Moufang lines defined by (generalized) Suzuki groups

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Abstract

We show that the automorphism group of a geometry defined by the generalized Suzuki groups is contained in the automorphism group of the corresponding Suzuki group. This shows that the study of these groups is equivalent with the study of those geometries. This completes, for the Suzuki groups as split BN-pairs of rank 1, a programme set up by Jacques Tits some years ago. We also provide a similar result for the generalized Suzuki-Tits inversive planes related to these groups.

1 Introduction

Technically, a building of rank one is just a set, endowed with all pairs of elements (which form the set of apartments). However, the buildings of rank one arising from higher rank (spherical or Moufang) buildings have a richer structure, induced by the larger rank building they are sitting in. Indeed, the Moufang condition satisfied by the larger building induces a particularly nice permutation group in these rank 1 buildings. As a standard example we mention the projective line over a field k, where every ordered quadruple gives a unique field elements (the cross-ratio), uniquely determined by the action of the group $\mathrm{PGL}_2(k)$ on the line. The presence and action of the unipotent subgroups allows one to speak here about a *Moufang line*. More generally, we will define a Moufang line related to any split BN-pair of rank one. Every algebraic group of relative rank one gives rise to a Moufang line, but those with root groups of nilpotency class two also give rise to an *additional* geometric structure on that Moufang line, according to Tits [5], and we will call the resulting geometry a *Moufang building of rank one*. Tits then asked whether this additional structure is rich enough to recover the algebraic group. More precisely, is the automorphism group of this geometric structure contained in the automorphism group of the corresponding algebraic group? Tits himself answered positively to that question for

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some classes of algebraic groups ($^2\mathrm{A}_2$, E_8). In the present paper we introduce a slightly more general notion of Moufang building of rank one, and we answer Tits' question in the case of (generalized) Suzuki groups.

The *Suzuki groups* were discovered by Suzuki [3]. In [4], Tits gave a geometric construction of these groups, with an extension to the infinite case, where the ground field (of characteristic 2) does not even have to be perfect (but it only has to admit a *Tits endomorphism*, see below). This construction uses certain Moufang generalized quadrangles of mixed type (called *Suzuki quadrangles* in [6]) defined over fields of characteristic 2. The Suzuki groups arise as (simple subgroups of the) centralizers of polarities in these quadrangles. In the non-perfect case, such quadrangles can have self-polar subquadrangles defined over certain vector spaces. This also gives rise to simple groups, which are natural generalizations of Suzuki groups. We call these *generalized Suzuki groups*. The results of the present paper are valid for generalized Suzuki groups.

The Moufang buildings of rank one corresponding to the Suzuki groups over perfect fields that we will define are in fact *inversive planes*, i.e., point-block geometries with the property that for every point x, the points distinct from x and the blocks through x (the so-called *derivation at x*) form an affine plane (and consequently every triplet of points is contained in a unique block). The Moufang buildings of rank one corresponding to the Suzuki groups over non-perfect fields and to generalized Suzuki groups are not inversive planes; they constitute point-block geometries with the property that the derivation at each point is a *net*, i.e., a point-line geometry with the property that there is at most one line incident with two points, and that every point is incident with exactly one line *parallel to* (i.e., disjoint from or equal to) any given line. Our First Main Result is in fact the Fundamental Theorem of these geometries: we prove that every automorphism of such a geometry is an automorphism of the corresponding (generalized) Suzuki group.

However, the point sets of the above geometries are subsets of the point sets of some 3-dimensional projective spaces $\mathrm{PG}(3,k)$, for a field k. These point sets have the property that no line intersects them in three or more points. Hence every triplet of points determines a unique plane — and a unique plane intersection, which we call a block. This way, we obtain a second point-block geometry related to any generalized Suzuki group. For perfect fields, this second geometry coincides with the above Moufang building of rank one; for imperfect fields and generalized Suzuku groups over imperfect fields, this second geometry has additional blocks compared to the Moufang building of rank one. Our Second Main Result yields a Fundamental Theorem for these second geometries.

One remark about our proofs. Our First Main Result for Suzuki groups (and *not* generalized Suzuki groups) follows more or less directly from a result by Tits [4] for the perfect case, generalized by the author for imperfect field in [6], stating that every collineation of the corresponding Suzuki generalized quadrangle that preserves the point set of a Suzuki-Tits ovoid over a field k with $|k| > 2$, belongs to the corresponding Suzuki group. Hence, one possible strategy would be to first generalize this result to the case of generalized Suzuki groups, and then the First Main Result would follow. However, this strategy does not work for the Second Main Result. So the most economical way seemed to us to avoid the above mentioned results by Tits and the author, and to provide a direct proof, large parts of which also can be used in the proof of the Second Main Result. The more that we get the generalization of the results of Tits and the author to the generalized Suzuki groups for free as a corollary.

Let us now get down to precise definitions and results.

2 Preliminaries and Statement of the Main Results

2.1 Moufang sets and Rank 1 buildings

Let X be a set, and let, for each $x \in X$, U_x be a group acting on X, fixing x. Then we say that $(X, (U_x)_{x\in X})$ is a *Moufang line* (for terminology, see Buekenhout [1]), if

(ML1) for every $x \in X$, U_x acts sharply transitively on $X \setminus \{x\}$, and

(ML2) the set $\{U_x \mid x \in X\}$ is normalized by the group $G^\dagger := \langle U_x \mid x \in X\rangle$.

The group $G^\dagger$ is usually referred to as the *little projective group*. If $G^\dagger$ is sharply 2-transitive, then we say that the Moufang line is *improper*; otherwise it is *proper*.

Now, for some $x \in X$, let $V_x \neq U_x$ be a nontrivial subgroup of U_x such that V_x is a normal subgroup of $G^\dagger_x$. We can then define V_y, $y \in X$ as the conjugate of V_x by an arbitrary element $g \in G^\dagger$ with $x^g = y$. Since $V_x \trianglelefteq G^\dagger_x$, this is well defined. The *Moufang building of rank one defined on X by $(U_x)_{x\in X}$ relative to $(V_x)_{x\in X}$* is the geometry (X, Λ), where Λ is a distinguished set of subsets of X obtained as follows: for each pair $x, y \in X$, the set $\{x\} \cup \{y^v \mid v \in V_x\}$ belongs to Λ.

We are especially interested in Moufang buildings of rank one defined on proper Moufang lines. Defining an automorphism of (X, Λ) as a permutation of X inducing a permutation of Λ, a fundamental question now is

($\bigstar$) Is $\mathrm{Aut}(X, \Lambda) \leq \mathrm{Aut}(G^\dagger)$?

A positive answer means that the study of the rank one Moufang building is essentially equivalent with the study of the corresponding group in that $\mathrm{Aut}(X, \Lambda)$ is then equal to the subgroup of $\mathrm{Aut}(G^\dagger)$ that preserves the set $\{U_x \mid x \in X\}$. This subgroup is referred to as the automorphism group of the corresponding Moufang set and denoted $\mathrm{Aut}(X, (U_x)_{x\in X})$.

2.2 Suzuki-Tits buildings of rank one and the First Main Result

The following description is based on Section 7.6 of [6]. Let k be a field with characteristic 2, and suppose that k admits a *Tits endomorphism* $\theta : x \mapsto x^\theta$; hence $(x^\theta)^\theta = x^2$ (but we do not necessarily have that θ is surjective). Let k^θ denote the image of k under θ. Let L be a vector space over k^θ contained in k and such that $k^\theta \subseteq L$ (note that this implies that $L \setminus \{0\}$ is closed under taking multiplicative inverses as $\ell^{-1} = (\ell^{-2})\ell$ and $\ell^{-2} \in k^2 \subseteq k^\theta$).

We also assume that L generates k as a ring. We now define the *Suzuki-Tits Moufang line* as follows.

Let X be the following set of points of $\mathrm{PG}(3,k)$, given with coordinates with respect to some given basis:

$$\begin{aligned} X &= \{k(1,0,0,0)\} \cup \{k(a^{2+\theta} + aa' + a'^{\theta}, 1, a', a) \mid a, a' \in L\}, \\ &= \{k(0,1,0,0)\} \cup \{k(1, a^{2+\theta} + aa' + a'^{\theta}, a, a') \mid a, a' \in L\}. \end{aligned}$$

We set $\infty = k(1,0,0,0)$ and $O = k(0,1,0,0)$. Let $(x,x')_\infty$ be the collineation of $\mathrm{PG}(3,k)$ determined by

$$k(x_0\ x_1\ x_2\ x_3) \mapsto k(x_0\ x_1\ x_2\ x_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^{2+\theta} + xx' + x'^{\theta} & 1 & x' & x \\ x & 0 & 1 & 0 \\ x^{1+\theta} + x' & 0 & x^{\theta} & 1 \end{pmatrix},$$

and let $(x,x')_O$ be the collineation of $\mathrm{PG}(3,k)$ determined by

$$k(x_0\ x_1\ x_2\ x_3) \mapsto k(x_0\ x_1\ x_2\ x_3) \begin{pmatrix} 1 & x^{2+\theta} + xx' + x'^{\theta} & x & x' \\ 0 & 1 & 0 & 0 \\ 0 & x^{1+\theta} + x' & 1 & x^{\theta} \\ 0 & x & 0 & 1 \end{pmatrix}.$$

Define the groups

$$U_\infty = \{(x,x')_\infty \mid x, x' \in L\} \text{ and } U_O = \{(x,x')_O \mid x, x' \in L\}.$$

Both groups U_∞ and U_O act on X, as an easy computation shows (for U_O use the second description of X above), and they act sharply transitively on $X \setminus \{k(1,0,0,0)\}$ and $X \setminus \{k(0,1,0,0)\}$, respectively. Moreover, one can check that $(U_O)^{(x,x')_\infty} = (U_\infty)^{(y,y')_O}$, with

$$y = \frac{x'}{x^{2+\theta} + xx' + x'^{\theta}} \text{ and } y' = \frac{x}{x^{2+\theta} + xx' + x'^{\theta}}.$$

It follows easily that X is a Moufang line, which we call a *Suzuki-Tits Moufang line* and denote by $\mathcal{M}(\mathrm{Sz}(k,L,\theta))$. The group $\mathrm{Sz}(k,L,\theta)$ is the (simple) *Suzuki group* generated by U_∞ and U_O. Note that $(x,x')_\infty(y,y')_\infty = (x+y, x'+y'+xy^{\theta})$. Also, we may identify the point $k(x^{2+\theta} + xx' + x'^{\theta}, 1, x', x)$ with the pair (x,x'), and $(1,0,0,0)$ with the symbol (∞). This way, the action of $(a,a')_\infty$ on X is given by

$$\begin{aligned} (a,a')_\infty \ :\ & (\infty) \mapsto (\infty), \\ & (x,x') \mapsto (x+a, x'+a'+xa^{\theta}). \end{aligned}$$

Now define $V_\infty = \{(0,x')_\infty \mid x' \in L\}$, then $V_\infty = [U_\infty, U_\infty] = Z(U_\infty)$. Hence V_∞ is normal in $\mathrm{Sz}(k,L,\theta)_{(\infty)}$ and, following the procedure explained before, we obtain a Moufang building (X,Λ) of rank one, which we call a *Suzuki-Tits Moufang building of rank one.*

In the finite case $k = L$, $|k| = 2^{2e+1}$, and (X, Λ) is the inversive plane corresponding to the Suzuki group $\mathrm{Sz}(2^{2e+1})$.

First Main Result. *Let k be a field with characteristic 2 admitting a Tits endomorphism θ. Let L be a vector space over k^θ contained in k and such that $k^\theta \subseteq L$. We also assume that L generates k as a ring. Let (X, Λ) be the Suzuki-Tits Moufang building of rank one corresponding to $\mathrm{Sz}(k, L, \theta)$, with $|k| > 2$. Then $\mathrm{Aut}(X, \Lambda)$ is generated by $\mathrm{Sz}(k, L, \theta)$ and by the permutations $m_{\ell,\sigma}$, where $\ell \in L$ and $\sigma \in \mathrm{Aut}(k)$ with $\sigma\theta = \theta\sigma$ and with the property that $\ell L = L^\sigma$, with*

$$m_{\ell,\sigma} : X \to X : k(x_0, x_1, x_2, x_3) \mapsto k(\ell^{2+\theta} x_0^\sigma, x_1^\sigma, \ell^{1+\theta} x_2^\sigma, \ell x_3^\sigma).$$

In particular we have $\mathrm{Aut}(X, \Lambda) = \mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$.

Remark. It is clear that the permutation $m_{\ell,\sigma}$ does not belong to $\mathrm{Sz}(k, L, \theta)$ if $\sigma \neq \mathsf{id}$. Also, if $\sigma = 1$, then $m_{\ell,\mathsf{id}}$ does not belong to $\mathrm{Sz}(k, L, \theta)$ whenever $\ell \notin k^\theta$. Indeed, one can show that the stabilizer of (∞) and $(0, 0)$ in $\mathrm{Sz}(k, L, \theta)$ is generated by the permutations $m_{\ell,\mathsf{id}}$ with $\ell = (x^{2+\theta} + xx' + x'^\theta)^\theta$, for some $x, x' \in L$ (this is proved in [4] for $L = k$; for $L \neq k$, one can use [2]. But since we do not need this result, we do not prove it). In particular, if $L = k$, then for every $\ell \in L$, the permutation $m_{\ell,\mathsf{id}}$ belongs to $\mathrm{Aut}(X, \Lambda)$, but only if $\ell \in k^\theta$ it could belong to $\mathrm{Sz}(k, L, \theta)$.

Example. We now show with an example that there exist such permutations $m_{\ell,\sigma}$ (and note that $m_{\ell,\sigma} = m_{1,\sigma} m_{\ell,\mathsf{id}}$) with the property that neither $m_{\ell,\mathsf{id}}$ nor $m_{1,\sigma}$ belongs to $\mathrm{Aut}(X, \Lambda) = \mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$.

Let k be the field $\mathbb{F}_2(X, Y, Z, U)$ of rational functions in 4 variables over the field $\mathbb{F}_2$ of two elements. Then the endomorphism $\theta : k \to k : f(X, Y, Z, U) \mapsto f(Y^2, X, U^2, Z)$ is a Tits endomorphism. Moreover the field automorphism $\sigma : k \to k : f(X, Y, Z, U) \mapsto f(Z, U, X, Y)$ commutes with θ. Put $L = k^\theta + Y \cdot k^\theta + YU \cdot k^\theta$. Then one easily checks that $L \neq L^\theta = YU \cdot L$, and L generates k as a ring.

2.3 Generalized Suzuki-Tits inversive planes and the Second Main Result

In our description of the geometry (X, Λ) above, it is easy to verify that each member of Λ is the intersection with X of a plane in $\mathrm{PG}(3, k)$ (remember that X is defined as a set of points in $\mathrm{PG}(3, k)$!). Also, on the one hand, one checks easily that, if k is perfect, then all nontrivial plane intersections occur in Λ (A plane intersection is *nontrivial* if it contains at least two points). On the other hand, if k is not perfect, then for fixed $\ell \in L$, the set $\{(\infty)\} \cup \{(x, \ell x) : x, \ell x \in L\}$ is a nontrivial plane intersection but does not belong to Λ if $\ell \in L \setminus k^\theta$. This motivates us the define the geometry (X, Ω), where Ω is the set of all nontrivial plane intersections of X in $\mathrm{PG}(3, k)$. If we call the elements of Ω *circles*, then every triplet of points defines a unique circle, and so we obtain a kind of a circle geometry. If $k = L$, then (X, Ω) is an inversive plane; if $k \neq L$, then the derivation at a point is just a linear space, and we call (X, Ω) a *generalized Suzuki-Tits inversive plane.*

Second Main Result. *Let k be a field with characteristic 2 admitting a Tits endomorphism θ. Let L be a vector space over k^θ contained in k and such that $k^\theta \subseteq L$. We also assume that L generates k as a ring. Let (X, Ω) be the generalized Suzuki-Tits inversive plane corresponding to* $\mathrm{Sz}(k, L, \theta)$*, with $|k| > 2$. Then* $\mathrm{Aut}(X, \Omega) = \mathrm{Aut}(X, \Lambda) = \mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$.

2.4 Two consequences

Let k, L, θ and X be as above. We define the following incidence geometry $\mathsf{W}(k, L, \theta) = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, where we call the members of $\mathcal{P}$ *points*, the ones of $\mathcal{L}$ *lines*, and where $\mathtt{I}$ is the incidence relation.

$$\mathcal{P} = \{(\infty)\} \cup \{(a) : a \in L\} \cup \{(m, b) : m, b \in L\} \cup \{(a, l, a') : a, a', l \in L\},$$

$$\mathcal{L} = \{[\infty]\} \cup \{[m] : m \in L\} \cup \{[a, l] : a, l \in L\} \cup \{[m, b, m'] : m, m', b \in L\},$$

and incidence is given by

$$[\infty] \ \mathtt{I} \ (\infty) \ \mathtt{I} \ [m] \ \mathtt{I} \ (m, b) \ \mathtt{I} \ [m, b, a^\theta m + l] \ \mathtt{I} \ (a, l, m^\theta a + b) \ \mathtt{I} \ [a, l] \ \mathtt{I} \ (a) \ \mathtt{I} \ [\infty],$$

for all $a, b, m, l \in L$. Then $\mathsf{W}(k, L, \theta)$ is a *generalized quadrangle*, i.e., a point-line incidence geometry whose incidence graph has diameter 4 and girth 8.

The involution ρ that interchanges round parentheses with square brackets is a duality of the geometry; the set of points incident with their images is given by $X^* = \{(\infty)\} \cup \{(a, a' + aa^\theta, a') : a, a' \in L\}$. This is a so-called *ovoid* of the quadrangle. By Chapters 3 and 7 of [6], one can represent $\mathsf{W}(k, L, \theta)$ in $\mathrm{PG}(3, k)$ in such a way that $X^* = X$ (and in that representation the point (∞) corresponds to $k(1, 0, 0, 0)$, and the point $(a, a^{1+\theta} + a', a')$ corresponds to $k(a^{2+\theta} + aa' + a'^\theta, 1, a', a))$. The group $\mathrm{Sz}(k, L, \theta)$ centralizes ρ. It is straightforward to check that a generic element of Λ is the set of points of $\mathsf{W}(k, L, \theta)$ collinear to an arbitrary point of $\mathsf{W}(k, L, \theta)$ not in X. Also, remember that Ω is the set of plane intersections of X.

Our main results now imply:

Corollary 1 *Every collineation of* $\mathsf{W}(k, L, \theta)$ *that preserves $X^* = X$ centralizes ρ, hence belongs to* $\mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$*. Consequently, every collineation of* $\mathsf{W}(k, L, \theta)$ *that preserves X also preserves X^ρ.*

This corollary, in the form of the last assertion, was proved by Tits in [4] for the case k perfect (and hence automatically equal to L), and for the case $k = L$ (but not necessarily perfect) in Theorem 7.6.10 of [6].

Corollary 2 *Every collineation of* $\mathrm{PG}(3, k)$ *that preserves X belongs to* $\mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$.

For perfect k, this corollary follows from the previous one; for all other cases, this is a new result.

3 Proof of the First Main Result

In this section, we let $\Delta := (X, \Lambda)$ be the Suzuki-Tits Moufang building of rank one corresponding to $\mathrm{Sz}(k, L, \theta)$, with k, L and θ as in Section 2, and with $|k| > 2$.

We first show that each permutation $m_{\ell,\sigma}$, $\ell \in L$ and $\sigma \in \mathrm{Aut}(k)$ with $\sigma\theta = \theta\sigma$ and $L^\sigma = \ell L$, as defined above, belongs to $\mathrm{Aut}(X, \Lambda)$. For this, it suffices to prove that it belongs to $\mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$. This, in turn, is equivalent with proving that it normalizes the groups U_∞ and U_O.

First note that $\ell x^\sigma \in L$, for all $x \in L$. Now, it is an easy exercise to calculate that, for all $a, a' \in L$, one has $(a, a')_\infty^{m_{\ell,\sigma}} = (\ell a^\sigma, \ell^{1+\theta} a'^\sigma)_\infty$, which belongs to U_∞ again. Likewise, a simple calculation reveals $(a, a')_O^{m_{\ell,\sigma}} = (\ell^{-1} a\sigma, \ell^{-1-\theta} a'^\sigma)_O \in U_O$.

We now turn to the converse. We let φ be an arbitrary permutation of X that induces a permutation of Λ. We first investigate the geometric structure of (X, Λ). As above, we view X as the set of pairs $(x, x') \in L \times L$ together with the symbol (∞). Remember that a general block $B \in \Lambda$ is defined as an orbit of $Z(U_{(x,x')})$ together with (x, x') itself, for $(x, x') \in L \times L$, or as an orbit of $Z(U_\infty)$ union $\{(\infty)\}$. In these cases, the points (x, x') and (∞), respectively, are called the *gnarls* of the blocks. It follows easily that there is a unique block with a given gnarl and containing a given point (distinct from the gnarl). Moreover, since the unique block with gnarl (∞) containing $(0, 0)$ is given by $\{(0, x') \mid x' \in L\} \cup \{(\infty)\}$, and the unique block with gnarl $(0, 0)$ containing (∞) is given by $\{(x, 0) \mid x \in L\} \cup \{(\infty)\}$, and since these are clearly distinct, we conclude that the gnarl of a block is unique (use also the doubly transitivity of $\mathrm{Sz}(k, L, \theta)$ on (X, Λ) to see this).

We now consider the derived block geometry $\Delta_\infty := (X \setminus \{(\infty)\}, \Lambda_\infty)$ consisting of the points different from (∞) and the intersections of the blocks of Λ containing (∞) with $X \setminus \{(\infty)\}$. There are two different kinds of blocks in Δ_∞: those coming from blocks of Δ with gnarl (∞) — and we call these *vertical blocks* — and the others — non-vertical blocks. We have the following lemma.

Lemma 3 *The block space Δ_∞ is a net, i.e., for each block B and each point $p \notin B$ there exists a unique block B' containing p and disjoint from B.*

Proof: Denote the block $\{(x, 0) \mid x \in L\}$ by $B_{(0,0)}$. Then, since every non-vertical block has a unique gnarl, and since the group U_∞ acts sharply transitively on $X \setminus \{(\infty)\}$, the map $U_\infty \to \Lambda_\infty : u \mapsto B_{(0,0)}^u$ is injective and surjective onto the subset of non-vertical blocks of Δ_∞. Hence we can define the block $B_{(a,b)}$ as the image of $B_{(0,0)}$ under $(a, b)_\infty \in U_\infty$. We have

$$B_{(a,b)} = \{(x + a, b + xa^\theta) \mid x \in L\},$$

for all $(a, b) \in L \times L$. Clearly, a general vertical block is given by $B_a = \{(a, x) \mid x \in L\}$, with $a \in L$. It is easy to see that the vertical blocks partition the set $X \setminus \{(\infty)\}$. Also, the block B_a meets the block $B_{(b,c)}$ in the point $(a, c + b^\theta(a + b))$. This proves the lemma for B vertical. Now let $B_{(a,b)}$ be an arbitrary non-vertical block, $(a, b) \in L \times L$. Then clearly the set of blocks $\{B_{(a,y)} \mid y \in L\}$, partitions $X \setminus \{(\infty)\}$. Also, if $a \neq c$, then the blocks $B_{(a,b)}$ and $B_{(c,d)}$, with $(c,d) \in L \times L$, intersect in the point

$$\left(\frac{b+d+aa^\theta+cc^\theta}{a^\theta+c^\theta}, \frac{c^\theta b+a^\theta d+(ac)^\theta(a+c)}{a^\theta+c^\theta}\right).$$

This completes the proof of the lemma. $\square$

As standard, we will call a set of blocks of Δ_∞ partitioning the point set a *parallel class of blocks*. The previous lemma implies that every block of Δ_∞ is contained in a unique such parallel class.

In the sequel, we will use the notation $B_{(a,b)}$ and B_a, $a, b \in L$, as introduced in the previous proof. We note that the gnarl of the block $B_{(a,b)}$ is exactly (a,b) and the gnarl of B_a is always (∞).

Note that $\mathrm{Aut}(\Delta)_{(\infty)}$ has at most two orbits on Λ_∞, namely, the set of vertical blocks, and the rest. But it is also easy to see that $\mathrm{Aut}\Delta_\infty$ is transitive on Λ_∞.

Our main aim is to prove that we can recognize the gnarl of each block in Δ. Therefore, it suffices to prove that $\mathrm{Aut}(\Delta)_{(\infty)}$ has exactly two orbits on Λ_∞. Notice that $Z(U_\infty)$ fixes every vertical block and acts sharply transitively on the set of points on any vertical block. Hence the following lemma proves our main aim.

Lemma 4 *No automorphism of Δ fixes (∞) and all blocks of Δ_∞ parallel to $B_{(0,0)}$, acts freely on the points on any such block, and maps $(0,0)$ to $(1,0)$.*

Proof: If some block of Δ_∞ is mapped onto a block of a different parallel class, then the intersection point is fixed, hence contradicting the free action. So all parallel classes are stabilized. This implies that B_0 is mapped onto B_1 and hence $(0,x')$ is mapped onto $(1,x')$, for all $x' \in L$. Let $a \in L$ be arbitrary. The block $B_{(1,a+1)}$ intersects $B_{(0,0)}$ in $(a,0)$, and B_0 in $(0,a)$. The latter is mapped onto $(1,a)$; hence $B_{(1,a+1)}$ is mapped onto $B_{(1,a)}$, which intersects $B_{(0,0)}$ in $(a+1,0)$. We have shown that $(a,0)$ is mapped onto $(a+1,0)$, and so (a,a') is mapped onto $(a+1,a')$. If we compose this mapping with $(1,1)_\infty$, then we obtain the automorphism $\psi : (x,x') \mapsto (x, x+x')$.

From the description in the previous section, it is clear that the map $\zeta : (x,x') \mapsto (x'/N, x/N)$, with $N = x^{2+\theta} + xx' + x'^\theta$, and $(0,0)^\zeta = (\infty)$, $(\infty)^\zeta = (0,0)$, is an automorphism of Δ. It maps the block $B_1 \cup \{(\infty)\} \in \Lambda$ onto the set

$$B' := \{(0,0)\} \cup \left\{\left(\frac{x}{1+x+x^\theta}, \frac{1}{1+x+x^\theta}\right) \mid x \in L\right\}.$$

Hence the set $(B'^\zeta)^\psi$ is also a block containing (∞). The elements of this block different from (∞) are

$$\left(\frac{(1+x+x^\theta)^{1+\theta}(1+x)}{1+x+x^3+x^4+x^\theta+x^{2\theta}+x^{3\theta}+x^{1+\theta}+x^{2+\theta}}, \frac{(1+x+x^\theta)^{1+\theta}x}{1+x+x^3+x^4+x^\theta+x^{2\theta}+x^{3\theta}+x^{1+\theta}+x^{2+\theta}}\right),$$

for x ranging through L. This block of Δ_∞ contains $(1,0)$ and $(0,1)$ (for the values $x=0$ and $x=1$, respectively). So this block is equal to $B_{(1,0)}$. This implies that

$$\frac{(1+x+x^\theta)^{1+\theta}(1+x)}{1+x+x^3+x^4+x^\theta+x^{2\theta}+x^{3\theta}+x^{1+\theta}+x^{2+\theta}}+\frac{(1+x+x^\theta)^{1+\theta}x}{1+x+x^3+x^4+x^\theta+x^{2\theta}+x^{3\theta}+x^{1+\theta}+x^{2+\theta}}=1,$$

for all $x \in L$. After an easy computation one obtains that $x^2+x^4=x^\theta+x^{3\theta}$, for all $x \in L$. This now easily implies, taking the injectivity of θ into account, that $(x+x^2)^\theta = x+x^3$, hence $x(x+x^2)^\theta = x^2+x^4$. Combining this with the equation in the previous sentence we obtain, for all $x \notin \{0,1\}$,

$$x = \left(\frac{x+x^3}{x+x^2}\right)^\theta = (1+x)^\theta.$$

So $x^\theta = 1+x$, implying $x^2=(x^\theta)^\theta = 1+x^\theta = x$, clearly a contradiction, since $|k|>2$.

The lemma is proved. $\square$

Hence we have shown that φ must preserve the gnarls of the blocks of Δ. Since the Suzuki group acts doubly transitively on the points of Δ, we may also assume that φ fixes the points (∞) and $(0,0)$. Consequently, φ fixes the blocks B_0 and $B_{(0,0)}$. It follows that there are two permutations α and β of L such that $(x,y)^\varphi = (x^\alpha, y^\beta)$. Since φ preserves gnarls, it maps the block $B_{(a,b)}$ onto the block $B_{(a^\alpha,b^\beta)}$. Now notice that the point (x,y) is contained in the block $B_{(a,b)}$ if and only if $y=b+a^\theta x$. A standard argument now shows that, for all $a,b,x \in L$,

$$(b+a^\theta x)^\beta = b^\beta + (a^\alpha)^\theta x^\alpha.$$

Put $\ell = 1^\alpha$, then setting $b=0$ and $a=1$ in the above, we see that $x^\beta = \ell^\theta x^\alpha$, for all $x \in L$. We now define the bijection $\sigma : L \to \ell^{-1}L : y \mapsto y^\sigma = \ell^{-1}y^\alpha$ and note that $1^\sigma = 1$. Plugging in these identities in the above equation yields

$$(b+a^\theta x)^\sigma = b^\sigma + (a^\sigma)^\theta x^\sigma,$$

for all $a,b,x \in L$. Putting $1=a$, we see that σ is additive; putting $b=0$ and $x=1$, we see that σ commutes with θ. Furthermore, it follows easily that $(xy)^\sigma = x^\sigma y^\sigma$ for $x \in L^\theta$ and $y \in k^\theta$. Since L generates k as a ring, and hence L^θ generates k^θ, this implies that σ stabilizes k^θ and is in fact an automorphism of k^θ. We may view σ as an automorphism of k by defining $x^\sigma = y$ if and only if $(x^\theta)^\sigma = t^\theta$ (and this is well defined and agrees on L). Now the action of φ on a point (x,y) is given by $(x,y)^\varphi = (\ell x^\sigma, \ell\ell^\theta y^\sigma)$, for all $x,y \in L$.

The proof of our First Main Result is complete. $\square$

4 Proof of the Second Main Result

In this section, we let $\Gamma := (X,\Omega)$ be the generalized Suzuki-Tits inversive plane corresponding to $\mathrm{Sz}(k,L,\theta)$, with $|k|>2$. Using $\Lambda \subseteq \Omega$, our Second Main Result will be proved when we show that every automorphism of Γ stabilizes Λ.

Similarly as before, one can define the derived block geometry $\Gamma_\infty = (X \setminus \{(\infty)\}, \Omega_\infty)$, which is a subgeometry of Δ_∞ (same point set, but one block set is contained in the other). In fact, the set $X \setminus \{(\infty)\}$ is given by the pairs (a, b) of elements $a, b \in L$, and so this set is a subset of the affine plane $\mathrm{AG}(2, k)$ over k. It is easy to see that the elements of Ω_∞ are the nontrivial intersections of lines of $\mathrm{AG}(2, k)$ with the point set of Γ_∞ (and we call such a nontrivial intersection a *trace*; "nontrivial" means that the intersection contains at least two elements). Our main aim is to show that we can recognize pairs of blocks of Ω_∞ that are traces of two parallel lines of $\mathrm{AG}(2, k)$. Note that there could be several parallel blocks through a point to a given block B; of course only one is the trace of a line parallel in $\mathrm{AG}(2, k)$ to the line with trace B.

We will also consider the derived geometries in points p different from (∞); these will be denoted by Γ_p.

We let k' be the set of elements m of k such that $mL = L$. Clearly k' is a subfield of k containing k^θ and being contained in L. Also, the mapping $h_m : (x, y) \mapsto (mx, my)$ is an automorphism of Γ_∞, if $m \in k'$. The set of all such mappings is a group $G_{[(0,0)]}$ of automorphisms of Γ_∞ fixing every block through $(0, 0)$.

Lemma 5 *The group $G_{[(0,0)]}$ is the set of all automorphisms of Γ_∞ that fix all blocks through $(0, 0)$.*

Proof: Suppose g is a nontrivial automorphism of Γ_∞ fixing all blocks through $(0, 0)$. We first claim that g does not fix any point on the block $Y := \{(0, y) : y \in L\}$. Indeed, suppose by way of contradiction that g fixes some point $(0, b)$, with $b \in L \setminus \{0\}$. Then g preserves the set of blocks of Γ_∞ that are incident with $(0, b)$ and that do not intersect the line $X := \{(x, 0) : x \in L\}$. One of these blocks is the block $X_b := \{(x, b) : x \in L\}$; any other block is the trace B_A of a line of $\mathrm{AG}(2, k)$ incident with $(0, b)$ and some point $(A, 0)$, with $A \in k \setminus L$. Let B be the block through $(0, 0)$ intersecting X_b in (b, b). Hence B is the trace of the line of $\mathrm{AG}(2, k)$ with equation $x + y = 0$. Let $A \in k \setminus L$ be such that it defines a (nontrivial) trace B_A. Then B_A is the trace of the line with equation $bx + Ay = Ab$. If g did not fix X_b, then, for some A, the system of equations

$$\begin{cases} x + y = 0 \\ bx + Ay = Ab \end{cases}$$

would have a solution (x_0, y_0) in $L \times L$ (the image of (b, b) under g). We easily calculate that $y_0 = Ab/(b + A)$. Since L is closed under multiplication with squares, this would imply that $Ab(b + A) \in L$, hence $b^2 A \in L$, so $A \in L$, a contradiction. Consequently g fixes X_b *pointwise*. A similar argument with the point (b, b), which is now fixed under g, and the block X reveals that the block $Y_b := \{(b, y) : y \in L\}$ must be fixed under g, and so must be fixed pointwise. Applying the mapping $(x, y) \mapsto (x + b, y + b)$, we obtain an automorphism h that fixes all points on both X and Y. Let (a, b) be an arbitrary point of Γ_∞. Then the traces of the lines with equations $x + y = a + b$ and $mx + y = ma + b$, with m an arbitrary element of k' distinct from 0 and 1, are blocks of Γ_∞ incident with (a, b). But these traces contain the points $(a + b, 0)$, $(0, a + b)$, and $(a + m^{-1}b, 0)$, $(0, ma + b)$, respectively, of Γ_∞, which are all fixed under h. Hence h fixes these blocks and the unique intersection point (a, b). This shows that h is trivial and hence so is g. Our claim is proved.

Now we claim that g maps the point $(0, 1)$ onto some point $(0, b)$, with $b \in k'$. Indeed, suppose not, then $(0, 1)$ is mapped onto some point $(0, b)$ with $b \in L \setminus k'$. First suppose that g maps X_1 onto X_b. Let $b' \in L$ be such that $bb' \notin L$ (such b' exists since $b \notin k'$). Then the image of $(b', 1)$ under g is the intersection of the blocks which are traces of the lines with equations $x + b'y = 0$ and $y = b$. But these lines intersect in the point (bb', b), which is not a point of Γ_∞. Consequently, the corresponding blocks do not meet and we have reached a contradiction, showing that g must map X_1 onto some block B_A (notation as above), with $A \in k \setminus L$. But then, similarly, the point $(1, 1)$ has got no image under g (since the lines with equations $x + y = 0$ and $bx + Ay = Ab$ define parallel blocks as shown above) and the claim is proved.

Hence g maps $(0, 1)$ onto some point $(0, m)$, with $m \in k'$. Now the mapping gh_m^{-1} preserves all lines through $(0, 0)$ but fixes $(0, 1)$, hence is the identity by our first claim. This shows $g = h_m$ and the lemma is proved. □

Now we define two *strongly parallel* blocks as two block that can be mapped onto each other by some automorphism fixing all blocks through some point of Γ_∞.

Lemma 6 *Any automorphism g of Γ_∞ maps strongly parallel blocks onto strongly parallel blocks.*

Proof: Let B and B' be strongly parallel. Without loss of generality we may assume that, for some $m \in k'$, B' is the image of B under h_m. Then the mapping $h_m^g = g^{-1}h_m g$ maps B^g onto B'^g. Since h_m^g fixes all lines through $(0, 0)^g$, the lemma follows. □

Lemma 7 *Two blocks are strongly parallel precisely when they are the traces of parallel lines of* $\mathrm{AG}(2, k)$.

Proof: It is clear that strongly parallel blocks are traces of parallel lines. It is the converse that requires some proof. Suppose two blocks B and B' are traces of parallel lines. Pick some arbitrary points (a, b) and (c, d) on B and B', respectively. Also, choose arbitrarily $m \in k' \setminus \{0, 1\}$ (which is always possible). Applying the translation $t : (x, y) \mapsto (x + (m + 1)a + mc, y + (m + 1)b + md)$ we obtain two blocks B^t and B'^t that are traces of parallel lines. The automorphism h_n, with $n = 1 + m^{-1}$, maps the point $(a, b)^t$ onto $(c, d)^t$, and preserves parallelism in $\mathrm{AG}(2, k)$, hence maps B^t onto B'^t. It follows that B^t and B'^t are strongly parallel, and so are B and B' by Lemma 6.

The lemma is proved. □

The next lemma finishes the proof of the Second Main Result.

Lemma 8 *Every automorphism of Γ stabilizes Λ.*

Proof: We already know that every member B of Λ has the property that a point $p \in B$ exists such that the group of automorphisms of (X, Ω) fixing all blocks through p that are strongly parallel to $B \setminus \{p\}$ in Γ_p, and such that all strong parallel classes in Γ_p are preserved, acts transitively on $B \setminus \{p\}$. Indeed, we can take for p the gnarl of B and then the group $Z(U_p)$ does the job. The proof of Lemma 4 tells us that p is unique. Hence the lemma will be proved once we have shown that for a member of $\Omega \setminus \Lambda$ this property does not hold. Equivalently, by transitivity, it suffices to show that no automorphism of (X, Ω) fixing (∞) induces a nontrivial automorphism of Γ_∞ that preserves all strong parallel classes and fixes all lines of one particular strong parallel class corresponding to a member B of $\Omega_\infty \setminus \Lambda_\infty$. We will actually only assume that B is not a vertical block (with terminology of the previous section; i.e., we assume that, if $B \in \Lambda_\infty$, then the gnarl of $B \cup \{(\infty)\}$ is not (∞)).

Without loss of generality, we may assume that B contains the point $(0, 0)$ and some point (a, b), with $a, b \in L$ and $a \neq 0$, and that an automorphism φ of Γ fixes (∞), stabilizes all strong parallel classes in Γ_∞, fixes all blocks of Γ_∞ strongly parallel to B, and maps $(0, 0)$ to (a, b). As in the first paragraph of the proof of Lemma 4 above, one calculates easily that φ maps (x, x') to $(x + a, x' + b)$. Hence it is straightforward to verify that $m_{a,\mathsf{id}}(a, b)_\infty \varphi m_{a,\mathsf{id}}^{-1}$ maps (x, x') to $(x, x+x')$. Now the rest of the proof of Lemma 4 applies, leading to a contradiction. $\square$

Now our Second Main Result follows directly from the First Main Result.

5 Proof of the consequences

First we note that any automorphism of $\mathsf{W}(k, L, \theta)$ that fixes all points of X^* necessarily is the identity. Indeed, as remarked in Subsection 2.4, every point of $\mathsf{W}(k, L, \theta)$ not in X^* is collinear precisely to the set of points of a block of (X, Λ) (identifying X^* with X again), and no two points are related to the same block, as this would mean that this block is, as a set of $\mathrm{PG}(3, k)$, contained in a line of $\mathrm{PG}(3, k)$, a contradiction.

Another immediate consequence of that remark is that every collineation of $\mathsf{W}(k, L, \theta)$ preserving $X^* = X$ preserves the set Λ. Hence every such collineation belongs to $\mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$.

We now claim that the mapping $m_{t,\sigma}$ induces a collineation of $\mathsf{W}(k, L, \theta)$ centralizing ρ (with $\sigma\theta = \theta\sigma$ and $tL = L^\theta$). Using the relation between the coordinates of points of X in $\mathrm{PG}(3, k)$ and their representation as points of $\mathsf{W}(k, L, \theta)$, we see that the mapping

$$\left\{ \begin{array}{ccc} (a, \ell, a') & \mapsto & (ta^\sigma, t^{1+\theta}\ell^\sigma, t^{1+\theta}a'^\sigma), \\ [m, b, m'] & \mapsto & [tm^\sigma, t^{1+\theta}b^\sigma, t^{1+\theta}m'^\sigma] \end{array} \right.$$

induces $m_{t,\sigma}$ in X and defines a collineation, say φ, of $\mathsf{W}(k, L, \theta)$. But since the prescription of the images of the lines is formally the same as that of the points, it follows immediately that it preserves the set $(X^*)^\rho$ (because (a, ℓ, a') belongs to X^* if and only if $[a, \ell, a']$ belongs to $X^{*\rho}$). Since it now follows easily that $p^{\varphi\rho} = p^{\rho\varphi}$ for every point p

of X^*, we conclude that $\varphi\rho\varphi^{-1}\rho^{-1}$ is the identity everywhere (remember a collineation of $\mathsf{W}(k, L, \theta)$ preserving X^* is determined by its action on X^* as we showed above). Similarly all elements of U_∞ and of U_O induce collineations of $\mathsf{W}(k, L, \theta)$ centralizing ρ.

Corollary 1 is proved.

For Corollary 2, we note that every collineation of $\mathrm{PG}(3, k)$ preserving X also preserves Δ, and that every member of $\mathrm{Aut}(\mathcal{M}(\mathrm{Sz}(k, L, \theta)))$ acts as a projective (semi-linear) transformation on X by the very definitions of U_∞, U_O and $m_{\ell,\sigma}$, see above. Hence all we have to show is that any collineation of $\mathrm{PG}(3, k)$ that fixes all elements of X is the identity. This can be shown using the theory of generalized quadrangles, but a direct proof goes as follows. The set X contains the points $k(1, 0, 0, 0)$, $k(0, 1, 0, 0)$, $k(1, 1, 1, 1)$, $k(1, 1, 1, 0)$ and $k(1, 1, 0, 1)$. We now view these points as vector lines in a 4-dimensional vector space over k. It is an elementary exercise to verify that any semi-linear transformation of that vector space preserving these five vector lines must have a scalar matrix, say with $c \in k$ on the diagonal, and with some companion field automorphism σ. Hence, for any $a, a', a'' \in k$, the vector $(a'', 1, a', a)$ is mapped onto $(ca''^\sigma, c, ca'^\sigma, ca^\sigma)$. For $a'' = a^{2+\theta} + aa' + a'^\theta$, the latter must be proportional to the former, and we conclude that $a^\sigma = a$ for all $a \in L$. Hence, since L generates k as a ring, σ must be the identity and Corollary 2 is proved.

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