Common point reguli of different generalized hexagons on $\mathsf{Q}(6, q)$

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Abstract

In this paper, we consider any two split Cayley generalized hexagons represented on the parabolic quadric $\mathsf{Q}(6, q)$ and determine their common point reguli. As an application of our results we investigate which 1-systems of $\mathsf{Q}(6, 3)$ that are a derivation of the exceptional spread of $\mathsf{H}(3)$, see [4], are a spread of some hexagon on this quadric.

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1 Introduction

From [6] we know that the intersection of the line sets of two generalized hexagons $\Gamma \cong \mathsf{H}(q)$ and $\Gamma' \cong \mathsf{H}(q)$ on the same quadric $\mathsf{Q}(6, q)$ is a dual ovoidal subspace in both these hexagons.

These dual ovoidal subspaces were introduced by Brouns and Van Maldeghem in [1] in order to characterize the finite generalized hexagon $\mathsf{H}(q)$ by means of certain regularity conditions. It follows from [1] that a dual ovoidal subspace, and hence the line intersection of two split Cayley generalized hexagons naturally embedded in the same $\mathsf{Q}(6, q)$, is either the set of lines at distance at most 3 from a given point, or the set of lines of an ideal non-thick subhexagon, or a distance-3 spread.

From [6] we know that, given a weak subhexagon of order $(1, q)$, a distance-3 spread, and a line set at distance at most 3 from a given point, there are respectively $q - 1$, $q + 1$ and q split Cayley generalized hexagons naturally embedded in $\mathsf{Q}(6, q)$ that contain this line set as a subset.

In [5] the authors showed that for a given split Cayley generalized hexagon naturally embedded in $\mathsf{Q}(6, q)$, q odd, there exists a suitable choice of a second hexagon

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such that their common point reguli define an incidence structure which is either a subdesign of ($q \equiv 2 \mod 3$) or isomorphic to ($q \not\equiv 2 \mod 3$) the Hölz design.

In the present paper we study, inspired by the results of [6] and [5], the common point reguli of two split Cayley generalized hexagons naturally embedded in the same $\mathsf{Q}(6,q)$.

The motivation for this study is two-fold. Firstly, the geometric information can be used in some specific situation to prove other results, for instance on spreads. We demonstrate this in the present paper by an application to the exceptional spreads of $\mathsf{H}(3)$. We determine by hand all isomorphism classes of 1-systems of $\mathsf{Q}(6,3)$ obtained by a derivation of this exceptional spread.

Secondly, the action of the stabilizer of a dual ovoidal subspace $\mathcal{S}$ of some $\mathsf{H}(q)$ on $\mathsf{Q}(6,q)$ inside the full group of collineations of $\mathsf{Q}(6,q)$ does not always act primitively on the set of generalized hexagons isomorphic to $\mathsf{H}(q)$ naturally embedded on $\mathsf{Q}(6,q)$ and containing $\mathcal{S}$. Indeed, in case $\mathcal{S}$ is related to a non-thick ideal subhexagon or a distance-3 spread, this stabilizer is roughly a dihedral group in its natural action, and so the generalized hexagons through $\mathcal{S}$ are paired up in a group-theoretical way. This pairing cannot be explained geometrically by looking at the intersections of line sets, but it can be recovered by considering the common point reguli. Hence, our geometric study provides a finer subdivision which explains the action of certain subgroups.

2 Preliminaries

2.1 Generalized hexagons and the split Cayley hexagon

A *generalized hexagon* Γ (of order (s,t)) is a point-line geometry the incidence graph of which has diameter 6 and girth 12 (and every line is incident with $s+1$ points; every point incident with $t+1$ lines). Note that, if $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set of Γ, then the *incidence graph* is the (bipartite) graph with set of vertices $\mathcal{P} \cup \mathcal{L}$ and adjacency given by incidence. The definition implies that, given any two elements a,b of $\mathcal{P} \cup \mathcal{L}$, either these elements are at distance 6 from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path from a to b. For two points a and b at distance four, there exists a unique point collinear with both, denoted by $a \bowtie b$. Finally, if two elements a and b are at distance $k < 6$, we denote the unique element at distance 1 from a and at distance $k-1$ from b by $\mathrm{proj}_a b$, and call this the *projection of b onto a*.

In this paper we are mostly interested in the split Cayley hexagons $\mathsf{H}(q)$. The standard model H of this hexagon, the construction of which is due to Tits [8], can be defined as follows (see [8]; also [9]). Choose coordinates in the projective space $\mathsf{PG}(6,q)$ in such a way that the points of $\mathsf{Q}(6,q)$ satisfy the equation $X_0X_4+X_1X_5+X_2X_6 = X_3^2$, and let the points of H be all points of $\mathsf{Q}(6,q)$. The lines of H are the lines on $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. A natural embedding of $\mathsf{H}(q)$ arises from the standard model by an automorphism of the full automorphism group of $\mathsf{Q}(6,q)$.

If q is even, the polar space $\mathsf{Q}(6,q)$ is isomorphic to the symplectic polar space $\mathsf{W}(5,q)$ (obtained by projection from the nucleus n of $\mathsf{Q}(6,q)$ – that is the intersection of all tangent hyperplanes of $\mathsf{Q}(6,q)$ – onto some hyperplane not containing n). This substantiality results in an embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, where all lines of $\mathsf{H}(q)$ are totally isotropic with respect to a certain symplectic polarity in $\mathsf{PG}(5,q)$ (here $n = (0,0,0,1,0,0,0)$ and choosing the hyperplane with equation $X_3 = 0$, the associated symplectic form is $X_0Y_4 + X_4Y_0 + X_1Y_5 + X_5Y_1 + X_2Y_6 + X_6Y_2$).

The generalized hexagon $\mathsf{H}(q)$ has the following property (see [9], 1.9.17 and 2.4.15). Let x, y be two opposite points and let L, M be two opposite lines at distance 3 from both x, y. All points at distance 3 from both L, M are at distance 3 from all lines at distance 3 from both x, y. Hence we obtain a set $\mathcal{R}(x,y)$ of $q+1$ points every member of which is at distance 3 from any member of a set $\mathcal{R}(L,M)$ of $q+1$ lines. We call $\mathcal{R}(x,y)$ a *point regulus*, and $\mathcal{R}(L,M)$ a *line regulus*. Any regulus is determined by two of its elements. The two above reguli are said to be *complementary*, i.e. every element of one regulus is at distance 3 from every element of the other regulus. Every regulus has a unique complementary regulus.

Now consider again our model H. We will call a line of the quadric $\mathsf{Q}(6,q)$ which does not belong to the hexagon H an *ideal line*. If L is an ideal line, then there is a unique point x collinear to every point of L. This point will be called the *focus* of L. On the quadric $\mathsf{Q}(6,q)$, every line regulus constitutes a hyperbolic quadric isomorphic to $\mathsf{Q}^+(3,q)$. Hence there is a unique *opposite regulus*, which is a set of $q+1$ ideal lines that intersect every line of the given regulus in a unique point. If q is odd, the quadric $\mathsf{Q}(6,q)$ is associated with a unique non-degenerate polarity ρ of $\mathsf{PG}(6,q)$ and the image under the polarity of the 3-space generated by a line regulus is a plane which meets $\mathsf{Q}(6,q)$ exactly in the complementary point regulus. Point reguli of H are thus simply (some) conics on $\mathsf{Q}(6,q)$. The plane in which all points of a point regulus $\mathcal{R}$ are contained will be referred to as the *regulus plane* $\alpha_{\mathcal{R}}$.

We now introduce some more notation and terminology.

Let $\mathcal{H}$ be a hyperplane in $\mathsf{PG}(6,q)$. Then exactly one of the following cases occurs.

(Tan) The points of $\mathsf{H}(q)$ in $\mathcal{H}$ are the points not opposite a given point x of $\mathsf{H}(q)$; in fact, $\mathcal{H}$ is the *tangent* hyperplane of $\mathsf{Q}(6,q)$ at x. We shall denote this hyperplane as $T_x\mathsf{Q}(6,q)$

(Sub) The lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of a subhexagon of $\mathsf{H}(q)$ of order $(1,q)$, the points of which are those points of $\mathsf{H}(q)$ that are incident with exactly $q+1$ lines of $\mathsf{H}(q)$ lying in $\mathcal{H}$. This subhexagon is uniquely determined by any two opposite points x, y it contains and will be denoted by $\Gamma(x,y)$. It contains exactly $2(q^2+q+1)$ points and if collinearity is called adjacency, then it can be viewed as the incidence graph of the Desarguesian projective plane $\mathsf{PG}(2,q)$

of order q. The lines of $\Gamma(x,y)$ can be identified with the incident point-line pairs of that projective plane. We denote $\Gamma(x,y)$ by $2\mathsf{PG}(2,q)$ and call it the *double* of $\mathsf{PG}(2,q)$. The q^2+q+1 points of $\Gamma(x,y)$ belonging to the same type of elements of $\mathsf{PG}(2,q)$, points or lines, are the points of a projective plane in $\mathsf{PG}(6,q)$. Hence $\mathcal{H} \cap \mathsf{Q}(6,q)$ contains two projective planes Π^+ and Π^-, the points of which are precisely the points of $\Gamma(x,y)$, and which we call the *hexagon twin planes of* $\mathcal{H}$. In this case, we call $\mathcal{H}$ a *hyperbolic* hyperplane. In fact, a hyperbolic hyperplane is a hyperplane that intersects $\mathsf{Q}(6,q)$ in a non-degenerate hyperbolic quadric.

(Spr) The lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of a distance-3 spread, called a *Hermitian* or *classical* distance-3 spread of $\mathsf{H}(q)$. In this case, we call $\mathcal{H}$ an *elliptic* hyperplane (as it intersects $\mathsf{Q}(6,q)$ in an elliptic quadric).

An *ovoidal subspace* in a generalized hexagon is a set of points $\mathcal{O}$ with the property that every point of that hexagon outside $\mathcal{O}$ is collinear with exactly one point of $\mathcal{O}$. Dually, one defines a *dual ovoidal subspace*.

An *ovoid*, short for distance-3 ovoid, of $\mathsf{H}(q)$ is a set of q^3+1 opposite points (Proposition 7.2.3 in [9]). Dually one defines a *spread*.

2.2 m-Systems

The notion of an *m-systems* on finite quadrics (and, more generally, on finite polar spaces) has been introduced by Shult and Thas [7]. The theory of m-systems has a lot of applications. Here, we only need the notion of a 1-system of $\mathsf{Q}(6,q)$. For the general definition, motivation and applications, we refer to [7].

A *1-system* $\mathcal{M}$ of $\mathsf{Q}(6,q)$ is a set of q^3+1 lines such that any plane of $\mathsf{Q}(6,q)$ containing an element of $\mathcal{M}$ does not intersect any other element of $\mathcal{M}$.

It is well known and easy to verify that, if a 1-system $\mathcal{M}$ contains a regulus $\mathcal{R}$ of lines (a set of $q+1$ lines of a hyperbolic quadric $\mathsf{Q}^+(3,q)$ entirely contained in $\mathsf{Q}(6,q)$), then we can *derive* $\mathcal{M}$ at that regulus — namely, we replace all elements of $\mathcal{R}$ by the elements of the opposite regulus (the other set of $q+1$ generators of $\mathsf{Q}^+(3,q)$) — and obtain a 1-system. If we do so at a number of disjoint reguli, then we call the obtained 1-system a *derivation* of the original one.

2.3 Main Result

In this paper we shall prove the following theorem:

Main Result. *Let H_1 and H_2 be two models of $\mathsf{H}(q)$ isomorphic to the model H as defined above. Denote by S, respectively Ω, the set of common lines, respectively point reguli of H_1 and H_2. For q odd, we have one of following situations:*

(i) S is the set of lines at distance at most 3 from a given point and $|\Omega| = q^3$.

(ii) S is the set of lines of an ideal subhexagon and $|\Omega| = q^2(q^2+q+1)$ or $|\Omega| = q^3(q^2+q+1)$.

(iii) S is a distance-3 spread and $|\Omega| = q^2(q^2-q+1)$ or $|\Omega| = q^2(q^2-q+1)(q+2)$.

Furthermore, in both situations (ii) and (iii) there exists, given H_1 and S, a unique hexagon H_2 such that Ω contains the maximal number of point reguli.

If, on the other hand, q is even then H_1 and H_2 share all their point reguli.

In the next section we determine the common point reguli of two distinct split Cayley hexagons embedded into the parabolic quadric $\mathsf{Q}(6,q)$ and prove the main result. In Section 4 we give an application of this result.

3 Proof of the Main Result

Let us start by proving that for even q any two hexagons embedded on the parabolic quadric $\mathsf{Q}(6,q)$ share all point reguli. This is equivalent with saying that in $\mathsf{W}(5,q)$ all point reguli are the same. In this representation of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$ a point regulus is the perp of a non-degenerate 3-space determined by the associated line regulus. Hence point reguli are lines only dependent on $\mathsf{W}(5,q)$.

From now on we shall be working with odd q.

Let H_1 and H_2 be two models of $\mathsf{H}(q)$ as described in Section 2.1 and choose coordinates of $\mathsf{Q}(6,q)$ such that H_1 is given in exactly the coordinates as in Section 2.1. Denote by S the intersection of the line sets and by Ω the set of all common point reguli of these two hexagons. From [6] we know that the subspace generated by S, say Π_S, is a hyperplane of $\mathsf{PG}(6,q)$.

Remember from the introduction that S has one of three types, say 0, + or $-$, corresponding to S being a set of lines at distance at most 3 from a given point, of an ideal non-thick subhexagon or of a distance-3 spread, respectively.

Lemma 1 *If $\mathcal{R}$ is an element of Ω, then either*

(a) the complement of $\mathcal{R}$ belongs to S or

(b) the complement of $\mathcal{R}$ in H_1 is opposite the one in H_2. Furthermore, every point of $\mathcal{R}$ is incident with a unique line of S.

Hence Ω is the union of two sets Ω_1 and Ω_2, which correspond to the sets of respective type (a) and type (b) point reguli.

Proof Consider $\alpha_{\mathcal{R}}$ the regulus plane of $\mathcal{R}$. The polar image of $\alpha_{\mathcal{R}}$, say Υ, is a 3-space which has to contain the complementary line regulus of $\mathcal{R}$ both in H_1 and in H_2. Hence these either coincide or are opposite, proving the first part of the lemma. Suppose they are opposite and M_i and L_i, $i \in \{0, \ldots, q\}$ denote the lines of Υ in H_1 and H_2 respectively.

Take a point p of $\mathcal{R}$. Inside H_1, p is collinear with all points of a certain ideal line L_i, $i \in \{0, \ldots, q\}$. In the same way, p is collinear with all points of say M_j, $j \in \{0, \ldots, q\}$, this time inside H_2. Hence pr_{ij}, with $L_i \mathtt{I} r_{ij} \mathtt{I} M_j$, is the unique line on p belonging to S. □

Note. Lemma 2(b) in fact states that if $\mathcal{R}$ is of type (b) then all of its points belong to Π_S, the hyperplane generated by all lines of S.

Lemma 2 *If S has type* 0 *then situation (b) of Lemma 1 does not occur.*

Proof Let x be the unique point that is at distance at most 3 from every line of S. Suppose by way of contradiction that $\mathcal{R}$ is a point regulus which is completely contained in $T_x\mathsf{Q}(6, q)$. As $\mathcal{R}$ belongs to $T_x\mathsf{Q}(6, q)$, which is the perp of x, we immediately find that x belongs to the three-space generated by $\mathcal{R}^c$. Hence x is incident with one of the lines, say L, of $\mathcal{R}^c$. By definition of S we know that L belongs to both hexagons. However, the complement of $\mathcal{R}$ in H_1 should be opposite the one determined by H_2, a contradiction as $L \in \mathcal{R}^c$.

□

To prove part (i) of the Main Result it now suffices to show that for given H_1 and S (the lines of H_1 in $T_x\mathsf{Q}(6, q)$ for some point $x \in \mathsf{Q}(6, q)$) the q other hexagons intersecting H_1 in S share q^3 point reguli with H_1. Say $\mathcal{R}$ belongs to Ω_1 and denote its complement by $\mathcal{R}^c$. Since all lines of $\mathcal{R}^c$ belong to S it is easy to see that $\mathcal{R}$ has to have x as one of its points (as $\mathcal{R}^c$ does not contain an element of $\Gamma_1(x)$, it belongs to $\Gamma_3(x)$, so $x \in \mathcal{R}$). Conversely, every point regulus on x has a complement which, by definition, consists of $q+1$ lines at distance 3 from x. In other words, these $q+1$ lines belong to S and Ω_1 is the set of all point reguli on x. Conclusion: if S is the line set at distance at most 3 from a point then $|\Omega| = q^3$.

In order to prove part (ii) of the Main Result we suppose that S has type $+$. Denote by Ω the set of point reguli of H_1 that are also point reguli of H_2. Clearly, Ω contains the subset Ω_1 of all point reguli complementary to the $q^2(q^2 + q + 1)$ line reguli in S.

Without loss of generality, we may assume $\Pi_S : X_3 = 0$ to be the hyperplane which determines all lines of S and denote the hexagon twin planes inside Π_S by Π^+ and Π^-. With this hyperplane we shall now determine a unique H_2 through S in the exact same way as described in [5].

Consider the point $p = \Pi_S^\rho$ with coordinates $(0, 0, 0, 1, 0, 0, 0)$ in $\mathsf{PG}(6, q)$. Every line through p and a point x of $\Pi_S \cap \mathsf{Q}(6, q)$ intersects $\mathsf{Q}(6, q)$ only in x, as x is the radical of that tangent line. Any other line through p and a point y on the quadric intersects $\mathsf{Q}(6,q)$ in a second point y'. The involution g interchanging y and y' and fixing all points of Π_S extends to an involutive collineation of $\mathsf{PG}(6,q)$, which we also denote by g. It is actually easy to see that g does not preserve H_1. Indeed, the set of lines of H_1 through a point x of $(\Pi_S \cap \mathsf{Q}(6,q)) \setminus (\Pi^+ \cup \Pi^-)$ fill up a plane of $\mathsf{Q}(6,q)$, and this plane is fixed under g only if it contains p or is contained in Π_S, clearly a contradiction.

Let H_2 be the image of H_1 under g, and note that $\mathsf{H}_1 \neq \mathsf{H}_2$. Henceforth, we shall use the convention of writing a point regulus of H_i with a subindex i, $i = 1,2$. So the point regulus determined by two points a, b in H_i is denoted by $\mathcal{R}_i(a,b)$. Since line reguli are determined by $\mathsf{Q}(6,q)$, such a notation for line reguli is superfluous.

We define the following set Ω_3 of point reguli common to H_1 and H_2. Consider two arbitrary but opposite lines L and M of S, let a be any point on $L \setminus (\Pi^+ \cup \Pi^-)$, and denote by Θ the 3-space generated by L and M. Let b be the point on M collinear with a on $\mathsf{Q}(6,q)$. Then b is at distance 4 from a in the incidence graph of both H_1 and H_2. Put $r = a \bowtie b$ (inside H_1) and denote r^g by r'. Obviously $r'a$ and $r'b$ are lines of H_2, implying r' belongs to Θ^ρ. Hence both r and r' belong to the point regulus in both H_1 and H_2 complementary to $\mathcal{R}(L,M)$. Therefore r' is collinear in H_1 with two points a' and b' (obviously distinct from a and b, respectively) on L and M respectively. Since g is an involution, the lines ra' and rb' of $\mathsf{PG}(6,q)$ are lines of H_2. The point regulus $\mathcal{R}_1(a,b')$ is complementary to $\mathcal{R}(rb, r'a')$ and the point regulus $\mathcal{R}_2(a,b')$ is complementary to $\mathcal{R}(ra', r'b)$. Since rb and $r'a'$ generate the same 3-space Υ in $\mathsf{PG}(6,q)$ as ra' and $r'b$, we conclude that $\mathcal{R}_1(a,b') = \mathcal{R}_2(a,b')$. Moreover, it is clear that Υ is invariant under g, and hence $p \in \Upsilon$. This implies now that $\mathcal{R}_1(a,b')$, which belongs to $\Upsilon^\rho \subseteq p^\rho$, is entirely contained in Π_S.

The set Ω_3 consist of all point reguli $\mathcal{R}_1(a,b')$, for all such choices of L, M and a (but a and b' determine L and M uniquely, so there is no need to include L and M in the notation). Remark that one can easily count the number of elements of Ω_3 to be $(q^2+q+1)q^2(q-1)$.

Lemma 3 *With the above notation, we have* $\Omega_3 = \Omega_2$ *and* $|\Omega| = (q^2+q+1)q^3$.

Proof From our discussion above we already have $\Omega_3 \subseteq \Omega_2$. Now suppose that $\Omega_3 \neq \Omega_2$ and let $\mathcal{R}$ be a point regulus of $\Omega_2 \setminus \Omega_3$. Let a and b be two points of $\mathcal{R}$. By Lemma 2 we know that both these points are incident with a unique line of S, say L_a and L_b respectively. Denote $\mathrm{proj}_{L_a} b$ and $\mathrm{proj}_{L_b} a$ by a' and b' respectively and let x and x' denote $a \bowtie b'$ and $a' \bowtie b$ respectively. We shall show that x' has to be the image of x under g. Indeed, since $\mathcal{R}$ belongs to the set of common point reguli of both hexagons we immediately find that xa' and $x'b'$ are lines of H_2. Suppose x^g is a point x'' distinct of x' then also xa'' (with $a'' = \mathrm{proj}_{L_a} x''$) belongs to the line set of H_2, a contradiction.

The lemma is proved. $\square$

Lemma 4 *Every other model of* $\mathsf{H}(q)$ *(not* H_2*) which also contains* S *as a subset of lines, intersects* H_1 *in a set of* $q^2(q^2+q+1)$ *point reguli. In other words, here* Ω *equals* Ω_1*.*

Proof Take a model $\mathsf{H}_3 \neq \mathsf{H}_2$ of $\mathsf{H}(q)$ through S. Obviously Ω, in the same way as before, contains Ω_1 as a subset. Suppose by way of contradiction that $\mathcal{R}(a,b)$ is a type (b) point regulus of both hexagons. As an immediate consequence of Lemma 1 we know that $\alpha_{\mathcal{R}}$, the regulus plane of $\mathcal{R}$, is a subspace of Π_S, the hyperplane containing all lines of S. By polarity we thus find $p = \Pi_S^{\rho}$ to be a point of $\Upsilon = \alpha_{\mathcal{R}}^{\rho}$. On $\mathsf{Q}(6,q)$ this 3-space constitutes a hyperbolic quadric isomorphic to $\mathsf{Q}^+(3,q)$ which we shall denote by Q^+_{Υ}.

Put L_a, respectively L_b, the unique element of S incident with a, respectively b. Inside H_1 we denote $\mathrm{proj}_{L_b} a$, $\mathrm{proj}_{L_a} b$ by b', a' and $a \bowtie b'$, $a' \bowtie b$ by x, x' respectively.

From our discussion above we know that there exists a unique point regulus $\mathcal{R}'$, on a and a point $b'' \mathrel{\mathtt{I}} L_b$, which belongs to both H_1 and H_2. The point b'' is the unique point of L_b which is collinear (within H_1) with $x^g = px \cap \mathsf{Q}(6,q) \setminus \{x\}$. However, since $p \in \Upsilon$ and $x \in \Upsilon$ we have that $x^g \in \Upsilon \cap (L_a L_b)^{\rho}$ and see that x^g should belong to a line of Q^+_{Υ} and hence equals x'. Furthermore, this implies that H_2 and H_3, apart from having all lines of S in common, share the complementary line regulus of $\mathcal{R}$ (i.e. $\mathcal{R}$ is of type (a) with respect to these two hexagons). In other words, we find that H_3 equals H_2 and we are done. $\square$

The proof of part (iii) is similar. Here we just point out the main difference.

First of all, the subset Ω_1 contains $q^2(q^2-q+1)$ point reguli which are complementary to the line reguli of S. There is, however, a crucial distinction between both situations (ii) and (iii) when it comes down to defining the set Ω_3. Before, we considered L and M two lines of S and a a point on $L \setminus (\Pi^+ \cup \Pi^-)$. In the current situation, with S a distance-3 spread, we lay no restriction on the choice of a on L and hence end up with

$$\frac{|S|(|S|-1)}{(q+1)q}(q+1)$$

elements of Ω_3 instead of

$$\frac{|S|(|S|-1)}{(q+1)q}(q-1)$$

as we did before.

In order to complete the proof we may carefully copy the above. Indeed, in the same way one can show that Ω_3 consists of all elements of type (b) and hence find $|\Omega| = q^2(q^2-q+1)(q+2)$. Finally by Lemma 4 we may conclude the main result to be proven.

Remark. If H_1 and H_2 share point reguli of type (b), meaning Ω_2 is a non-empty set, then (while the set Ω_1 determines all lines of S) the set Ω_2 determines all lines of $\mathsf{H}_1 \setminus S$ (and consequently also all lines of H_2), as we shall show.

Denote the set of lines that are in a complementary regulus of some element of Ω_2 by $\mathcal{L}$ and suppose $\mathcal{L}$ has cardinality A. First of all, we determine, given a line L of $\mathcal{L}$ the number n_1 of point reguli $\omega_2 \in \Omega_2$, such that $L \in \omega_2^c$. Since L belongs to $\mathsf{H}_1 \setminus S$, it is concurrent with a unique line L_1 of S (as Π_S is a hyperplane). A point x on L and a line M through x then completely fix an element $\omega_2 \in \Omega_2$, such that $L \in \omega_2^c$ (this is true because M is concurrent with a unique line L_2 of S and because x and x^g determine two points a and b, on L_1 and L_2, of ω_2). Hence by an easy counting argument one obtains that $n_1 = q$ (consider the triple $(x, M, \mathcal{R})$, with $L_1 \not\mathrel{\mathtt{I}} x \mathrel{\mathtt{I}} L$, $L_1 \neq M \mathrel{\mathtt{I}} x$, $L \in \mathcal{R}$ and $\mathcal{R}^c \in \Omega_2$).

We are now ready to apply a double counting on the couples (ω_2, L), $\omega_2 \in \Omega_2$ and $L \in {\omega_2}^c$. Here we treat the case where S has type $-$ and omit the counting for S of type $+$, which is similar. Since $|\Omega_2| = (q^3 + 1)q^2$ we find

$$(q^3 + 1)q^2(q + 1) = Aq$$

and hence

$$A = (q^3 + 1)q(q + 1)$$

This number of lines together with all spread lines add up to

$$A + (q^3 + 1) = \frac{q^6 - 1}{q - 1}$$

the total amount of lines in H_1.

4 A transitive 1-system of $\mathsf{Q}(6, 3)$

In this section we will be working with the exceptional spread $\mathcal{S}_{\mathrm{E}}$ of $\mathsf{H}(3)$ as constructed in [4]. However, to clarify further reading we admit a short introduction to this particular type of spreads.

Let $\mathcal{H}$ be an elliptic hyperplane in $\mathsf{PG}(6, 3)$ and let $\mathcal{S}_{\mathrm{H}}$ be the corresponding spread of $\mathsf{H}(3)$. We extend $\mathsf{PG}(6, 3)$, $\mathsf{Q}(6, 3)$ and $\mathsf{H}(3)$ to $\mathsf{PG}(6, 9)$, $\mathsf{Q}(6, 9)$ and $\mathsf{H}(9)$, respectively, as projective varieties. By [10], the hyperplane $\mathcal{H}$ viewed as a hyperplane of $\mathsf{PG}(6, 9)$, defines in $\mathsf{H}(9)$ the subhexagon Γ of $\mathsf{H}(9)$ and $\Gamma \cap \mathsf{H}(3) = \mathcal{S}_{\mathrm{H}}$. If Π^+ and Π^- are the two hexagon twin planes of Γ then according to [10] the lines of $\mathcal{S}_{\mathrm{H}}$ meet $\Pi^\pm$ in the points of a hermitian curve, which we will call $\mathcal{U}^\pm$. Put $\Phi^\pm$ be the corresponding isomorphism between the lines of $\mathcal{S}_{\mathrm{H}}$ and the points of $\mathcal{U}^\pm$. The image of $\mathcal{R}_0$, a fixed line regulus of $\mathcal{S}_{\mathrm{H}}$, under $\Phi^\pm$ is then the intersection of $\mathcal{U}^\pm$ with a line $L_0^\pm$.

Roughly, the construction of $\mathcal{S}_{\mathrm{E}}$ goes as follows. There are three Hermitian spreads containing $\mathcal{R}_0$. In each of these, we choose appropriately two additional reguli in such a way that, together with $\mathcal{R}_0$, these three reguli form, viewed as blocks of $\mathcal{U}^\pm$, a polar triangle.

Apart from this definition of $\mathcal{S}_{\mathrm{E}}$, we shall also be using the observation that when considering the set of all line reguli of $\mathcal{S}_{\mathrm{E}}$ (seven in total) as a point set and the sets of three reguli contained in a common elliptic hyperplane as line set one obtains a geometry isomorphic to the projective plane of order 2.

Let Γ be the geometry, isomorphic to $\mathsf{PG}(2,2)$, with as point set the reguli of $\mathcal{S}_{\mathrm{E}}$ and where three points are on a line if they –as line reguli of $\mathcal{S}_{\mathrm{E}}$– belong to the same Hermitian spread (see [4]). From now on we shall denote the line reguli of $\mathcal{S}_{\mathrm{E}}$ by $\mathcal{R}_0, \ldots, \mathcal{R}_6$ and use the standard description of Γ as a difference set. Namely, for each $i \in \mathbb{Z} \mod 7$, there is a line $\{\mathcal{R}_i, \mathcal{R}_{i+1}, \mathcal{R}_{i+3}\}$.

Note that it now makes sense to talk about switching points of Γ when meaning to switch the corresponding line reguli.

By [3] we know that switching any number of line reguli of $\mathcal{S}_{\mathrm{E}}$ yields a 1-system of $\mathsf{Q}(6,3)$. It is now straightforward to see that the 1-system obtained by a "full" derivation of $\mathcal{S}_{\mathrm{E}}$ (i.e. switch all seven reguli) admits $2^3.\mathsf{SL}_3(2)$ acting transitively.

We start by determining when a derivation belongs to some $\mathsf{H}_2 \neq \mathsf{H}_1$.

Take H_1 and H_2, a model of $\mathsf{H}(3)$ on $\mathsf{Q}(6,3)$, and let $\mathcal{S}_{\mathrm{E}}$ be an exceptional spread of this hexagon. Denote a derivation of $\mathcal{S}_{\mathrm{E}}$ by $\mathcal{S}'_{\mathrm{E}}$, the common line set of those two hexagons by S and the set of switched line reguli by χ. Since $\mathcal{S}'_{\mathrm{E}}$ is a proper derivation of $\mathcal{S}_{\mathrm{E}}$, the set Ω_2 will be non-empty. By Lemma 4 we may thus conclude that $\mathsf{H}_2 = \mathsf{H}_1^g$, where g is the involutive collineation of previous section linked to Π_S, the hyperplane generated by S. Furthermore we know that p, the polar point of Π_S, belongs to every 3-space generated by any one of the switched reguli. In other words, if the lines of $\mathcal{R}_i$ determine the 3-space Υ_i then p belongs to Υ_i when $\mathcal{R}_i$ is one of the line reguli that we switch.

Lemma 5 *As soon as χ contains the points of a line of Γ as a subset, the thus obtained derivation of $\mathcal{S}_{\mathrm{E}}$ can never be a spread of any other hexagon H_2.*

Proof Suppose $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_3$ are three such line reguli. These three determine, by definition of the exceptional spread, a polar triangle Δ^+ in Π^+. In the exact same way they also determine a polar triangle Δ^- in Π^-. Denote a point of these respective polar triangles by $r_{ij}^{\pm}$ if it is the intersection point of $L_i^{\pm}$ and $L_j^{\pm}$, where $L_i^{\pm}$ is the line containing all points of $\mathcal{R}_i^{\Phi^{\pm}} \cap \mathcal{U}^{\pm}$.

With this notation at hand it easy to see that $\Upsilon_0 \cap \Upsilon_1 = r_{01}^+ r_{01}^-$. However, these two points (r_{01}^+ and r_{01}^-) belong to Υ_3^{ρ} which is a plane disjoint of Υ_3. Hence Υ_0, Υ_1 and Υ_3 have an empty intersection, which is contradictory to the fact that p belongs to this intersection. □

Lemma 6 *If there is a line of Γ on which exactly one point is switched, then $\mathcal{S}'_{\mathrm{E}}$ can never be a spread of any other hexagon.*

Proof Suppose by way of contradiction that there exists a line L on which we switch a single point t and suppose that the hexagon H_2 contains $\mathcal{S}'_{\mathrm{E}}$. The two remaining points on L correspond to two reguli of $\mathcal{S}_{\mathrm{E}}$ that are not switched and are hence entirely contained in Π_S, as defined above. Since these two reguli generate Π_S, the lines of t belong to S and to H_2, a contradiction. □

As an immediate consequence of Lemma 5 and 6 we see that the only possibility for a derivation of $\mathcal{S}_{\mathrm{E}}$ to be a spread of H_2 is obtained by not switching all points on a line of Γ and switching the rest.

Proposition 7 *The* 1*-system obtained by not switching all points on a line of* Γ *and switching the rest is a spread of* H_2*.*

Proof Before starting the actual proof of this theorem, we shall give some more information concerning the construction of $\mathcal{S}_{\mathrm{E}}$. As stated above we start with a fixed line regulus $\mathcal{R}_0$. A crucial property shall be that this regulus is contained in three distinct Hermitian spreads. This phenomenon can be explained by looking at the extension of $\mathsf{PG}(6,3)$ to $\mathsf{PG}(6,9)$. Indeed, take any two lines M and N of $\mathcal{R}_0$ and consider these lines in $\mathsf{H}(9)$. On those lines we have three pairs of conjugate points and each of these pairs determines a hyperbolic hyperplane of $\mathsf{Q}(6,9)$ (the focus of each transversal to M and N in these two points, together with $\mathcal{R}_0$ completely determines this hyperplane), which is an elliptic hyperplane of $\mathsf{Q}(6,3)$.

For instance, if $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_3$ are the three reguli contained in $\mathcal{S}_{\mathrm{H}}$, the Hermitian spread we started from, then the intersection of $L_0^{\pm}$, where $L_0^{\pm} = \mathcal{R}_0^{\Phi^{\pm}} \cap \mathcal{U}^{\pm}$, with M and N gives us the corresponding conjugate pair of points. Furthermore r_{13}^{-}, respectively r_{13}^{+}, is the point of $\mathsf{H}(9)$ collinear to both intersection points of L_0^{+}, respectively L_0^{-}, with M and N.

Let α be the polar image of Υ_0 and put $\overline{\alpha}$ the extension of α over $\mathsf{GF}(9)$. The intersection of α with $\mathsf{Q}(6,3)$ is a conic, say C. The extension of C over $\mathsf{GF}(9)$, denoted by $\overline{C}$, contains six additional points of which r_{13}^{+} and r_{13}^{-} are a conjugate pair. As this first pair corresponds to $\mathcal{S}_{\mathrm{H}} = \mathcal{S}_{\mathrm{H}}^{0}$ we shall, from now on, denote them by r_0^{+} and r_0^{-}. In general we put (r_i^{+}, r_i^{-}) as the conjugate pair of points corresponding to the Hermitian spread $\mathcal{S}_{\mathrm{H}}^{i}$, with $i = 0, 1, -1$, on $\mathcal{R}_0$.

We are now ready to start the actual proof of this theorem.

Suppose $\chi = \{\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_3\}$ and switch all other points of Γ. Without loss of generality we may suppose $\mathcal{R}_2$, $\mathcal{R}_6$ and $\mathcal{R}_4$, $\mathcal{R}_5$ together with $\mathcal{R}_0$ to be reguli of the Hermitian spreads $\mathcal{S}_{\mathrm{H}}^{1}$ and $\mathcal{S}_{\mathrm{H}}^{-1}$, respectively. If $p = \Pi_S^{\rho}$, with $\Pi_S = \langle \Upsilon_0, \Upsilon_1, \Upsilon_3 \rangle$, is a point of every one of the 3-spaces $\Upsilon_2, \Upsilon_4, \Upsilon_5, \Upsilon_6$ then the obtained 1-system is a spread of $\mathsf{H}_2 (= \mathsf{H}_1^{g})$.

First of all, one can easily see that $\Upsilon_2 \cap \Upsilon_6 = r_1^{+} r_1^{-}$ while $\Upsilon_4 \cap \Upsilon_5 = r_{-1}^{+} r_{-1}^{-}$. As both these lines contain two conjugate points they are lines which belong to α and hence intersect in a point, say $t \in \alpha$. It now suffices to show that t equals p to complete the proof.

As we already know two lines of α on t and as every line on t and a point of C is either a tangent or an intersection line, the two remaining lines of α through t are intersection lines of C.

Since $r_0^+r_0^-$ is a line of α, it does not pass through t. Hence tr_0^+ and tr_0^- are tangent lines of $\overline{C}$, meaning $r_0^+r_0^-$ belongs to t^ρ. On the other hand, a simple application of the polarity gives us, as t belongs to Υ_0^ρ $(=\alpha)$, that also Υ_0 is a subspace of t^ρ. Hence the span of this 3-space and the line $r_0^+r_0^-$ belongs to the polar hyperplane of t. As both Υ_0 and $r_0^+r_0^-$ are disjoint subspace of Π_S, they in fact span this particular hyperplane. In other words, we have that $t^\rho = p^\rho$ or that t equals p and we are done.

□

A nice consequence of this theorem is that there are only 4 equivalence classes of derived 1-systems of $\mathcal{S}_{\rm E}$ on $\mathsf{Q}(6,q)$, as we shall show. We can also determine the automorphism groups in each case. Suppose $\mathcal{S}_{\rm E}$ is an exceptional spread of H_1.

Corollary 8 *If $\mathcal{S}'_{\rm E}$ is a derivation of $\mathcal{S}_{\rm E}$, then – up to isomorphism – either*

(i) the set χ is empty;

(ii) the set χ contains a unique point;

(iii) the set χ contains two points;

(iv) the set χ contains three points on a line of Γ.

In cases (i) and (iv), the automorphism group of $\mathcal{S}'_{\rm E}$ has the structure $2^3 \cdot \mathsf{SL}_3(2)$; in the other two cases we have $2^3.\mathsf{S}_4$, where S_4 is a maximal subgroup of $\mathsf{SL}_3(2)$ corresponding to a point stabilizer in $\mathsf{PG}(2,2)$.

Proof The first three equivalence classes are obtained trivially (the automorphism group acts 2-transitively on the set of line reguli).

Now suppose χ contains three points which are not on a line. Without loss of generality we may assume $\mathcal{R}_0, \mathcal{R}_3, \mathcal{R}_6$ to be these three points (with the labelling of Γ as described above). By additionally switching $\mathcal{R}_5$, we obtain (as a result of Theorem 7) a derivation of $\mathcal{S}_{\rm E}$ which is contained in another hexagon H_2. Mapping H_1 to H_2 and switching only $\mathcal{R}_5$ shows the equivalence between this situation and situation (ii).

When χ contains three points on a line a similar technique always results in the switching of three points on a line in the new hexagon. Hence this is an isomorphism class which is non-equivalent with the classes (i) up to (iii).

For the remaining situations, the cardinality of χ will be greater than or equal to 4 and it will be advisable to look at the complement of this set of points, denoted by $\overline{\chi}$.

Suppose $\overline{\chi}$ contains three points in a triangle. Again without loss of generality we may assume this triangle to contain the points $\mathcal{R}_0, \mathcal{R}_3, \mathcal{R}_6$. This situation is equivalent to situation (iii), as switching $\mathcal{R}_2$ and $\mathcal{R}_3$ gives us a derivation contained in some H_2.

Not switching the points on a line and switching the rest is just how we obtain a derivation which yields a spread of H_2, hence this is equivalent with situation (i).

Switching the unique third point on the line containing both non-switched points of $\overline{\chi}$ shows the equivalence between $|\chi| = 5$ and situation (ii).

While switching two points on any line through the unique element of $\overline{\chi}$ demonstrates the equivalence between $|\chi| = 6$ and situation (iii).

Finally, suppose we have switched all points of Γ. Switching back three points on a line results in the spread of Theorem 7. Hence switching all points of Γ can be reduced to situation (iv) and we are done.

Regarding the automorphism groups, the one of case (i) follow from [4]. It is clear that, if all reguli are switched, then we obtain again the same automorphism group; whence case (iv). In case (ii), the regulus that can be switched in order to obtain a spread in a hexagon is unique; clearly the automorphisms of the bottom group 2^3 also act on every derivation. This proves the case (ii). For case (iii), it is similarly enough to remark that the third point on the line joining the two points that are switched is unique with respect to the following geometric property: if it is switched, then we obtain a spread of type (iv).

This completes the proof of the corollary.

□

The following theorem states a general result concerning similar questions starting from a Hermitian spread $\mathcal{S}_{\mathrm{H}}$ of H_1, a model of $\mathsf{H}(q)$.

Proposition 9 *If we switch disjoint blocks of the Hermitian spread $\mathcal{S}_{\mathrm{H}}$ of H_1, then the following statements are equivalent:*

(i) The obtained set of lines is a spread of some hexagon $\mathsf{H}_2 \neq \mathsf{H}_1$.

(ii) The obtained set of lines is isomorphic to $\mathsf{H}_2(q^2)$ on $\mathsf{Q}^-(5, q)$.

(iii) All blocks conjugate to some given block B are switched.

Proof We start by proving the equivalence between (i) and (ii). Suppose $\mathcal{S}'_{\mathrm{H}}$, the derived set of lines, is a spread of H_2. This fact, together with the knowledge that switching lines does not alter the space they are in, easily implies that $\mathcal{S}'_{\mathrm{H}}$ is a Hermitian spread. Thus situation (i) implies situation (ii).

If $\mathcal{S}'_{\mathrm{H}}$ is isomorphic to $\mathsf{H}_2(q^2)$ on $\mathsf{Q}^-(5, q)$, then there are $q + 1$ hexagons – hence at least one – containing this set of lines as a subset. Hence the equivalence between the first two cases is shown.

In order to prove the equivalence of (ii) and (iii), we dualize the situation and consider ovoids of the Hermitian generalized quadrangle $\mathsf{H}(3, q^2)$. Note that an ovoid of $\mathsf{H}(3, q^2)$ is Hermitian if and only if all points of it are contained in a plane of the ambient projective space $\mathsf{PG}(3, q^2)$. Also, a regulus is here a set of points on a secant line, and "switching a regulus" corresponds to "substituting a secant line all of whose points are contained in the ovoid by the conjugate line with respect to the (unitary) polarity of $\mathsf{PG}(3, q^2)$ defined by $\mathsf{H}(3, q^2)$"; we will briefly say that we "replace a block (by its conjugate)". It is now clear that, in a Hermitian ovoid $\mathcal{O}$ generating the plane π, replacing one block by its conjugate does not produce a Hermitian ovoid because the unchanged points still span π. Also, if we replace two block B_1 and B_2 by their conjugates B_1' and B_2', respectively, then clearly all conjugates of lines in the plane π' spanned by B_1' and B_2' are incident with the intersection point z of the lines defined by the blocks B_1 and B_2. Hence, in order to obtain an ovoid contained in π', we have to get rid of every point x not in the intersection of π and π' by replacing the unique block on the line xz. The block B defined by $\pi \cap \pi'$ must remain unchanged, and that is exactly the block all of whose points are conjugate to z with respect to the polarity in π corresponding to the Hermitian curve $\mathcal{O}$. The equivalence between (ii) and (iii) now follows. □

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