition, see in Section 2) has a certain dimension d, and whenever $d > 2$, an abstract projective space is isomorphic to a projective space over some skew field. However, in dimension 2 (where we speak of projective *planes*) there are a lot of examples which do not arise from a vector space. In fact, as a side remark, we note that “being over some skew field” is equivalent with “satisfying the Axiom of Desargues”. This is the reason why projective spaces and planes over a skew field are sometimes called *Desarguesian* spaces and planes. We will also use this terminology in this Introduction. Recall that the Axiom of Desargues says that, if two triangles $A_0B_0C_0$ and $A_1B_1C_1$ are in perspective from a point X (i.e., if the lines A_0A_1, B_0B_1 and C_0C_1 all contain the point X), then they are also in perspective from some line L (i.e., the points $A_0A_1 \cap B_0B_1$, $B_0B_1 \cap C_0C_1$ and $C_0C_1 \cap A_0A_1$ lie on L).

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Let us give an example to fix the ideas. Let K be any field, and let V be a three-dimensional vector space over K. The projective plane $PG(V)$, also denoted by $PG(2, K)$ because it has dimension 2, consists of all one-and two-dimensional subspaces of V (these are indeed exactly the proper non-trivial subspaces of V). For dimension reasons we call the one-dimensional subspaces *points* and the two-dimensional ones *lines*. Note the following properties of $PG(V)$.

Considering two points A and B, there is a unique line L containing both A and B. Indeed, A and B are just two different one-dimensional subspaces of V, and there is exactly one two-dimensional subspace L spanned by A and B.

Also, considering two lines L and M, there is a unique point A contained in both L and M. Indeed, L and M are two different two-dimensional subspaces of V, and these intersect in a unique one-dimensional subspace A by dimension considerations.

Finally, there are four points in such a position that they pairwise define six different lines. Indeed, we can take the vector lines generated by the vectors with coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$, respectively (after introducing coordinates in V).

The previous three properties are taken as the axioms for an abstract projective plane. Hence, an abstract projective plane consists of a set $\mathcal{P}$ of *points*, and a set $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, such that every pair of points is contained in exactly one line, every two lines intersect in exactly one point and there exist four points in such a position that they pairwise define six different lines.

The smallest example of a projective plane is the Desarguesian projective plane over the field $GF(2)$ with two elements. We denote it by $PG(2, 2)$—replacing $GF(2)$ by 2 in $PG(2, GF(2))$. It has seven points and seven lines, and every line has exactly three points. It is easy to see and is well known that there is, up to isomorphism, only one abstract projective plane where each line carries exactly three points, and it has seven points in total and seven lines in total. This plane is usually called the *Fano plane*. Since vector lines in a vector space over $GF(2)$ only have one non-trivial vector, we may identify every vector line (which is a one-dimensional subspace) with its unique non-trivial vector. Hence the Fano plane consists of the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$. What are the lines? Well, three vectors form a line if they are contained in a vector plane (a two-dimensional subspace). Together with the zero vector, these three vectors form a four group (where addition is in $GF(2)$). For example, the three vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$ form, together with $(0, 0, 0)$ a group of order 4, and hence, viewed as points of the Fano plane, the triple $\{(1, 1, 0), (0, 1, 0), (1, 1, 0)\}$ forms a line. Further on below, we will give another construction of the Fano plane.

The reason for introducing projective spaces from vector spaces lies in the theory of groups. The projective linear group $PGL(V)$, with V any vector space, acts primitively on the set of vectors of V, but it acts imprimitively on the set of points of the projective space $PG(V)$ (and even two-transitively). Hence, the projective space provides a better permutation module for the projective linear group $PGL(V)$ and the special linear group $PSL(V)$.

We remark that by going from a vector space to the associated projective space, the dimension drops by one unit. Hence an $(n + 1)$-dimensional vector space gives rise to an n-dimensional projective space (this is related to what happens in real life when one projects the universe from its eyes onto the sky). From now on, when we speak of "projective space" and "projective subspace", we deal with projective dimensions, too. So the projective subspaces of dimension 1 of a projective plane are its lines, while those of dimension 0 are the points.

There is a very tight connection of projective geometry with *affine geometry*. An affine space is, in fact, the same thing as a projective space where one subspace of maximal dimension, together with all its subspaces, have been deleted. For instance, an affine plane is a projective plane with one line and all its points removed. One can again axiomatize this situation to obtain abstract affine planes. Such axioms were already studied by Euclid in his famous *Elements*. The difference, however, is that the modern axioms do not necessarily lead to the *real* affine plane, but include affine planes over arbitrary skew fields, and in particular allow for finite planes. The usual axioms on the set of points of an affine plane are (1) every pair of points determine a unique line, (2) through a point of a given line passes a unique line disjoint from the given one, (3) there exist three points not contained in a common line. By introducing parallelism one can define "points at infinity" and a "line at infinity" and add this to the affine plane to obtain a projective plane. So we can go back and forth from affine to projective planes, and similarly for affine and projective spaces.

One way to construct projective planes that are not Desarguesian is to use the notion of a "spread" of a projective space. Let V be an $(n + 1)$-dimensional vector space over some skew field K. Let $j < n$ be a natural number. A j-*spread* of the projective space $PG(V) = PG(n, K)$ is a set of j-dimensional projective subspaces which, viewed as set of points contained in it, partition the point set of $PG(V)$. Let us give an example. Put $n = 3$ and $j = 1$. So we consider a 1-spread of a projective space $PG(3, K)$ of dimension 3. Such a 1-spread is also called a *line spread*, since it consists of lines. Put $K = GF(2)$ again, then $PG(3, 2)$ has exactly 15 points, given with coordinates by 4-tuples (a, b, c, d), with $a, b, c, d \in \{0, 1\}$, and not $(0, 0, 0, 0)$, similarly as above for the Fano plane. A spread is, for example, given by the five lines

$$\begin{aligned}
&\{(1, 0, 0, 0), (0, 1, 0, 0), (1, 1, 0, 0)\},\\
&\{(0, 0, 1, 0), (0, 0, 0, 1), (0, 0, 1, 1)\},\\
&\{(1, 0, 1, 0), (0, 1, 0, 1), (1, 1, 1, 1)\},\\
&\{(1, 1, 1, 0), (0, 1, 1, 1), (1, 0, 0, 1)\},\\
&\{(1, 0, 1, 1), (1, 1, 0, 1), (0, 1, 1, 0)\}.
\end{aligned}$$

Now, how do we construct a projective plane with a spread? Let us explain and illustrate this in the case of a line spread $\mathcal{S}$ in $PG(3, K)$, for any skew field K. What we do is consider $PG(3, K)$ as a three-dimensional projective subspace of $PG(4, K)$, and then we define the following structure $\mathcal{A}$. The point set $\mathcal{P}$ of $\mathcal{G}$ is the set of points of $PG(4, K)$ not contained in $PG(3, K)$. The lines of $\mathcal{A}$ are the subsets of points of $\mathcal{P}$ that are contained in fixed planes β of $PG(4, K)$ with the property that β and $PG(3, K)$ have exactly an element of $\mathcal{S}$ in common. A routine check now shows that the structure $\mathcal{A}$ is an affine plane. It turns out that the corresponding projective plane is Desarguesian if and only if the spread $\mathcal{S}$ is a so-called *regular* spread. We will not explain what exactly this means, since it is not relevant for the present paper. The bottom line is that there are many non-regular spreads and there is a vast research body devoted to these structures. This should convince the reader that spreads are interesting objects worth to be studied.

Projective planes have been fuzzified by Kuijken et al., see [2]. In the present paper we fuzzify the notion of a spread. As an application, we extend the classical connection of statistics with combinatorial geometry. Let us now explain what this classical connection is.

Let $\mathcal{P}$ be a set of v objects or people that must be tested in small groups of size k for statistical purposes. The aim of the tests are in general to see how one object behaves when certain others are in its company. Hence, it is important that in the various tests there is a certain regularity of objects being together with the other objects. For instance, it could be preferable that each set of t objects must be together in λ tests with each other. A scheme with these constraints is called a *design*, in particular a $t - (v, k, \lambda)$-design. In many cases, one does not mention the parameters v, k, λ and simply talks about a t-design. The higher t is, the more difficult it is to find such designs. A 1-design just consists of a set of points endowed with a set of subsets of constant size such that each point is contained in a constant number of subsets. It is not difficult to construct such designs, and they exist in abundance. For $t = 2$, the situation is already more involved. However, there are still a lot of 2-designs. One notable class of examples are the finite projective spaces. If we take as set of objects the points of the projective space $PG(n, K)$, with K a finite field, and as set of subsets all j-dimensional projective spaces, then we obtain a 2-design. In particular, if the subsets are the lines, then $\lambda = 1$.

Also the spreads have an application here. Indeed, a problem that arises is that, if an object is in ℓ different groups (sets), then the complete testing lasts for at least ℓ times the duration d of a single test. In the optimal situation, it is exactly equal to ℓ times d. But clearly, this can only be achieved if in time d *all objects* can be tested in various groups, which means that the groups of these particular tests form a partition of the whole set, hence a spread if we deal with projective spaces. A further problem is then to partition the set of *all subsets* into such partitions. In the case of a projective space and j-spreads, this is called a j-fence. Let us again give an example.

Suppose we must test 15 objects in groups of three and each two objects must be together in exactly one of the test groups. A design for that is the point set of $PG(2, 3)$ endowed with all its lines. There are 35 lines. So for optimal testing, we should find seven line spreads such that each line is contained in exactly one such spread. It is well known that this can indeed be done. So, projective geometry gives an answer to a combinatorial problem in statistics. A good reference for all the above is [1].

Our aim is now to define the fuzzy counterpart of a spread in fuzzy projective spaces, and to apply this in a more involved question about test groups in statistics. The paper is organized as follows. In Section 2, we recall the formal definitions of the (well-known) notions we need. In Section 3, we introduce the new notion of a fuzzy spread of a fuzzy projective space and give some immediate examples. In Section 4, we apply our definition to the smallest projective space, the Fano plane, and classify fuzzy spreads in the Fano plane. Although at first glance, fuzzy spreads are not very restrictive, our investigations in the Fano plane show that not all possibilities with respect to the various base points of the fuzzy lines of the fuzzy spread are possible (for a definition of base point, see later). We also prove a general result about fuzzy line spreads in arbitrary fuzzy projective planes. In Section 5, we present an application to statistics. As already mentioned, this application is a generalization of a classical link between the theory of designs and the theory of test procedures in statistics.

2. Preliminaries

We now get down to definitions. The following definitions and theorems concerning the basic concepts of the subject have been taken from [2,1].

First recall that fuzzy sets were introduced by Zadeh in the fundamental paper [5]. A *fuzzy set* λ of a set X is a function $\lambda : X \longrightarrow [0, 1]$.

Definition 2.1. We now define the *n-dimensional projective space* $PG(n, K)$ for $n > 0$ and K any (skew) field. Let V be any vector space of dimension $n + 1$ over K. Then $PG(n, K)$, the *n-dimensional projective space over* K, is the set of all subspaces of V distinct from the trivial subspaces $\{\overrightarrow{o}\}$ and V. The one-dimensional subspaces of V are called the *points* of $PG(n, K)$, the two-dimensional subspaces are called the (*projective*) *lines* and the three-dimensional ones are called (*projective*) *planes*. A non-trivial $(k + 1)$-dimensional subspace of V is also called a *k-subspace* of $PG(n, K)$, or simply, a *subspace*. Since every subspace of V is itself a vector space, we may view any subspace of $PG(n, K)$ as a projective space. For two subspaces U, U' of $PG(n, K)$, we write $U \leqslant U'$ if U is contained in U'.

We use common projective terminology such as a set of points are *collinear* when they are contained in a common line. Also, if $|K| = q + 1$ is finite, we call q the *order* of the projective space.

Remark 2.2. In Definition 2.1 we only defined the so-called *Desarguesian* projective spaces. This does not make any difference for n-dimensional projective spaces with $n \geqslant 3$. But for $n = 2$, there exist non-Desarguesian projective planes. The definition of an abstract projective plane is given in the Introduction. Here, for completeness' sake, we mention one of the axiom systems for an abstract projective space of arbitrary dimension (including infinite dimensions). First we note that such a system only needs to axiomatize the set of lines of a projective space, because the other subspaces can be reconstructed from the set of lines (for instance, a plane is the union of the sets of points on the lines meeting all of three fixed lines that pairwise meet in three different points). The following axioms characterize projective spaces. Let $\mathcal{P}$ be a set of points, and let there be given a set $\mathcal{L}$ of subsets, called lines. Then $\mathcal{P}$ and $\mathcal{L}$ are the point set and line set, respectively, of a projective space of dimension at least three if (1) every pair of points is contained in a unique line, (2) every line contains at least three points and (3) if A and B are two intersecting lines (i.e., lines that have a point X in common) and C and D are two lines that both intersect both A and B in points different from X, then C and D have a point in common.

We now recall the definition of a crisp spread. A (crisp) *k-spread* S, or simply *spread* of the projective geometry $PG(n, K)$ is a partition of the point set of $PG(n, K)$ into k-spaces, for some k, $1 \leqslant k \leqslant n - 1$, see [1].

Definition 2.3. Suppose $\mathcal{P}$ is an n-dimensional projective space. A fuzzy set λ on the point set of $\mathcal{P}$ is a *fuzzy n-dimensional projective space* on $\mathcal{P}$ if $\lambda(p) \geqslant \lambda(q) \wedge \lambda(r)$, for all collinear points p, q, r of $\mathcal{P}$. We denote as $(\lambda, \mathcal{P})$. The projective space $\mathcal{P}$ is called the *underlying* (*crisp*) *projective space* of $(\lambda, \mathcal{P})$. If $\mathcal{P}$ is a fuzzy point, line, plane, etc., we use underlying point, underlying line, underlying plane, etc., respectively. We will sometimes briefly write λ instead of $(\lambda, \mathcal{P})$.

This definition has been taken from [2] and was inspired by and deduced from the definition of a fuzzy vector space (see e.g., [3]). In fact, what is meant is that in the set of points of a line, all elements have the same membership degree, except possibly one. Also, and more generally, this implies that in any subspace U, all points have the same membership degree, except possibly in a subspace U' of U, where again all points have the same membership degree, except possibly in a subspace U'' of U', etc. The exceptions always have a larger membership degree. The reason is what we could call the *principle of level substructures*, which says that in a fuzzy structure the set of points with a membership degree exceeding a given number r, $0 \leqslant r \leqslant 1$, must be a substructure "of the same nature". This is, for instance, also true for fuzzy groups, see [4], and encoded in the definition. An obvious advantage of this principle is that crisp substructures of such fuzzy structures inherit in a natural way a fuzzy structure. This makes, for instance, inductive arguments possible.

In the case of projective spaces, we do not only have the fuzzy structure inherited by a given subspace, but a subspace can also have a fuzzy structure on its own.

Definition 2.4. Let $(\lambda, \mathcal{P})$ be a fuzzy projective space and let U be a subspace of $\mathcal{P}$. Then (λ_U, U) is called a *fuzzy subspace* of $(\lambda, \mathcal{P})$ if $\lambda_U(x) \leqslant \lambda(x)$ for $x \in U$, and $\lambda_U(x) = 0$ for $x \notin U$.

As already alluded to above, the following proposition gives the structure of a fuzzy projective line.

Proposition 2.5. *Let $(\lambda, \mathcal{L})$ be a fuzzy projective line. Then there are constants $a, b \in]0, 1]$, $a \leqslant b$, and a point z of $\mathcal{L}$ such that*

(i) $\lambda(z) = b$,
(ii) $\lambda(x) = a$, *for all* $x \neq z$.

By the previous proposition, every fuzzy projective line admitting points with different membership degrees contains a unique point with maximal membership degree. We will refer to such a point as the *base point* of the fuzzy line.

More generally, the structure of a fuzzy projective space looks as follows (see [2]).

Let $(\lambda, \mathcal{P})$ be a fuzzy projective space of dimension n. Then there are constants $a_i \in]0, 1]$, $i = 0, 1, \dots n$, with $a_i \leqslant a_{i+1}$, and a chain of subspaces $(U_i)_{0 \leqslant i \leqslant n}$, with $U_i \leqslant U_{i+1}$ and $\dim U_i = i$, such that

$$\begin{aligned} \lambda : \ & \mathcal{P} \to [0, 1], \\ & x \to a_0 \quad \text{for } x \in U_0, \\ & x \to a_i \quad \text{for } x \in U_i \backslash U_{i-1}, \quad i = 1, 2, \dots, n. \end{aligned}$$

3. Fuzzy projective spreads of fuzzy projective spaces

We now introduce the notion of a fuzzy spread of a fuzzy projective space.

Definition 3.1. Let λ be a fuzzy projective space on $\mathcal{P}$. A set S of fuzzy projective subspaces of fixed dimension such that for every point x of the projective space $\mathcal{P}$ we have

$$\lambda(x) = \sum_{\mu \in S} \mu(x)$$

is called a fuzzy spread of λ.

Note that the crisp set S is not necessarily a crisp spread of the underlying projective space $\mathcal{P}$ (indeed, see the next example below). This property is not surprising since otherwise the study of fuzzy spreads would rapidly be reduced to the study of crisp spreads. Exactly this remark makes the fuzzy spreads richer objects than the crisp spreads. Of course, if only crisp sets and subsets are allowed, then a fuzzy spread is equivalent to a crisp set (see the last remark below of the present section), which implies that crisp spreads are special cases of fuzzy spreads. This property is most desirable.

Example 3.2. We take a projective plane $\mathcal{P}$ of order s. Let us take all lines of $\mathcal{P}$. Is it possible to find fuzzy sets λ_i of all lines constituting a fuzzy projective spread in the fuzzy projective plane?

A fuzzy projective plane (λ, P) is of the following form (see above and [2]):

$$\begin{aligned} \lambda : \ & \mathcal{P} \to [0, 1], \\ & p \to a_0 \quad \text{for } p = q, \\ & p \to a_1 \quad \text{for } p \in L_\infty \backslash \{q\}, \\ & p \to a_2 \quad \text{for } p \in \mathcal{P} \backslash L_\infty \end{aligned}$$

for a certain point–line pair (q, L_∞), with $q \in L_\infty$, in $\mathcal{P}$, and for some reals $a_0 \geqslant a_1 \geqslant a_2$ in $]0, 1]$.

Then one can give a fuzzy projective spread in $(\lambda, \mathcal{P})$ as follows:

$$\lambda_i(x) = \frac{\lambda(x)}{s+1}.$$

Indeed, since every point is on equally many lines, namely on $s+1$ lines, and since every projective line inherits a natural fuzzy line, the fuzzy set obtained from that natural one by dividing by $s+1$ is a fuzzy projective line, and the membership degrees in any point add up to $s+1$ times the original membership degree divided by $s+1$, i.e., the original membership degree emerges.

This example can be seen as a kind of trivial spread. Intuitively, it is somehow clear that reduction of the number of lines used in a fuzzy spread makes a construction more difficult and might even lead to non-existence. We illustrate and emphasize this in the next section.

Remark 3.3. In the crisp case $\lambda = 1$, a spread $S = \{\lambda_L : \dim L = k\}$, where each λ_L is the characteristic function of L, clearly amounts to an ordinary spread of $\mathcal{P}$. Note that in the crisp case there are restrictions on k and n for the existence of a spread; in the fuzzy case this is no longer the case. Indeed, similar to our example above, one can construct a spread for every k and n.

4. Fuzzy spreads of finite projective planes

The goal of this section is to illustrate the definition of a fuzzy spread in the case of a very concrete example, namely the Fano plane (see the Introduction and below). Further, the theorems we prove show that the notion of a fuzzy spread is not a trivial one, and some interesting mathematics arises, raising new questions. Finally, the projective spaces form a class of so-called *designs* which are useful in statistics, and the results in this section prepare an application of fuzzy spreads to statistics. As explained in the Introduction, a design is a combinatorial blueprint of how to group together test persons or objects of an experiment for non-simultaneous testing if certain homogeneous boundary conditions must be satisfied. Typically such a condition reads *every t persons must eventually be contained in exactly λ groups of k persons*, for given natural numbers t, k, λ. For instance, for projective planes of order s, we have $t = 2$, $\lambda = 1$ and $k = s+1$. A vast body of literature consists in trying to determine the possible values of t, k, λ. In the last section we will present a generalization to simultaneous testing of groups of objects using fuzzy spreads, and so the results of the present section can be seen as the first counterparts for determining the possible parameters of such tests. For instance, from our results it can be concluded that a minimal number of tests must be done (this is the interpretation of Proposition 4.1). We will also interpret Theorem 4.6 in the next section.

On the theoretic level, we deduce from the results below that fuzzy spreads are not “random structures”, but they give rise to some interesting combinatorial questions (e.g., the possibilities for the configurations of base points and special points for fuzzy spreads).

4.1. The Fano plane

In this subsection we investigate the fuzzy spreads of the smallest non-trivial projective space, namely the Fano plane. This projective plane, denoted by $PG(2,2)$, consists of seven points and seven lines and can, alternatively, be defined as follows: the points are the integers modulo 7 and the lines are the sets $\{i, i+1, i+3\}$, for all i modulo 7. One can see this with the following identification (using the description with coordinates in the Introduction):

$(1,0,0) \mapsto 1 \bmod 7$,
$(0,1,0) \mapsto 2 \bmod 7$,
$(0,0,1) \mapsto 3 \bmod 7$,
$(1,1,0) \mapsto 4 \bmod 7$,
$(0,1,1) \mapsto 5 \bmod 7$,
$(1,1,1) \mapsto 6 \bmod 7$,
$(1,0,1) \mapsto 7 \bmod 7$.

We will denote a fuzzy line with the pair (a, b), $a \geqslant b$, where a is the highest membership degree on that line (and the corresponding point is called *base point*) and b the membership degree of the other points on that line.

Proposition 4.1. *There are no fuzzy spreads with at most three lines.*

Proof. Since the lines of a spread must cover all points of the plane, each line contains three points, and there are seven points in total, we see that there must be at least three lines in the spread. If there are exactly three lines, then they must meet in one point x (otherwise there are three intersection points counted double, so we only cover six points). But then the membership degrees of the points different from x on these lines must be 1 and so the point x has 3 times membership degree 1 (by definition of fuzzy line), a contradiction. □

For the sequel, it is convenient to define the notions of *weight* and of *special point*.

The number of subspaces of a fuzzy spread in a point is called the *weight* of that point. If a point has maximum weight then it is called a *special point*.

Proposition 4.2. *If a fuzzy projective spread has exactly four lines, then there are constants $a, b, a \geqslant b, 2a+b = 1$, such that the spread consists of three concurrent fuzzy lines $(1, a)$, $(1, a)$ and $(1, b)$ and one other fuzzy line $(1 - b, 1 - a)$.*

Proof. Assume, by way of contradiction, that there is no point with weight 3. Since the total weight must be $4 \times 3 = 12$, either there are six points with weight 2 and a unique point with weight 0 (but this contradicts the definition of a spread) or there are five points with weight 2 and two points with weight 1. In the latter case we consider the line L containing the two points A, B with weight 1. Let C be the third point on L. If L belonged to the fuzzy spread, then the three other lines must contain C, contradicting the fact that C has weight 2. Hence, there are unique lines L_A and L_B of the spread containing A and B, respectively. Let D be the intersection point of L_A and L_B. Since C has weight 2, there are two lines of the spread through C, and these lines differ from L. These two lines contain in total all four points not on L, hence also D is on one of these lines. But then the weight of D is equal to 3, a contradiction.

Hence there are a point with weight 3 and a unique fuzzy line of the spread not containing that point. Writing this fuzzy line as $(1 - a, 1 - b)$, the rest follows in a straightforward way. □

Proposition 4.3. *If a fuzzy projective spread has five lines, then the following cases are possible*:

(i) *no base point coincides with a special point*,
(ii) *exactly one base point coincides with a special point*,
(iii) *two base points coincide with the same special point*.

On the other hand the following cases are impossible:

(iv) *exactly one base point coincides with each of the two special points*,
(v) *two base points coincide with one special point and one other base point coincides with the other special point*,
(vi) *each special point coincides with two different base points*.

Proof. (i) Let b_1, b_2, b_3 and b_4 be the membership degrees of the respective base points x_1, x_2, x_3, x_4 of four lines and 1 be the membership degree of the base point of the fifth line (the latter is the unique line that is incident with two special points—points with weight 3—and one point with weight (1). We may assume that x_1, x_2 are the base points of two lines through one special point and that x_3, x_4 are those of two lines through the other special point. It is easy to calculate that

$$\sum b_i \geqslant 2 \quad \text{and} \quad b_1 + b_2 = b_3 + b_4$$

holds and, conversely, when these relations hold, then one can get a fuzzy spread S with five lines such that no base point coincides with any special point. Note that, if $x_1 = x_2$ and $x_3 = x_4$ (and then all base points are incident with the same line), then necessarily $b_1 + b_2 = b_3 + b_4 = 1$.

(ii) Let again b_1, b_2, b_3 and b_4 be the membership degrees of the respective base points x_1, x_2, x_3, x_4 of the four lines and 1 be membership degree of base point of fifth line (as above). We may assume that x_1, x_2 are the base points of two lines through one special point and x_3, x_4 are those of two lines through the other special point. Also, x_1 is a special point. There are essentially three different non-equivalent ways to choose the points x_1, x_2, x_3, x_4 according to the above restrictions. We give one example. Let all of x_1, x_2, x_3, x_4 be distinct and such that x_1, x_2, x_4 are collinear (there is a unique such case as is easily verified). Then $(b_1, 1 - b_3)$, $(b_2, 1 - b_4)$ and $(1, b_4 - b_1)$ are the fuzzy lines concurrent at the special point x_1. Also, one easily computes the relation

$$b_2 + 2b_3 + 3b_4 \geqslant 3.$$

Conversely, when this relation holds, then one can get a fuzzy spread S with five lines such that one base point coincides with a special point and such that all of x_1, x_2, x_3, x_4 are distinct with x_1, x_2, x_4 collinear. The other two cases can be treated similarly.

(iii) Let again b_1, b_2, b_3, b_4 and x_1, x_2, x_3, x_4 be as before. Now we assume that $x_1 = x_2$ is a special point. There are two inequivalent cases: $x_1 = x_2, x_3, x_4$ all collinear or not. We treat the first case. Then, $(b_1, 1 - b_3)$, $(b_2, 1 - (b_1 + b_2)/2)$ and $(1, 1-b_1-b_2)$ are the fuzzy lines of the spread concurrent at the special point x_1 and $(b_4, (b_1 + b_2)/2)$, $(b_3, (b_1 + b_2)/2)$ and $(1, 1 - b_1 - b_2)$ are the fuzzy lines of the spread concurrent at the other special point. Moreover, we have

$$\sum b_i \geqslant 2 \quad \text{and} \quad b_3 = b_4$$

and, conversely, if the above relation holds, then one can get a fuzzy spread S with five lines and above properties.

(iv) With the same notation as above and assuming that x_1 is a special point and x_4 is the other special point, then we find the following inequality:

$$2 > \sum b_i \geqslant 2,$$

which is clearly a contradiction.

(v) This is completely similar to the previous case (including the same contradiction).

(vi) Again similar. □

Proposition 4.4. *If a fuzzy projective spread has six lines, then at most four base points can coincide with special points, but at least one base point coincides with a special point.*

Proof. Suppose first that six base points $x_1, \ldots, x_6$ coincide with special points. Let b_1, b_2, b_3, b_4, b_5 and b_6 be the membership degrees of these respective base points. After a straightforward computation, one finds the equality which is clearly a contradiction. Similarly for five base points coinciding with special points.

Now suppose that no base point is a special point. Remark that the three non-special points are collinear, and that, in this case, each non-special point is 2 times a base point. Then one can calculate that this situation leads to the equality

$$\sum b_i = 0,$$

where the b_i are again the membership degrees of the respective base points, $1 \leqslant i \leqslant 6$. □

Proposition 4.5. *If a fuzzy projective spread has seven lines, then given any mapping ρ from the set of lines to the set of points, it is always possible to find a fuzzy spread such that each fuzzy line (λ, L) has a unique base point $\rho(L)$ (in other words, the membership degree of each base point is strictly larger than the membership degree of the other points on the line).*

Proof. This can be shown similarly as the previous propositions; in the next theorem we present a general proof (for arbitrary finite projective planes). □

4.2. A general existence result

We generalize the last proposition of the previous subsection to all finite projective planes.

Theorem 4.6. *If a fuzzy projective spread of a projective plane $\mathcal{P}$ of order q has $q^2 + q + 1$ lines, then given any mapping ρ from the set of lines to the set of points, it is always possible to find a fuzzy spread such that each fuzzy line (λ, L) has a unique base point $\rho(L)$ (in other words, the membership degree of each base point is strictly larger than the membership degree of the other points on the line).*

Proof. Let $\{L_i : i = 1, 2, \ldots, q^2 + q + 1\}$ be the set of lines of $\mathcal{P}$. Let b_i be the membership degree of the base point x_i of the fuzzy line (λ_i, L_i) and let a_i be the membership degree of the other points on (λ_i, L_i). Expressing that the sum of the membership degrees in an arbitrary point add up to 1, we obtain a system of $q^2 + q + 1$ equations with the $2(q^2 + q + 1)$ unknowns $a_1, a_2, \ldots, a_{q^2+q+1}, b_1, \ldots, b_{q^2+q+1}$. Now we put $\varepsilon_i > 0$ equal to $b_i - a_i$ and consider it as a constant. Then the above system of equations reduces to a system of $q^2 + q + 1$ equations with $q^2 + q + 1$ unknowns, of which the matrix is exactly the incidence matrix M of $\mathcal{P}$. It is well known that such a matrix is non-singular (indeed, it is easy to check that the eigenvalues of the matrix MM^t are equal to q and $(q + 1)^2$, with respective multiplicities $q^2 + q$ and 1), hence for each choice of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{q^2+q+1})$, there is a unique solution $(a_1, a_2, \ldots, a_{q^2+q+1})$. This mapping is obviously continuous. Expressing continuity of this mapping in the point $(0, 0, \ldots, 0)$, which is mapped onto $(1/(q + 1), 1/(q + 1), \ldots, 1/(q + 1))$, we find $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{q^2+q+1})$, $0 < \varepsilon_i < 1/(q + 1)$, with corresponding solution $(a_1, a_2, \ldots, a_{q^2+q+1})$ satisfying $0 < a_i < 1$, for all i. This proves the theorem. $\square$

5. Application

5.1. Description

The origin of designs lies in statistics. Projective planes and projective spaces are a special type of designs (2-designs). Suppose we must test the behaviour of seven different objects when three objects are put together. If the behaviour of two such objects in a group of three is more or less independent of the third one, then we may form seven groups of three in such a way that two given objects are never together in two or more groups. A model to achieve this is given by the Fano plane, where the objects are the points, and the groups are the lines.

We now consider a slightly more complicated test procedure.

Set-up and context: Suppose we have given a family $\mathcal{F}$ of sets of objects. For instance, $\mathcal{F}$ consists of seven sets of 30 pupils, each set characterized by the fact that the pupils excel in a certain ability.

Objective: We must perform tests to see the behaviour of these objects when put together in one environment with objects of different sets. We have to do a careful mixing of the sets, because we should recognize different behaviour for the interaction of different pairs of sets. In our example, we want to test how the abilities develop in mixed courses. But we cannot mix the pupils arbitrarily, we have to do it in systematic way putting together pupils from about three sets. In this way the correlation between the different sets will be better visible.

Strategy: We define a geometry with point set, the sets of objects, and the lines the different classes of objects that we mix together. In our example, we make seven classes with pupils from three sets each time, in such a way that we have combined every two different abilities in some class (this is possible using the Fano plane). We can do this in a uniform way (each class consists of 30 pupils, 10 out of three different sets), but it might be more interesting to have some *dominating* groups. By this we mean that in each class one set could be over-presented, while the others are equally represented. The question is: can we do this, and how can we do this if every set must be dominating in exactly one class?

The answer is given by Theorem 4.6. Indeed, if we denote the number of people of one group present in one test by a fraction between 0 and 1, each test amounts to a fuzzy line and the set of tests corresponds with a fuzzy spread consisting of seven fuzzy lines (if every combination of groups must be tested) or less (if not). Our results in the previous section now show for instance, that certain choices for the dominating classes are impossible. Theorem 4.6 asserts that we can always perform such tests when the groups correspond to the lines of a finite projective plane, and the proof of that theorem even allows to essentially choose the numbers ε_i.

Of course, the numbers ε_i do not have an immediate physical interpretation. But one can replace these numbers by the quotients b_i/a_i. The matrix of the system of equations will be different, and not guaranteed to be non-zero (although it seems that is almost always is), but the computations remain essentially the same. We will illustrate this in the next paragraph.

Also, in practical situation, we might want to use the exact number of objects instead of fractions. The right-hand side of our system of equations will then show the number of objects in each class.

5.2. Numerical example

Denote the seven sets of 30 pupils by $A_0, A_1, A_2, \ldots, A_6$. We must form seven classes and each class contains pupils of three sets. In each class one set of pupils is over-represented. We can use the Fano plane to form classes C_i, $i \in \{0, 1, 2, 3, 4, 5, 6\}$, and C_i contains pupils of sets A_i, A_{i+1} and A_{i+3}, where we read the subscripts modulo 7 (and we recognize the lines of the Fano plane in the subscripts). It is convenient to let the pupils of set A_i be over-represented in class C_i. Now suppose that we want, as an example, that the pupils of A_0 are not over-represented in their class, that the pupils of A_1 and A_2 are twice in numbers compared to the pupils of the other sets in classes C_1 and C_2, respectively, that there are 3 times as much pupils of sets A_3 and A_4 as there are pupils of any other set in classes C_3 and C_4, respectively, and that there are 4 times as much pupils of sets A_5 and A_6 as there are pupils of any other set in classes C_5 and C_6, respectively. In other words, if b_i denotes the number of pupils of set A_i in class C_i, and if a_i denotes the number of pupils of the respective sets A_{i+1} and A_{i+3} in C_i, then $b_0 = a_0$, $b_1 = 2a_1$, $b_2 = 2a_2$, $b_3 = 3a_3$, $b_4 = 3a_4$, $b_5 = 4a_5$ and $b_6 = 4a_6$. Since every set contains exactly 30 pupils, we obtain the following equations:

$$\begin{aligned}
&b_0 + a_6 + a_4 = 30,\\
&b_1 + a_0 + a_5 = 30,\\
&b_2 + a_1 + a_6 = 30,\\
&b_3 + a_2 + a_0 = 30,\\
&b_4 + a_3 + a_1 = 30,\\
&b_5 + a_4 + a_2 = 30,\\
&b_6 + a_5 + a_3 = 30,
\end{aligned}$$

which can be written as

$$\begin{aligned}
&a_0 + a_6 + a_4 = 30,\\
&2a_1 + a_0 + a_5 = 30,\\
&2a_2 + a_1 + a_6 = 30,\\
&3a_3 + a_2 + a_0 = 30,\\
&3a_4 + a_3 + a_1 = 30,\\
&4a_5 + a_4 + a_2 = 30,\\
&4a_6 + a_5 + a_3 = 30.
\end{aligned}$$

In order to obtain natural numbers (whole people!), we can solve the last system of equations, choose a approximate integer solution and calculate the numbers b_i from the first system of equations. In this way, we do not even have to be careful about our choices for the a_i, since it always results in integer solutions for b_i that are exact. But some choices may approximate better the boundary conditions. Let us illustrate this.

In the above example, an approximate integer solution is $(a_0, a_1 \ldots, a_6) = (16, 6, 9, 2, 7, 3, 7)$. In fact, in the exact solution, we have $a_4 = 7.44$, and we can approximate this by either 7 or 8. So, another approximation would be $(a'_0, a'_1, \ldots, a'_6) = (16, 6, 9, 2, 8, 3, 7)$. For the corresponding b_i's, we obtain easily

$$(b_0, b_1, \ldots, b_6) = (16, 11, 17, 5, 22, 14, 25)$$

and

$$(b'_0, b'_1, \ldots, b'_6) = (16, 11, 17, 5, 22, 13, 25).$$

The second solution seems slightly better since 13 is closer to 4 times 3 than 14 (looking at the boundary condition $b_5 = 4a_5$), and for the condition $b_4 = 3a_4$, it does not make so much difference (22 being approximately either 3 times 7 or 3 times 8). Anyway, the obtained solution seems to be satisfactory. There is a small class, namely C_3, only consisting of nine pupils. But this is due to our boundary conditions, which were in fact a bit extreme.

5.3. Final remarks

Other questions might be in order. For instance, for budget reasons, we might be interested in the least possible number of classes. Then Proposition 4.1 tells us that 4 is a lower bound, for the Fano case. In other projective planes, similar bounds can be deduced (using the theory of *blocking sets*, but we will not pursue this idea here further).

With this application in mind, we think fuzzy spreads could be of use and we hope the present paper gives a foundation to study them.

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