Unitals in the Hölz-design on 28 points

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Abstract

In this paper, we provide a theoretical proof of the fact that the only unitals contained in the 2 − (28, 4, 5) Hölz design are Hermitian and Ree unitals (confer a computer search by Tonchev, [8]).

1 Introduction

In 1981, Hölz [3] constructed a family of 2 − (q^3 + 1, q + 1, q + 2)-designs whose point set coincides with the point set of the Hermitian unital over the field GF(q), and with an automorphism group containing PGU_3(q). Here, q is any odd prime power. Two years later, Thas [7] proved that these designs are one-point extensions of the Ahrens-Szekeres generalized quadrangles AS(q) of order (q − 1, q + 1) (see [1]).

In a previous paper [9] the authors gave an alternative construction of the Hölz design, for q ≠ 2 mod 3 making use of two hexagons embedded in the parabolic quadric Q(6, q).

In 1991, Tonchev [8] shows “by a computer search” that the only unitals contained in the 2 − (28, 4, 5) Hölz design are Hermitian and Ree unitals. Using our findings from [9], we came across the following construction of that particular Hölz design.

Take Γ the unique generalized quadrangle of order (2, 4). We define D = (P, B, Σ) with point and block set deduced from Γ. Define P as the point set of Γ to which we add a new point α. The set B contains two types of blocks: blocks of type (a) contain the point α together with three points of any line of Γ (Line-block) and those of type (b) contain the four points of any two intersecting lines of Γ, which are distinct of the intersection point (Vee-block). It is now easy to see that D is a 2 − (28, 4, 5) design.

In the present note we use previous findings to give a computer-free proof of the result in [8].

2 Preliminaries

A t − (v, k, λ) design, for integers t, v, k and λ with v > k > 1 and k ≥ t ≥ 1, is an incidence structure D satisfying following axioms: D contains v points; each of its blocks

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is incident with $k$ points; any $t$ points are incident with exactly $\lambda$ common blocks. For further information on designs we refer to [4].

The following class of 2-designs is due to G. Hölz [3]. Let $\mathcal{U}$ be a hermitian curve of $\text{PG}(2, q^2)$ [2]. A Baer subplane [5] $\text{PG}(2, q) = D$ is said to satisfy property $(H)$ if for each point $x \in D \cap \mathcal{U}$ the tangent line $L_x$ to $\mathcal{U}$ at $x$ is a line of $D$ (i.e. $|L_x \cap D| = q + 1$). If $D$ satisfies this property $(H)$ then one can show that if $|D \cap \mathcal{U}| \geq 3$ then $|D \cap \mathcal{U}| = q + 1$, for $q$ even the points of $D \cap \mathcal{U}$ are collinear, and for $q$ odd the points of $D \cap \mathcal{U}$ are collinear or form an oval in $D$. If $D_1$ and $D_2$ are Baer subplanes satisfying property $(H)$ and if $|D_1 \cap D_2 \cap \mathcal{U}| \geq 3$, then $D_1 \cap \mathcal{U} = D_2 \cap \mathcal{U}$. If moreover $D_1 \cap \mathcal{U}$ is an oval of $D_1$, then $D_1 = D_2$.

Let $q$ be odd. If $x$ and $y$ are distinct points of $\mathcal{U}$, then (1) there are exactly $q + 1$ Baer subplanes $D$ in $\text{PG}(2, q^2)$ which satisfy property $(H)$ and for which $D \cap \mathcal{U} = xy \cap \mathcal{U}$, and (2) there are exactly $q + 1$ Baer subplanes $D$ in $\text{PG}(2, q^2)$ which satisfy property $(H)$ and for which $D \cap \mathcal{U}$ is an oval of $D$ through $x$ and $y$. Let $B_1$ be the set of all intersections $L \cap \mathcal{U}$ with $L$ a non tangent line of $\mathcal{U}$, and let $B'$ be the set of all intersections $D \cap \mathcal{U}$ with $D$ a Baer subplane of $\text{PG}(2, q^2)$ satisfying property $(H)$ and containing at least three points of $\mathcal{U}$. Finally, let $B^* = B' - B_1$.

Clearly $S_1 = (\mathcal{U}, B_1, \in)$ is a $2 - (q^3 + 1, q + 1, 1)$ design and $S' = (\mathcal{U}, B', \in)$ is a $2 - (q^3 + 1, q + 1, q + 2)$ design and $S^* = (\mathcal{U}, B^*, \in)$ is a $2 - (q^3 + 1, q + 1, q + 1)$ design. Moreover any two distinct blocks of these designs have at most two points in common.

A generalized quadrangle $\Gamma$ (of order $(s, t)$) is a point-line geometry the incidence graph of which has diameter 4 and girth 8 (and every line is incident with $s + 1$ points; every point incident with $t + 1$ lines). Note that, if $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set of $\Gamma$, then the incidence graph is the (bipartite) graph with set of vertices $\mathcal{P} \cup \mathcal{L}$ and adjacency given by incidence. The definition implies that, given any two elements $a, b$ of $\mathcal{P} \cup \mathcal{L}$, either these elements are at distance 4 from one another in the incidence graph, in which case we call them opposite, or there exists a unique shortest path from $a$ to $b$. In other words, given any non-incident point-line pair, say $(p, L)$, there exists a unique point on the line $L$ which is collinear with $p$.

A spread of the generalized quadrangle $\Gamma$ is a set of lines of $\Gamma$ partitioning the point set into lines. In other words, every point of $\Gamma$ is incident with a unique line of the spread.

Let $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a generalized quadrangle of order $(s, t)$ with $|\mathcal{P}| = v$ and $|\mathcal{B}| = b$. The $(i + 1) - (v + i, s + 1 + i, t + 1)$ design $S' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ is said to be an $i$-th extension of $S$ if for any $i$ distinct points $x_1, \ldots, x_i$ of $\mathcal{P}'$ the derived structure of $S'$ with respect to $x_1, \ldots, x_i$ is isomorphic to $S$. Recall that the derived structure of $S'$ with respect to $x_1, \ldots, x_i$ is the 1-design $S'(x_1, \ldots, x_i) = (\mathcal{P}'(x_1, \ldots, x_i), \mathcal{B}'(x_1, \ldots, x_i), \mathcal{I}'(x_1, \ldots, x_i))$ with $\mathcal{P}'(x_1, \ldots, x_i) = \mathcal{P}' \setminus \{x_1, \ldots, x_i\}$, $\mathcal{B}'(x_1, \ldots, x_i)$ the set of all blocks of $\mathcal{B}'$ incident with $x_1, \ldots, x_i$, and $\mathcal{I}'(x_1, \ldots, x_i)$ the incidence induced by $\mathcal{I}'$. A first extension of $S$ is shortly called an extension of $S$.  

2
3 Main Result

In this section we shall use the construction of the $2 - (28, 4, 5)$ design, $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, as given in Section 1. More explicit, we take a $\mathcal{Q}^- (5, 2)$ (say $\mathcal{Q}$) in $\text{PG}(5, 2)$ and embed this 5-space as a hyperplane, $\mathcal{H}$, into $\text{PG}(6, 2)$. Let $\alpha$ be any point of $\text{PG}(6, 2)$ not in $\mathcal{H}$. Then the points of $\mathcal{D}$ can also be seen as a set of affine points in $\text{AG}(6, 2)$, namely all points on the affine lines $\alpha x$, with $x$ a point of $\mathcal{Q}$. In the mean time the blocks of $\mathcal{D}$ contain 4 points which determine affine planes in $\text{AG}(6, 2)$. First of all, we have all affine planes through $\alpha$ intersecting $\mathcal{H}$ in a line of the generalized quadrangle and secondly, the four points in the disjoint union of two intersecting lines of $\mathcal{Q}$ determine two affine lines of $\text{AG}(6, 2)$ which intersect $\mathcal{H}$ in the same point (namely that intersection point). Hence a Vee-block also determines four points of an affine plane. To simplify notation we shall denote the affine point on $\alpha x$, distinct of $\alpha$, by $X$.

A unital $\mathcal{U}$ of $\mathcal{D}$ is by definition a subset of $\mathcal{B}$ such that any two points of $\mathcal{D}$ are contained in a unique block of $\mathcal{U}$. Hence, for every point $x$ in $\mathcal{D}$ such a unital defines a spread, denoted by $S_x$, in the derived quadrangle $\mathcal{D}_x$ of this particular point. It is well known that the generalized quadrangle of order $(2, 4)$ has two non-isomorphic spreads: first, the Hermitian spread (in which any two lines determine a line regulus completely contained in it) and second a spread obtained by switching one of the line reguli of this previous spread. For further reference we shall denote this latter spread by non-Hermitian.

3.1 One Hermitian spread implies all Hermitian spreads

In this subsection we will be working under the assumption that $\mathcal{U}$ determines at least one Hermitian spread and we will show that in this particular case all other derived spreads have to be Hermitian as well.

Without loss of generality we may assume that we obtain such a Hermitian spread by a one-point derivation in $\alpha$. Hence, $S_\alpha$ determines 9 lines of $\mathcal{Q}$.

Take $L$ any one of those lines and say $x$, $y$ and $z$ are the points incident with this line in $\mathcal{Q}$. A what turns out to be very useful property of a non-Hermitian spread is that it is the union of three disjoint line reguli and these are in fact the only reguli it contains.

Meaning, every line of a non-Hermitian spread is contained in a unique regulus of the spread.

Lemma 1 $S_x$ is completely fixed by the line regulus on the line $L$ of $S_\alpha$.

PROOF. Take $\mathcal{R}_x$ a line regulus of $S_x$ on $L$ (this is possible, independent of $S_x$ being Hermitian or not).

Corresponding to the two types of blocks of $\mathcal{D}$ on $X$ we have two types of lines in $\mathcal{D}_x$.

The first type corresponds to the Line-blocks, the second to the Vee-blocks. As $L$ results from a block containing $\alpha$ and $X$ the spread $S_x$ cannot contain any more lines of this first type. Suppose $\mathcal{R}_x$ contains $L$ and a Vee-line through the line $xst$ of $\mathcal{Q}$. There are then four grids $\mathcal{R}_i, i = a, b, c, d$, in $\mathcal{Q}$ containing both lines $xyz$ and $xst$ and in each of
these grids we denote the unique point collinear with * and *‘ by \(i_{**’}\), where * \(\in\) \{s, t\} and *‘ \(\in\) \{y, z\}. If \(\mathcal{R}_a\) is the grid containing the spread lines of \(S_a\) then by transitivity we may assume \(sb_{sy}b_{tz}\) and consequently also \(tb_{sy}b_{sz}\) to be a line of \(\mathcal{R}_x\) (\(sa_{ty}a_{tz}\) is impossible as otherwise \(A_{ty}\) and \(A_{tz}\) would be in two distinct blocks of \(\mathcal{U}\)).

Every point \(b_{**’}\) is incident with a unique line of \(S_a\). As none of these four points are on the same spread line they determine four distinct elements of \(S_a\). These four lines together with the spread lines in \(R_a\) add up to seven of the nine lines of that particular spread. Meaning there are only two spread lines left which are in a grid with \(L_x\) as we shall show.

Indeed, as \(S_a\) is a Hermitian spread every one of those four lines is in a regulus with \(L_x\). Considering the regulus on \(L\) and the line incident with, for instance, the point \(b_{sy}\) we can see that this regulus necessarily contains the spread line through the point \(b_{ty}\) as well. In the same way \(L\) is in a regulus with the lines incident with \(b_{sz}\) and \(b_{tz}\). Hence also \(L\) and the two remaining lines are three lines of a grid \(R’\). Consider the second non-spread line, say \(xuv\), of \(R’\) on \(x\). We shall now look at the possible spread lines of \(S_x\) through \(u\) and \(v\) respectively.

Such a spread line in fact corresponds to a block of \(\mathcal{U}\) through \(X\) and \(U\) (respectively \(V\)). The block through \(X\) and \(U\) is completely determined by a line through \(v\).

Suppose the non-spread line on \(y\) of \(R’\) belongs to \(R_c\), then consequently the one on \(z\) belongs to \(R_d\). We will now show that the remaining lines on \(y\) and \(z\) (namely in \(R_d\) and \(R_c\) respectively) are in a grid, say \(R”\), with \(L\) and \(xuv\). Suppose by way of contradiction that the line \(ud_{syz}\) intersects either \(R_a\) or \(R_b\), say \(R_j\) with \(j \in\) \{\(a, b\}\). Not allowing triangles in \(Q\) the third point on this line necessarily has to be the point \(j_{*2z}\), with \(\{*1, *2\} = \{s, t\}\).

This, on its turn, implies that \(vd_{syz}j_{*1z}\) is a quadric line. Hence, since \(x \sim *2\) and \(v \sim d_{syz}\) we find that \(u \sim d_{syz}\) and in the same way that \(u \sim j_{*1y}\). However, given the definition of these points, it is obvious that \(d_{syz} \sim j_{*1z}\) and thus we have our contradiction (\(ud_{syz}\) contains a point of \(R_c\) as we know from \(R’\)).

By the defining property of a generalized quadrangle every point outside a grid is collinear with exactly three lines of the grid. So if we consider the grid \(R_b\) and the point \(u\) then the spread line on \(u\) and the line \(L_u\) (which is the line of \(R”\) on \(u\) distinct of \(xuv\)) are the only two lines not intersecting this particular grid. However a block on \(X\) and \(V\) is determined by such a line on \(u\) which does not equal \(xuv\) and does not contain any of the points \(b_{**’}\) (as \(X\) is already in a block with each of the points \(B_{**’}\)). Hence this block and consequently also the block on \(X\) and \(U\) is determined by lines of \(R”\).

In \(S_x\) we now have two reguli on a line. Meaning, \(S_x\) has to be a Hermitian spread and as we can see in the dual situation in \(H(3, 3)\), this Hermitian spread is completely fixed (in \(H(3, 4)\) this translates into two lines through a point fixing the plane which determines the ovoid).

\[\square\]

As \(L\) and \(x\) were chosen arbitrary this holds for every point \(x\) in \(Q\).

### 3.2 Hermitian spreads imply uniqueness

In this section we show that
Theorem 2 The $2-(28,4,5)$ Hölz design contains - up to isomorphism - a unique unital which intersects all derived subdesigns in Hermitian spreads.

PROOF. Without loss of generality we may fix a Hermitian spread in the derived generalized quadrangle $\mathcal{D}_a$. If the construction of a unital containing this spread is hereby fixed - up to isomorphism - then the above stated is proven.

Suppose $L = xyz$ is a spread line of $S_a$. Consider $xst$, another line of the quadric and denote the grids through $L$ and this line by $R_i$, $i = a, b, c, d$ and use the same notation of the point set as introduced in previous section. Furthermore, say $R_a$ is the unique grid containing three spread lines of $S_a$.

With these definitions we shall prove two short lemmas. First, we have

Lemma 3 Suppose all $S_p$, for $P \in \mathcal{D}$, are Hermitian spreads. Then if the block of $\mathcal{U}$ on $P \in \{X, Y, Z\}$ and $S$ is determined by a line of $R_i$ then so is the block on $P$ and $T$.

PROOF. To prove this we have to consider two distinct cases. Either $P$ equals $X$ or $P$ equals $Y$ or $Z$ (which are equivalent situations). First of all, we know that $XYZ\alpha$ is a block of the so-called unital.

If $P$ equals $X$ then the block on $P$ and $S$ contains the points $I_{ty}$ and $I_{tz}$. As in a Hermitian spread any two lines have to determine a regulus of which all lines belong to the spread, we consider the two lines $\alpha YZ$ and $SI_{ty}I_{tz}$ of $\mathcal{D}_p$ and determine the unique grid they both are part of. First of all, since $xst$ is a line of $\mathcal{Q}$ the line $\alpha ST$ is an element of $\mathcal{D}_p$. Secondly, $y$ (respectively $z$) is in a Vee-line with $x$ and $i_{az}$, $i_{tz}$ ($i_{sy}$, $i_{ty}$ respectively). Hence we find $\alpha ST$, $YI_{iz}I_{sz}$ and $ZI_{ty}I_{sy}$ and consequently $TI_{sz}I_{sy}$ as lines of that particular grid.

For $P$ equal to $Y$ the block on $Y$ and $S$ is given by $YSI_{sz}I_{ty}$. By similar arguments as used above we then find $PTI_{iz}I_{sy}$ as a block of the unital and we are done.

Using previous notations we can stipulate the second lemma as follows:

Lemma 4 Let $S_\alpha$ be a fixed Hermitian spread in $\mathcal{D}_a$. Take $p$ a point of $\mathcal{Q}$ which, by definition of a spread, uniquely determines a line $L_p$ of $S_\alpha$. Then a second block of a hypothetical unital on $P$ (next to the one corresponding to $L_p$) completely fixes $S_p$.

PROOF. The findings from the previous Section tell us that $S_p$ has to be a Hermitian spread. Therefore a block on $P$ determines a second line and hence a regulus $\mathcal{R}_p$ on $L$ in $S_p$. The statement in Lemma 1 now completes the proof of this lemma.

As any hypothetical unital containing this subset (corresponding to $S_\alpha$) already contains a block through $A_{ty}$ and $A_{tz}$ a block on $X$ and $S$ will be determined by a quadric line on $t$ off $R_a$. However, by transitivity, we may choose this line to be the line $tb_{ty}b_{tz}$. Since $S_x$ has to be a Hermitian spread we thus find

\[XSB_{ty}B_{tz}\]
\[ XTB_{sy}B_{sz} \]

as elements of the yet to be completed unital.

We now want to determine a block through \( Y \) and \( S \). As \( y \) and \( s \) are opposite points in \( Q \) we need to determine a path from the one to the other and hence find a Vee-block containing both. This path cannot contain the point \( x \), as otherwise \( Y \) and \( Z \) are in two distinct blocks of the unital, nor can \( a_{sy} \) (respectively \( b_{sy} \)) be on that path (two distinct blocks on \( S \) and \( A_{sz} \) (\( S \) and \( B_{ty} \))). Hence this path has to be contained in either \( R_c \) or \( R_d \). Nevertheless, transitivity shows that these two situations are equivalent. In other words, we may assume

\[ YSC_{ty}C_{sz} \]

and consequently (Lemma 3) also

\[ YTC_{sy}C_{tz} \]

to be blocks of the unital.

Finally, considering the points \( Z \) and \( S \) leads to the uniqueness of our unital, as we shall see. Indeed, by similar arguments as used above we may exclude the lines of \( R_a \) and \( R_b \) to be in the determining path from \( z \) to \( s \). On the other hand, the lines of \( S_y \) imply that it cannot be a path in \( R_c \) neither (as otherwise we would have two blocks on \( C_{tz} \) and \( C_{sy} \)). Meaning the choice for a block through \( Z \) and \( S \) and henceforward by Lemma 4 also \( S_z \) is fixed. This same Lemma also yields that \( S_x \) and \( S_y \) are completely determined. To complete the proof of the above stated it suffices to take a general point \( p \) of \( Q \) and show that the spread \( S_p \) is fixed. Call \( L_p \) the spread line on \( p \). On \( L \) there is a unique point \( u \in \{ x, y, z \} \) collinear to \( p \). As \( S_u \) is fixed, we thereby obtain a block on \( P \) and \( U \). By Lemma 4 we obtain that \( S_p \) is fixed and we are done.

\[ \square \]

3.3 Non-Hermitian spreads imply uniqueness

This section, compared to the previous ones, might seem a bit more technical. However, we shall start from the same grids \( R_i, \ i = a, b, c, d \), as defined above. As before, we shall choose \( R_u \) as the grid containing three lines of the, in this section non-Hermitian, spread \( S_\alpha \) (transitivity again yields the choice of \( \alpha \)). One can easily read between the lines of Section 3.1 that starting the construction of the unital with a non-Hermitian spread implies that all other derived spreads have to be non-Hermitian as well.

We are now ready to show that

**Theorem 5** The \( 2-(28, 4, 5) \) Hölz design contains - up to isomorphism - a unique unital which intersects all derived subdesigns in non-Hermitian spreads.

**PROOF.** Let us begin by considering the points \( X \) and \( S \) and look at the unital block they could determine. Despite of the fact that the group at hand is by far as transitive
as before, we are still able to choose the determining line on \( t \) in the grid \( R_b \) and hence find the block 

\[ XSB_{ty}B_{tz} \]

on these two points.

Taking into account that \( S_x \) has to be non-Hermitian the line \( L \) can either determine a regulus of \( S_x \) with \( sb_{ty}B_{tz} \) or not. The first case, however, leads to a contradiction as we shall prove further on.

Before doing so, we will assemble some useful information. By construction of \( R_i \), \( i = a, b, c, d \), it is easy to see that a point \( i_1 \) is collinear to every \( j_{s_1 t_2} \), where \( \{ s_1, s_2 \} = \{ s, t \}, \{ t_1, t_2 \} = \{ y, z \} \) and \( i, j \in \{ a, b, c, d \} \). Denote the third point on the line \( b_{tz}i_{sy} \) (respectively \( b_{ty}i_{sz} \)) by \( i_1 \) (respectively \( i_2 \)). One can now easily determine following set of lines (which will come in handy later on).

\[
\begin{array}{cccccccc}
 a_1b_{tz}a_{sy} & a_2b_{ty}a_{sz} & a_1b_{sy}a_{tz} & a_2b_{sz}a_{ty} & a_1d_{sz}c_{ty} & a_2d_{sy}c_{tz} & a_1d_{ty}c_{sz} & a_2d_{tz}c_{sy} \\
\end{array}
\]

As \( S_\alpha \) will remain fixed throughout the whole of this section we wrote down the lines it contains in following table:

<table>
<thead>
<tr>
<th>Points of ( Q )</th>
<th>Blocks in ( U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x ) ( y ) ( z ) ( \alpha )</td>
<td>( X ) ( Y ) ( Z )</td>
</tr>
<tr>
<td>( s ) ( a_{sy} ) ( a_{sz} ) ( \alpha )</td>
<td>( S ) ( A_{sy} ) ( A_{sz} )</td>
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<tr>
<td>( t ) ( a_{ty} ) ( a_{tz} ) ( \alpha )</td>
<td>( T ) ( A_{ty} ) ( A_{tz} )</td>
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<tr>
<td>( a_1 ) ( d_{sz} ) ( c_{ty} ) ( \alpha )</td>
<td>( A_1 ) ( D_{sz} ) ( C_{ty} )</td>
</tr>
<tr>
<td>( a_2 ) ( d_{sy} ) ( c_{tz} ) ( \alpha )</td>
<td>( A_2 ) ( D_{sy} ) ( C_{tz} )</td>
</tr>
<tr>
<td>( c_1 ) ( b_{tz} ) ( c_{sy} ) ( \alpha )</td>
<td>( C_1 ) ( B_{tz} ) ( C_{sy} )</td>
</tr>
<tr>
<td>( c_2 ) ( b_{ty} ) ( c_{sz} ) ( \alpha )</td>
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<td>( d_1 ) ( b_{sy} ) ( d_{tz} ) ( \alpha )</td>
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<td>( d_2 ) ( b_{sz} ) ( d_{ty} ) ( \alpha )</td>
<td>( D_2 ) ( B_{sz} ) ( D_{ty} )</td>
</tr>
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</table>

Table 1: \( S_\alpha \)

**Note** that a non-Hermitian spread is completely determined by a regulus and a single line.

We are now ready to start proving the statement of this section.

Suppose \( L \) is in a regulus with \( sb_{ty}B_{tz} \). Then we have

\[ XSB_{ty}B_{tz} \]
\[ XTB_{sy}B_{sz} \]

as blocks of \( U \).
This regulus together with the lines of \( S_\alpha \) determine a unique non-Hermitian spread \( S_x \)

<table>
<thead>
<tr>
<th>Blocks cfr ( S_x )</th>
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<tbody>
<tr>
<td>( \alpha )</td>
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as we immediately obtain the line \( a_1 d_{tz} c_{sy} \) in addition to the given regulus. We now look at a block on \( Y \) and \( S \) and hence determine a suitable path from \( y \) to \( s \). A similar argument as used in previous section leads to the fact that such a path cannot belong to the grid \( R_\alpha \) nor to the grid \( R_b \). Therefore we have either \( YSC_{ty}C_{sz} \) or \( YSD_{ty}D_{sz} \) as a unital block. In the same way we find \( YTC_{tz}C_{sy} \) or \( YTD_{tz}D_{sy} \) as possible blocks. However, non of the four combinations will suffice.

First, say \( YSC_{ty}C_{sz} \) and \( YTC_{tz}C_{sy} \) are blocks of \( U \). These two blocks determine two lines which together with \( S_\alpha \) immediately force \( d_2 b_{sy} d_{sz} \) and \( d_1 b_{tz} d_{ty} \) to be elements of any spread \( S_y \) containing those two lines. As we already noted above a regulus and a single line determine all lines of a non-Hermitian spread, hence \( S_y \) is fixed. Looking at the Vee-lines on \( y \) and \( c_2 \) shows that \( c_2 d_{tz} a_{ty} \) is a spread line and hence \( YC_2 D_{tz} A_{ty} \) is a block of what ought to be a unital. Nevertheless, this alleged unital already contains the block \( XC_2 A_{ty} D_{sz} \), a contradiction.

When \( YSD_{ty}D_{sz} \) and \( YTD_{tz}D_{sy} \) are unital blocks we find a similar contradiction: with this combination corresponds a unique block on \( Y \) and \( C_2 \), namely \( YC_2 B_{sz} C_{sy} \). This thereby fixes \( S_y \) and consequently all blocks on \( Y \). The block on \( Y \) and \( D_1 \), namely \( YD_1 A_{sy} C_{sz} \), yields a contradiction with the block on \( X \) and \( D_1 \) (two blocks on \( D_1 \) and \( A_{sy} \)).

If, on the other hand, \( YSD_{ty}D_{sz} \) and \( YTC_{tz}C_{sy} \) would be in \( U \) then there is no block on \( Y \) and \( D_1 \).

Finally, say \( YSC_{ty}C_{sz} \) and \( YTD_{tz}D_{sy} \) are elements of \( U \). Then one can easily see that \( ZSC_{tz}C_{sy} \) and \( ZTD_{sz}D_{ty} \) also have to be. These blocks, however, can never be in a unital with \( S_\alpha \) and \( S_x \) since there will be no block on \( S \) and \( D_{sy} \), such a block is determined by a quadric line through \( d_{sz} \), which give contradictions with

\[
\begin{align*}
a_1 d_{tz} c_{ty} \\
z d_{sz} d_{tz} \\
c_1 a_{ty} d_{sz} \\
d_2 b_{ty} d_{sz}
\end{align*}
\]

\( S_\alpha \), \( S_y \), \( S_z \) and \( S_x \) respectively.
In other words, there exists no $S_y$ compatible with $S_\alpha$ and $S_x$ as given in Table 2. Hence we have to re-evaluate the choices we made to construct $S_x$.

Before $S_x$ contained $XTB_{sy} B_{sz}$, next to the fixed block $XSB_{ty} B_{tz}$, as a block on $X$ and $T$. This, however, leads to a contradiction. The block on $X$ and $T$ will therefore be given by

$$XTD_{sy} D_{sz}$$

or by

$$XTC_{sy} C_{sz}$$

corresponding to lines of the grids $R_d$ and $R_e$ respectively. To prove that the latter block cannot occur in a unital containing the block $XSB_{ty} B_{tz}$ and all blocks of Table 1 we first prove following lemma:

**Lemma 6** Take $S$ a non-Hermitian spread of the generalized quadrangle of order $(2,4)$. Suppose $R$ is one of the three reguli on $S$. Any regulus on a line $M$ of the complementary regulus $R^c$ contains no or two lines of the spread.

**PROOF.** To prove this lemma we consider the dual situation, i.e. an ovoid $H(2,4)$ of $H(3,4)$ in which we replace the points on a line $L$ by the points on $L^\perp$, the polar line of $L$. Let the lines of $R$ correspond to the points on $L^\perp$, then $M$ corresponds to one of the points, say $m$, on $L$. A regulus on $M$ translates into a line containing the point $m$, which obviously intersects $H(3,4)$ in no $(L)$ or two points.

\[\square\]

An immediate consequence of this lemma is that every line on $t$ (not in $R_\alpha$) determines a regulus, and consequently also a grid of the quadric, on $xst$ containing two lines of $S_\alpha$. The line $tb_{ty} b_{tz}$ determines such a grid with $xst$ containing $Sc_{sy}C_{sz}$ as a line and this will be the reason why $XTC_{sy} C_{sz}$ cannot be in a unital with $XSB_{ty} B_{tz}$ and $S_\alpha$.

Suppose, by way of contradiction, that the opposite is true and consider $a_1$ and $a_2$ as introduced above. A block on $X$ and $A_1$ is determined by

\[
\begin{bmatrix}
  a_2b_{ty}a_{sz} \\
  a_2b_{sz}a_{ty} \\
  a_2d_{sy}c_{tz} \\
  a_2d_{tz}c_{sy}
\end{bmatrix}
\]

one of these lines on $a_2$. As, in this particular case, $X$ is already in a block of $U$ with $B_{ty}$, with $C_{sy}$ and $a_2d_{sy}c_{tz}$ is an element of $S_\alpha$, we conclude that

$$XA_1 B_{sz} A_{ty}$$

is the only possible block of $U$ on these two points. In the same way we find that

$$XA_2 B_{sy} A_{tz}$$
has to belong to \( \mathcal{U} \). However the corresponding set of four lines in \( \mathcal{D}_x \) cannot be completed into a spread, as we shall show. First of all, one can easily see that the blocks

\[ XC_1 A_{sy} D_{tz} \]
\[ XC_2 A_{sz} D_{ty} \]

automatically determine two other spread lines. There are now no possibilities left for spread lines on \( d_1 \) and \( d_2 \).

Conclusion, given the fixed non-Hermitian spread \( S_\alpha \) and the block on \( X \) and \( S \) any unital containing corresponding blocks will also contain the block \( XTD_{sy}D_{sz} \). If so, then considering the lines on \( d_2 \) (\( d_1 \) respectively) yields two distinct possibilities for blocks on \( X \) and \( D_1 \) (\( D_2 \) respectively). Two out of four combinations, however, lead to a contradiction and the remaining two combinations will be shown to be isomorphic. The block on \( X \) and \( D_1 \) can either be determined by \( d_1 a_{tz} c_{sy} \) or by \( d_2 a_{sz} c_{ty} \), as where the one on \( X \) and \( D_2 \) is by \( d_1 a_{ty} c_{sz} \) or by \( d_1 a_{sz} c_{ty} \). In chronological order these situations will be denoted by increasing numbers 1 to 4.

A combination of situation 1 with situation 3 leads to a contradiction as there remains no acceptable block on \( X \) and \( A_2 \). In the same way the second and fourth situation allow no block on \( X \) and \( C_1 \).

Situation 1 and 3 and situation 2 and 4, on the other hand, lead to unique non-Hermitian spreads \( S_x \) and \( S'_x \) respectively.

<table>
<thead>
<tr>
<th>Blocks cfr ( S_x )</th>
<th>Blocks cfr ( S'_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X ) ( S ) ( B_{ty} ) ( B_{tz} )</td>
<td>( X ) ( S ) ( B_{ty} ) ( B_{tz} )</td>
</tr>
<tr>
<td>( X ) ( T ) ( D_{sy} ) ( D_{sz} )</td>
<td>( X ) ( T ) ( D_{sy} ) ( D_{sz} )</td>
</tr>
<tr>
<td>( X ) ( A_1 ) ( D_{tz} ) ( C_{sy} )</td>
<td>( X ) ( A_1 ) ( B_{sz} ) ( A_{ty} )</td>
</tr>
<tr>
<td>( X ) ( A_2 ) ( B_{sy} ) ( A_{tz} )</td>
<td>( X ) ( A_2 ) ( D_{ty} ) ( C_{sz} )</td>
</tr>
<tr>
<td>( X ) ( C_1 ) ( B_{sz} ) ( C_{ty} )</td>
<td>( X ) ( C_1 ) ( B_{sy} ) ( C_{tz} )</td>
</tr>
<tr>
<td>( X ) ( C_2 ) ( A_{sz} ) ( D_{ty} )</td>
<td>( X ) ( C_2 ) ( A_{sy} ) ( D_{tz} )</td>
</tr>
<tr>
<td>( X ) ( D_1 ) ( A_{sy} ) ( C_{tz} )</td>
<td>( X ) ( D_1 ) ( A_{sz} ) ( C_{ty} )</td>
</tr>
<tr>
<td>( X ) ( D_2 ) ( A_{ty} ) ( C_{sz} )</td>
<td>( X ) ( D_2 ) ( A_{sz} ) ( C_{ty} )</td>
</tr>
<tr>
<td>( X ) ( \alpha ) ( Y ) ( Z )</td>
<td>( X ) ( \alpha ) ( Y ) ( Z )</td>
</tr>
</tbody>
</table>

Nevertheless, while fixing \( S_\alpha \) we can map \( S_x \) onto \( S'_x \) by applying the group element which fixes \( x \), \( s \) and \( t \) and switches all pairs \( (i,j) \) with \( xi j \), \( si j \) or \( tij \) a line of \( \mathcal{Q} \). Hence it suffices to proceed using \( S_x \) as the non-Hermitian spread of \( \mathcal{D}_x \) in \( \mathcal{U} \).

To complete the proof of Theorem 5 we shall, just as in the previous section, consider the blocks on \( Y \) and \( Z \). First of all, taking into account the blocks on \( \alpha \), \( X \) and \( S \), \( T \) a block of \( \mathcal{U} \) on \( Y \) and \( S \) (respectively \( T \)) and also on \( Z \) and \( S \) (respectively \( T \)) has to be determined by lines of \( R_c \) or \( R_d \) (respectively \( R_b \) or \( R_c \)).

Suppose we have a block on \( Y \) and \( S \) in \( R_c \) and the one on \( Y \) and \( T \) in \( R_b \) (namely \( YSC_{ty}C_{sz} \), \( YTB_{sy}B_{tz} \)). This choice of blocks immediately forces (in this order)

\[ YA_2 A_{sz} B_{tz}, YC_2 D_{tz} A_{ty}, YA_1 A_{tz} B_{ty} \]
and

\[ YC_1 D_{sy} A_{sz} \]

to be elements of the unital. However, this leaves us no choice for a block on \( Y \) and \( D_2 \), a contradiction. In the same way

\[ YSD_{ty} D_{sz} \]

together with

\[ YTC_{sy} C_{tz} \]

leads to a situation where there is no acceptable block on \( Y \) and \( D_1 \).

If both the block on \( Y \) and \( S \) and the one on \( Y \) and \( T \) are determined by \( R_c \), then we are able to complete this set of blocks on \( Y \) into a spread \( S_y \). Nevertheless, these two blocks force us to take

\[ ZSD_{tz} D_{sy} \]
\[ ZTB_{ty} B_{sz} \]

as blocks on \( Z \) and this combination of blocks can never be in a unital of \( D \), as we shall show. Indeed, a block on \( S \) and \( B_{sy} \) is determined by one of the non-spread lines on \( b_{sz} \)

\[
\begin{array}{c}
  b_{sz} a_{ty} a_2 \\
  b_{sz} c_{ty} c_2 \\
  b_{sz} b_{tz} z
\end{array}
\]

and each of these lines give a contradiction with the known blocks of \( S_x \), \( S_z \) and \( S_y \) respectively. Hence the blocks of \( U \) on \( Y \) and \( S \) and \( T \) respectively are uniquely determined by \( R_d \) and \( R_b \). Finally, considering the possible blocks on \( Z \) and these two points we find

\[ ZSC_{sy} C_{tz} \]
\[ ZTC_{ty} C_{sz} \]

as the only plausible combination (all other combinations allow no block on \( Z \) and \( D_2 \)). These two sets can be completed into non-Hermitian spreads \( S_y \) and \( S_z \), which are compatible with \( S_\alpha \), and this in a unique way.
Note that the spread lines in previous table are denoted in order that they are forced to be so-called spread lines.

To end the proof of this theorem it suffices to take a general point $p$ of $Q$ and show that the spread $S_p$ is fixed. Before coming to this part we claim that $S_p$ is fixed for all $p \in x^\perp$.

First, take $p$ equal to $a_1$. From previous findings we already know four out of the nine spread lines of $S_{a_1}$, say $L_1, \ldots, L_4$. Showing that these four lines are as such that both $L_1$ and $L_2$ are not in a regulus contained in $S_{a_1}$ with $L_3$ and $L_4$; nor is $L_3$ with $L_4$ implies the uniqueness of $S_{a_1}$. Indeed, if this is the case then $L_1$ and $L_2$ necessarily determine a regulus of the spread and hence $S_{a_1}$ is fixed. Take

$$
L_1 = y d_{sy} c_{sz} \\
L_2 = z a_{sz} b_{sy} \\
L_3 = \alpha d_{sz} c_{ty} \\
L_4 = x d_{sz} c_{ty}
$$

as the four known lines. After some calculations we find following lines $M_{ij}$

$$
M_{13} = s d_{ty} c_{sy} \\
M_{14} = c_{tz} d_{sz} t \\
M_{23} = a_{tz} d_{tz} d_1 \\
M_{24} = b_{ty} c_{ty} d_2 \\
M_{34} = a_2 y z \\
M_{12} = a_2 c_1 c_2
$$

as third lines in $R(L_i, L_j)$. Since $S$ and $C_{sy}$, $T$ and $D_{sz}$, $A_1$ and $D_{tz}$, $B_{ty}$ and $D_2$ and finally $Y$ and $Z$ are already in blocks of $S_x$, $S_y$, $S_z$, $S_{a_1}$ and $S_{a_2}$ respectively, we find on the one hand that $A_1A_2C_1C_2$ is a block of the unital and on the other hand that $S_{a_1}$ is fixed.

For $p$ equal to $a_2$ we immediately find a regulus of $S_{a_2}$, namely

$$
\begin{align*}
\alpha & d_{sy} c_{tz} \\
x & b_{sy} a_{tz} \\
a_1 & c_1 c_2
\end{align*}
$$

and hence also $S_{a_2}$ is fixed.

For $p \in \{s, t, d_1, d_2\}$ we know that $STD_1D_2$ determines a first line of the spread $S_p$. Apart from this line we have six other, two by two distinct, lines (corresponding to $S_x$, $S_y$, $S_z$, $S_{a_1}$ and $S_{a_2}$) and obviously seven out of nine lines of the spread completely determine the spread.

If $p$ equals $c_1$ or $c_2$ we obtain at least seven distinct lines of $S_p$ whens considering all previous constructed spreads. Hence $S_p$ is fixed.

Finally, consider $p$ any point of $Q$ which is non-collinear to $x$. Then $S_p$ is determined by the unique elements of $S_x$, $S_y$ and $S_{M_i^p}$ (with $M_i, i \in \{1, \ldots, 5\}$, a line on $x$ and $M_i^p$ the projection of $p$ onto $M_i$) it belongs to. On can easily see that we thus establish a line set which uniquely determines all lines of $S_p$ and we are done.

\hfill \square
References


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