Unitals in the Hölz-design on 28 points

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Abstract

In this paper, we provide a theoretical proof of the fact that the only unitals contained in the $2-(28,4,5)$ Hölz design are Hermitian and Ree unitals (confer a computer search by Tonchev, [8]).

1 Introduction

In 1981, Hölz [3] constructed a family of $2-(q^3+1, q+1, q+2)$-designs whose point set coincides with the point set of the Hermitian unital over the field $\mathbf{GF}(q)$, and with an automorphism group containing $\mathbf{PGU}_3(q)$. Here, q is any odd prime power. Two years later, Thas [7] proved that these designs are one-point extensions of the Ahrens-Szekeres generalized quadrangles $AS(q)$ of order $(q-1, q+1)$ (see [1]).
In a previous paper [9] the authors gave an alternative construction of the Hölz design, for $q \not\equiv 2 \bmod 3$ making use of two hexagons embedded in the parabolic quadric $\mathsf{Q}(6,q)$.
In 1991, Tonchev [8] shows "by a computer search" that the only unitals contained in the $2-(28,4,5)$ Hölz design are Hermitian and Ree unitals. Using our findings from [9], we came across the following construction of that particular Hölz design.
Take Γ the unique generalized quadrangle of order $(2,4)$. We define $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ with point and block set deduced from Γ. Define $\mathcal{P}$ as the point set of Γ to which we add a new point α. The set $\mathcal{B}$ contains two types of blocks: blocks of type (a) contain the point α together with three points of any line of Γ (*Line-block*) and those of type (b) contain the four points of any two intersecting lines of Γ, which are distinct of the intersection point (*Vee-block*). It is now easy to see that $\mathcal{D}$ is a $2-(28,4,5)$ design.
In the present note we use previous findings to give a computer-free proof of the result in [8].

2 Preliminaries

A $t-(v,k,\lambda)$ *design*, for integers t, v, k and λ with $v > k > 1$ and $k \geq t \geq 1$, is an incidence structure $\mathcal{D}$ satisfying following axioms: $\mathcal{D}$ contains v points; each of its blocks

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is incident with k points; any t points are incident with exactly λ common blocks. For further information on designs we refer to [4].

The following class of 2-designs is due to G. Hölz [3]. Let $\mathcal{U}$ be a hermitian curve of $\mathbf{PG}(2, q^2)$ [2]. A Baer subplane [5] $\mathbf{PG}(2, q) = D$ is said to satisfy property (H) if for each point $x \in D \cap \mathcal{U}$ the tangent line L_x to $\mathcal{U}$ at x is a line of D (i.e. $|L_x \cap D| = q+1$). If D satisfies this property (H) then one can show that if $|D \cap \mathcal{U}| \geq 3$ then $|D \cap \mathcal{U}| = q+1$, for q even the points of $D \cap \mathcal{U}$ are collinear, and for q odd the points of $D \cap \mathcal{U}$ are collinear or form an oval in D. If D_1 and D_2 are Baer subplanes satisfying property (H) and if $|D_1 \cap D_2 \cap \mathcal{U}| \geq 3$, then $D_1 \cap \mathcal{U} = D_2 \cap \mathcal{U}$. If moreover $D_i \cap \mathcal{U}$ is an oval of D_i, then $D_1 = D_2$.
Let q be odd. If x and y are distinct points of $\mathcal{U}$, then (1) there are exactly $q+1$ Baer subplanes D in $\mathbf{PG}(2, q^2)$ which satisfy property (H) and for which $D \cap \mathcal{U} = xy \cap \mathcal{U}$, and (2) there are exactly $q+1$ Baer subplanes D in $\mathbf{PG}(2, q^2)$ which satisfy property (H) and for which $D \cap \mathcal{U}$ is an oval of D through x and y. Let B_1 be the set of all intersections $L \cap \mathcal{U}$ with L a non tangent line of $\mathcal{U}$, and let B' be the set of all intersections $D \cap \mathcal{U}$ with D a Baer subplane of $\mathbf{PG}(2, q^2)$ satisfying property (H) and containing at least three points of $\mathcal{U}$. Finally, let $B^* = B' - B_1$
Clearly $S_1 = (\mathcal{U}, B_1, \in)$ is a $2-(q^3+1, q+1, 1)$ design and $S' = (\mathcal{U}, B', \in)$ is a $2-(q^3+1, q+1, q+2)$ design and $S^* = (\mathcal{U}, B^*, \in)$ is a $2-(q^3+1, q+1, q+1)$ design. Moreover any two distinct blocks of these designs have at most two points in common.

A *generalized quadrangle* Γ (of order (s,t)) is a point-line geometry the incidence graph of which has diameter 4 and girth 8 (and every line is incident with $s+1$ points; every point incident with $t+1$ lines). Note that, if $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set of Γ, then the *incidence graph* is the (bipartite) graph with set of vertices $\mathcal{P} \cup \mathcal{L}$ and adjacency given by incidence. The definition implies that, given any two elements a, b of $\mathcal{P} \cup \mathcal{L}$, either these elements are at distance 4 from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path from a to b. In other words, given any non-incident point-line pair, say (p, L), there exists a unique point on the line L which is collinear with p.

A *spread* of the generalized quadrangle Γ is a set of lines of Γ partitioning the point set into lines. In other words, every point of Γ is incident with a unique line of the spread.

Let $S = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ be a generalized quadrangle of order (s,t) with $|\mathcal{P}| = v$ and $|\mathcal{B}| = b$. The $(i+1)-(v+i, s+1+i, t+1)$ design $S' = (\mathcal{P}', \mathcal{B}', \mathtt{I}')$ is said to be an *i-th extension* of S if for any i distinct points $x_1, \ldots x_i$ of $\mathcal{P}'$ the derived structure of S' with respect to $x_1, \ldots, x_i$ is isomorphic to S. Recall that the derived structure of S' with respect to $x_1, \ldots x_i$ is the 1-design $S'(x_1, \ldots x_i) = (\mathcal{P}'(x_1, \ldots x_i), \mathcal{B}'(x_1, \ldots x_i), \mathtt{I}'(x_1, \ldots x_i))$ with $\mathcal{P}'(x_1, \ldots x_i) = \mathcal{P}' \setminus \{x_1, \ldots x_i\}$, $\mathcal{B}'(x_1, \ldots x_i)$ the set of all blocks of $\mathcal{B}'$ incident with $x_1, \ldots x_i$, and $\mathtt{I}'(x_1, \ldots x_i)$ the incidence induced by $\mathtt{I}'$. A *first extension* of S is shortly called an *extension* of S.

3 Main Result

In this section we shall use the construction of the $2-(28,4,5)$ design, $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$, as given in Section 1. More explicit, we take a $\mathcal{Q}^-(5,2)$ (say $\mathcal{Q}$) in $\mathbf{PG}(5,2)$ and embed this 5-space as a hyperplane, H, into $\mathbf{PG}(6,2)$. Let α be any point of $\mathbf{PG}(6,2)$ not in H. Then the points of $\mathcal{D}$ can also be seen as a set of affine points in $\mathbf{AG}(6,2)$, namely all points on the affine lines αx, with x a point of $\mathcal{Q}$. In the mean time the blocks of $\mathcal{D}$ contain 4 points which determine affine planes in $\mathbf{AG}(6,2)$. First of all, we have all affine planes through α intersecting H in a line of the generalized quadrangle and secondly, the four points in the disjoint union of two intersecting lines of $\mathcal{Q}$ determine two affine lines of $\mathbf{AG}(6,2)$ which intersect H in the same point (namely that intersection point). Hence a Vee-block also determines four points of an affine plane. To simplify notation we shall denote the affine point on αx, distinct of α, by X.

A unital $\mathcal{U}$ of $\mathcal{D}$ is by definition a subset of $\mathcal{B}$ such that any two points of $\mathcal{D}$ are contained in a unique block of $\mathcal{U}$. Hence, for every point x in $\mathcal{D}$ such a unital defines a spread, denoted by S_x, in the derived quadrangle $\mathcal{D}_x$ of this particular point. It is well known that the generalized quadrangle of order $(2,4)$ has two non-isomorphic spreads: first, the Hermitian spread (in which any two lines determine a line regulus completely contained in it) and second a spread obtained by switching one of the line reguli of this previous spread. For further reference we shall denote this latter spread by non-Hermitian.

3.1 One Hermitian spread implies all Hermitian spreads

In this subsection we will be working under the assumption that $\mathcal{U}$ determines at least one Hermitian spread and we will show that in this particular case all other derived spreads have to be Hermitian as well.

Without loss of generality we may assume that we obtain such a Hermitian spread by a one-point derivation in α. Hence, S_α determines 9 lines of $\mathcal{Q}$.
Take L any one of those lines and say x, y and z are the points incident with this line in $\mathcal{Q}$. A what turns out to be very useful property of a non-Hermitian spread is that it is the union of three disjoint line reguli and these are in fact the only reguli it contains. Meaning, every line of a non-Hermitian spread is contained in a unique regulus of the spread.

Lemma 1 *S_x is completely fixed by the line regulus on the line L of S_α.*

PROOF. Take $\mathcal{R}_x$ a line regulus of S_x on L (this is possible, independent of S_x being Hermitian or not).
Corresponding to the two types of blocks of $\mathcal{D}$ on X we have two types of lines in $\mathcal{D}_x$. The first type corresponds to the Line-blocks, the second to the Vee-blocks. As L results from a block containing α and X the spread S_x cannot contain any more lines of this first type. Suppose $\mathcal{R}_x$ contains L and a Vee-line through the line xst of $\mathcal{Q}$. There are then four grids R_i, $i = a,b,c,d$, in $\mathcal{Q}$ containing both lines xyz and xst and in each of

these grids we denote the unique point collinear with $*$ and $*'$ by $i_{**'}$, where $* \in \{s,t\}$ and $*' \in \{y,z\}$. If $\mathcal{R}_a$ is the grid containing the spread lines of S_α then by transitivity we may assume $sb_{ty}b_{tz}$ and consequently also $tb_{sy}b_{sz}$ to be a line of $\mathcal{R}_x$ ($sa_{ty}a_{tz}$ is impossible as otherwise A_{ty} and A_{tz} would be in two distinct blocks of $\mathcal{U}$).
Every point $b_{**'}$ is incident with a unique line of S_α. As non of these four points are on the same spread line they determine four distinct elements of S_α. These four lines together with the spread lines in R_a add up to seven of the nine lines of that particular spread. Meaning there are only two spread lines left which are in a grid with L, as we shall show. Indeed, as S_α is a Hermitian spread every one of those four lines is in a regulus with L. Considering the regulus on L and the line incident with, for instance, the point b_{sy} we can see that this regulus necessarily contains the spread line through the point b_{ty} as well. In the same way L is in a regulus with the lines incident with b_{sz} and b_{tz}. Hence also L and the two remaining lines are three lines of a grid R'. Consider the second non-spread line, say xuv, of R' on x. We shall now look at the possible spread lines of S_x through u and v respectively.
Such a spread line in fact corresponds to a block of $\mathcal{U}$ through X and U (respectively V). The block through X and U is completely determined by a line through v.

Suppose the non-spread line on y of R' belongs to R_c, then consequently the one on z belongs to R_d. We will now show that the remaining lines on y and z (namely in R_d and R_c respectively) are in a grid, say R'', with L and xuv. Suppose by way of contradiction that the line ud_{*_1y} intersects either R_a or R_b, say R_j with $j \in \{a,b\}$. Not allowing triangles in $\mathcal{Q}$ the third point on this line necessarily has to be the point j_{*_2z}, with $\{*_1,*_2\} = \{s,t\}$. This, on its turn, implies that $vd_{*_2y}j_{*_1z}$ is a quadric line. Hence, since $x \sim *_2$ and $v \sim d_{*_2y}$ we find that $u \sim d_{*_2z}$ and in the same way that $u \sim j_{*_1y}$. However, given the definition of these points, it is obvious that $d_{*_2z} \sim j_{*_1y}$ and thus we have our contradiction (ud_{*_2z} contains a point of R_c as we know from R').

By the defining property of a generalized quadrangle every point outside a grid is collinear with exactly three lines of the grid. So if we consider the grid R_b and the point u then the spread line on u and the line L_u (which is the line of R'' on u distinct of xuv) are the only two lines not intersecting this particular grid. However a block on X and V is determined by such a line on u which does not equal xuv and does not contain any of the points $b_{**'}$ (as X is already in a block with each of the points $B_{**'}$). Hence this block and consequently also the block on X and U is determined by lines of R''.

In S_x we now have two reguli on a line. Meaning, S_x has to be a Hermitian spread and as we can see in the dual situation in $\mathsf{H}(3,4)$, this Hermitian spread is completely fixed (in $\mathsf{H}(3,4)$ this translates into two lines through a point fixing the plane which determines the ovoid).

□

As L and x were chosen arbitrary this holds for every point x in $\mathcal{Q}$.

3.2 Hermitian spreads imply uniqueness

In this section we show that

Theorem 2 *The* $2-(28,4,5)$ *Hölz design contains - up to isomorphism - a unique unital which intersects all derived subdesigns in Hermitian spreads.*

PROOF. Without loss of generality we may fix a Hermitian spread in the derived generalized quadrangle $\mathcal{D}_\alpha$. If the construction of a unital containing this spread is hereby fixed - up to isomorphism - then the above stated is proven.

Suppose $L = xyz$ is a spread line of S_α. Consider xst, another line of the quadric and denote the grids through L and this line by R_i, $i = a, b, c, d$ and use the same notation of the point set as introduced in previous section. Furthermore, say R_a is the unique grid containing three spread lines of S_α.

With these definitions we shall prove two short lemmas. First, we have

Lemma 3 *Suppose all* S_p*, for* $P \in \mathcal{D}$*, are Hermitian spreads. Then if the block of* $\mathcal{U}$ *on* $P \in \{X, Y, Z\}$ *and* S *is determined by a line of* R_i *then so is the block on* P *and* T*.*

PROOF. To prove this we have to consider two distinct cases. Either P equals X or P equals Y or Z (which are equivalent situations). First of all, we know that $XYZ\alpha$ is a block of the so-called unital.
If P equals X then the block on P and S contains the points I_{ty} and I_{tz}. As in a Hermitian spread any two lines have to determine a regulus of which all lines belong to the spread, we consider the two lines αYZ and $SI_{ty}I_{tz}$ of $\mathcal{D}_p$ and determine the unique grid they both are part of. First of all, since xst is a line of $\mathcal{Q}$ the line αST is an element of $\mathcal{D}_p$. Secondly, y (respectively z) is in a Vee-line with x and i_{sz}, i_{tz} (i_{sy}, i_{ty} respectively). Hence we find αST, $YI_{tz}I_{sz}$ and $ZI_{ty}I_{sy}$ and consequently $TI_{sz}I_{sy}$ as lines of that particular grid.

For P equal to Y the block on Y and S is given by $YSI_{sz}I_{ty}$. By similar arguments as used above we then find $PTI_{tz}I_{sy}$ as a block of the unital and we are done.

□

Using previous notations we can stipulate the second lemma as follows:

Lemma 4 *Let* S_α *be a fixed Hermitian spread in* $\mathcal{D}_\alpha$*. Take* p *a point of* $\mathcal{Q}$ *which, by definition of a spread, uniquely determines a line* L_p *of* S_α*. Then a second block of a hypothetical unital on* P *(next to the one corresponding to* L_p*) completely fixes* S_p*.*

PROOF. The findings from the previous Section tell us that S_p has to be a Hermitian spread. Therefore a block on P determines a second line and hence a regulus $\mathcal{R}_p$ on L in S_p. The statement in Lemma 1 now completes the proof of this lemma.

□

As any hypothetical unital containing this subset (corresponding to S_α) already contains a block through A_{ty} and A_{tz} a block on X and S will be determined by a quadric line on t off R_a. However, by transitivity, we may choose this line to be the line $tb_{ty}b_{tz}$. Since S_x has to be a Hermitian spread we thus find

$$XSB_{ty}B_{tz}$$

$$XTB_{sy}B_{sz}$$

as elements of the yet to be completed unital.

We now want to determine a block through Y and S. As y and s are opposite points in $\mathcal{Q}$ we need to determine a path from the one to the other and hence find a Vee-block containing both. This path cannot contain the point x, as otherwise Y and Z are in two distinct blocks of the unital, nor can a_{sy} (respectively b_{sy}) be on that path (two distinct blocks on S and A_{sz} (S and B_{ty})). Hence this path has to be contained in either R_c or R_d. Nevertheless, transitivity shows that these two situations are equivalent. In other words, we may assume

$$YSC_{ty}C_{sz}$$

and consequently (Lemma 3) also

$$YTC_{sy}C_{tz}$$

to be blocks of the unital.

Finally, considering the points Z and S leads to the uniqueness of our unital, as we shall see. Indeed, by similar arguments as used above we may exclude the lines of R_a and R_b to be in the determining path from z to s. On the other hand, the lines of S_y imply that it cannot be a path in R_c neither (as otherwise we would have two blocks on C_{tz} and C_{sy}). Meaning the choice for a block through Z and S and henceforward by Lemma 4 also S_z is fixed. This same Lemma also yields that S_x and S_y are completely determined. To complete the proof of the above stated it suffices to take a general point p of $\mathcal{Q}$ and show that the spread S_p is fixed. Call L_p the spread line on p. On L there is a unique point $u \in \{x, y, z\}$ collinear to p. As S_u is fixed, we thereby obtain a block on P and U. By Lemma 4 we obtain that S_p is fixed and we are done.

$\square$

3.3 Non-Hermitian spreads imply uniqueness

This section, compared to the previous ones, might seem a bit more technical. However, we shall start from the same grids R_i, $i = a, b, c, d$, as defined above. As before, we shall choose R_a as the grid containing three lines of the, in this section non-Hermitian, spread S_α (transitivity again yields the choice of α). One can easily read between the lines of Section 3.1 that starting the construction of the unital with a non-Hermitian spread implies that all other derived spreads have to be non-Hermitian as well.

We are now ready to show that

Theorem 5 *The $2-(28, 4, 5)$ Hölz design contains - up to isomorphism - a unique unital which intersects all derived subdesigns in non-Hermitian spreads.*

PROOF. Let us begin by considering the points X and S and look at the unital block they could determine. Despite of the fact that the group at hand is by far as transitive as before, we are still able to choose the determining line on t in the grid R_b and hence find the block

$$XSB_{ty}B_{tz}$$

on these two points.

Taking into account that S_x has to be non-Hermitian the line L can either determine a regulus of S_x with $sb_{ty}b_{tz}$ or not. The first case, however, leads to a contradiction as we shall prove further on.

Before doing so, we will assemble some useful information. By construction of R_i, $i = a, b, c, d$, it is easy to see that a point $i_{*_1 *'_1}$ is collinear to every $j_{*_2 *'_2}$, where $\{*_1, *_2\} = \{s, t\}$, $\{*'_1, *'_2\} = \{y, z\}$ and $i, j \in \{a, b, c, d\}$. Denote the third point on the line $b_{tz}i_{sy}$ (respectively $b_{ty}i_{sz}$) by i_1 (respectively i_2). One can now easily determine following set of lines (which will come in handy later on).

$a_1b_{tz}a_{sy}$	$a_2b_{ty}a_{sz}$	$c_1b_{tz}c_{sy}$	$c_2b_{ty}c_{sz}$	$d_1b_{tz}d_{sy}$	$d_2b_{ty}d_{sz}$
$a_1b_{sy}a_{tz}$	$a_2b_{sz}a_{ty}$	$c_1b_{sy}c_{tz}$	$c_2b_{sz}c_{ty}$	$d_1b_{sy}d_{tz}$	$d_2b_{sz}d_{ty}$
$a_1d_{sz}c_{ty}$	$a_2d_{sy}c_{tz}$	$c_1a_{ty}d_{sz}$	$c_2a_{tz}d_{sy}$	$d_1a_{ty}c_{sz}$	$d_2a_{tz}c_{sy}$
$a_1d_{ty}c_{sz}$	$a_2d_{tz}c_{sy}$	$c_1a_{sz}d_{ty}$	$c_2a_{sy}d_{tz}$	$d_1a_{sz}c_{ty}$	$d_2a_{sy}c_{tz}$

As S_α will remain fixed throughout the whole of this section we wrote down the lines it contains in following table:

Points of $\mathcal{Q}$			**Blocks in $\mathcal{U}$**			
x	y	z	α	X	Y	Z
s	a_{sy}	a_{sz}	α	S	A_{sy}	A_{sz}
t	a_{ty}	a_{tz}	α	T	A_{ty}	A_{tz}
a_1	d_{sz}	c_{ty}	α	A_1	D_{sz}	C_{ty}
a_2	d_{sy}	c_{tz}	α	A_2	D_{sy}	C_{tz}
c_1	b_{tz}	c_{sy}	α	C_1	B_{tz}	C_{sy}
c_2	b_{ty}	c_{sz}	α	C_2	B_{ty}	C_{sz}
d_1	b_{sy}	d_{tz}	α	D_1	B_{sy}	D_{tz}
d_2	b_{sz}	d_{ty}	α	D_2	B_{sz}	D_{ty}

Table 1: S_α

Note that a non-Hermitian spread is completely determined by a regulus and a single line.

We are now ready to start proving the statement of this section.

Suppose L is in a regulus with $sb_{ty}b_{tz}$. Then we have

$$XSB_{ty}B_{tz}$$

$$XTB_{sy}B_{sz}$$

as blocks of $\mathcal{U}$.

This regulus together with the lines of S_α determine a unique non-Hermitian spread S_x

Blocks cfr S_x			
α	X	Y	Z
X	S	B_{ty}	B_{tz}
X	T	B_{sy}	B_{sz}
X	A_1	D_{tz}	C_{sy}
X	A_2	D_{ty}	C_{sz}
X	C_1	A_{tz}	D_{sy}
X	C_2	A_{ty}	D_{sz}
X	D_1	A_{sy}	C_{tz}
X	D_2	A_{sz}	C_{ty}

as we immediately obtain the line $a_1d_{tz}c_{sy}$ in addition to the given regulus. We now look at a block on Y and S and hence determine a suitable path from y to s. A similar argument as used in previous section leads to the fact that such a path cannot belong to the grid R_a nor to the grid R_b. Therefore we have either $YSC_{ty}C_{sz}$ or $YSD_{ty}D_{sz}$ as a unital block. In the same way we find $YTC_{tz}C_{sy}$ or $YTD_{tz}D_{sy}$ as possible blocks. However, non of the four combinations will suffice.

First, say $YSC_{ty}C_{sz}$ and $YTC_{tz}C_{sy}$ are blocks of $\mathcal{U}$. These two blocks determine two lines which together with S_α immediately force $d_2b_{sy}d_{sz}$ and $d_1b_{tz}d_{ty}$ to be elements of any spread S_y containing those two lines. As we already noted above a regulus and a single line determine all lines of a non-Hermitian spread, hence S_y is fixed. Looking at the Vee-lines on y and c_2 shows that $c_2d_{tz}a_{ty}$ is a spread line and hence $YC_2D_{tz}A_{ty}$ is a block of what ought to be a unital. Nevertheless, this alleged unital already contains the block $XC_2A_{ty}D_{sz}$, a contradiction.

When $YSD_{ty}D_{sz}$ and $YTD_{tz}D_{sy}$ are unital blocks we find a similar contradiction: with this combination corresponds a unique block on Y and C_2, namely $YC_2B_{sz}C_{sy}$. This thereby fixes S_y and consequently all blocks on Y. The block on Y and D_1, namely $YD_1A_{sy}C_{sz}$, yields a contradiction with the block on X and D_1 (two blocks on D_1 and A_{sy}).

If, on the other hand, $YSD_{ty}D_{sz}$ and $YTC_{tz}C_{sy}$ would be in $\mathcal{U}$ then there is no block on Y and D_1.

Finally, say $YSC_{ty}C_{sz}$ and $YTD_{tz}D_{sy}$ are elements of $\mathcal{U}$. Then one can easily see that $ZSC_{tz}C_{sy}$ and $ZTD_{sz}D_{ty}$ also have to be. These blocks, however, can never be in a unital with S_α and S_x since there will be no block on S and D_{sy}: such a block is determined by a quadric line through d_{sz}, which give contradictions with

$a_1d_{sz}c_{ty}$
$zd_{sz}d_{tz}$
$c_1a_{ty}d_{sz}$
$d_2b_{ty}d_{sz}$

S_α, S_y, S_x and S_x respectively.

In other words, there exists no S_y compatible with S_α and S_x as given in Table 2. Hence we have to re-evaluate the choices we made to construct S_x.

Before S_x contained $XTB_{sy}B_{sz}$, next to the fixed block $XSB_{ty}B_{tz}$, as a block on X and T. This, however leads to a contradiction. The block on X and T will therefore be given by

$$XTD_{sy}D_{sz}$$

or by

$$XTC_{sy}C_{sz}$$

corresponding to lines of the grids R_d and R_c respectively. To prove that the latter block cannot occur in a unital containing the block $XSB_{ty}B_{tz}$ and all blocks of Table 1 we first prove following lemma:

Lemma 6 *Take S a non-Hermitian spread of the generalized quadrangle of order (2,4). Suppose $\mathcal{R}$ is one of the three reguli on S. Any regulus on a line M of the complementary regulus $\mathcal{R}^c$ contains no or two lines of the spread.*

PROOF. To prove this lemma we consider the dual situation, i.e. an ovoid $\mathsf{H}(2,4)$ of $\mathsf{H}(3,4)$ in which we replace the points on a line L by the points on $L^\perp$, the polar line of L. Let the lines of $\mathcal{R}$ correspond to the points on $L^\perp$, then M corresponds to one of the points, say m, on L. A regulus on M translates into a line containing the point m, which obviously intersects $\mathsf{H}(3,4)$ in no (L) or two points.

□

An immediate consequence of this lemma is that every line on t (not in R_a) determines a regulus, and consequently also a grid of the quadric, on xst containing two lines of S_α. The line $tb_{ty}b_{tz}$ determines such a grid with xst containing $sc_{sy}c_{sz}$ as a line and this will be the reason why $XTC_{sy}C_{sz}$ cannot be in a unital with $XSB_{ty}B_{tz}$ and S_α.
Suppose, by way of contradiction, that the opposite is true and consider a_1 and a_2 as introduced above. A block on X and A_1 is determined by

$$\boxed{\begin{array}{c} a_2b_{ty}a_{sz} \\ a_2b_{sz}a_{ty} \\ a_2d_{sy}c_{tz} \\ a_2d_{tz}c_{sy} \end{array}}$$

one of these lines on a_2. As, in this particular case, X is already in a block of $\mathcal{U}$ with B_{ty}, with C_{sy} and $a_2d_{sy}c_{tz}$ is an element of S_α, we conclude that

$$XA_1B_{sz}A_{ty}$$

is the only possible block of $\mathcal{U}$ on these two points. In the same way we find that

$$XA_2B_{sy}A_{tz}$$

has to belong to $\mathcal{U}$. However the corresponding set of four lines in $\mathcal{D}_x$ cannot be completed into a spread, as we shall show. First of all, one can easily see that the blocks

$$XC_1A_{sy}D_{tz}$$

$$XC_2A_{sz}D_{ty}$$

automatically determine two other spread lines. There are now no possibilities left for spread lines on d_1 and d_2.

Conclusion, given the fixed non-Hermitian spread S_α and the block on X and S any unital containing corresponding blocks will also contain the block $XTD_{sy}D_{sz}$. If so, then considering the lines on d_2 (d_1 respectively) yields two distinct possibilities for blocks on X and D_1 (D_2 respectively). Two out of four combinations, however, lead to a contradiction and the remaining two combinations will be shown to be isomorphic. The block on X and D_1 can either be determined by $d_2a_{tz}c_{sy}$ or by $d_2a_{sy}c_{tz}$, as where the one on X and D_2 is by $d_1a_{ty}c_{sz}$ or by $d_1a_{sz}c_{ty}$. In chronological order these situations will be denoted by increasing numbers 1 to 4.
A combination of situation 1 with situation 3 leads to a contradiction as there remains no acceptable block on X and A_2. In the same way the second and fourth situation allow no block on X and C_1.

Situation 1 and 3 and situation 2 and 4, on the other hand, lead to unique non-Hermitian spreads S_x and S'_x respectively.

Blocks cfr S_x				**Blocks cfr S'_x**			
X	S	B_{ty}	B_{tz}	X	S	B_{ty}	B_{tz}
X	T	D_{sy}	D_{sz}	X	T	D_{sy}	D_{sz}
X	A_1	D_{tz}	C_{sy}	X	A_1	B_{sz}	A_{ty}
X	A_2	B_{sy}	A_{tz}	X	A_2	D_{ty}	C_{sz}
X	C_1	B_{sz}	C_{ty}	X	C_1	B_{sy}	C_{tz}
X	C_2	A_{sz}	D_{ty}	X	C_2	A_{sy}	D_{tz}
X	D_1	A_{sy}	C_{tz}	X	D_1	A_{tz}	C_{sy}
X	D_2	A_{ty}	C_{sz}	X	D_2	A_{sz}	C_{ty}
X	α	Y	Z	X	α	Y	Z

Nevertheless, while fixing S_α we can map S_x onto S'_x by applying the group element which fixes x, s and t and switches all pairs (i, j) with xij, sij or tij a line of $\mathcal{Q}$. Hence it suffices to proceed using S_x as the non-Hermitian spread of $\mathcal{D}_x$ in $\mathcal{U}$.

To complete the proof of Theorem 5 we shall, just as in the previous section, consider the blocks on Y and Z. First of all, taking into account the blocks on α, X and S, T a block of $\mathcal{U}$ on Y and S (respectively T) and also on Z and S (respectively T) has to be determined by lines of R_c or R_d (respectively R_b or R_c).

Suppose we have a block on Y and S in R_c and the one on Y and T in R_b (namely $YSC_{ty}C_{sz}$, $YTB_{sy}B_{tz}$). This choice of blocks immediately forces (in this order)

$$YA_2A_{sy}B_{sz},\ YC_2D_{tz}A_{ty},\ YA_1A_{tz}B_{ty}$$

and

$$YC_1D_{sy}A_{sz}$$

to be elements of the unital. However, this leaves us no choice for a block on Y and D_2, a contradiction. In the same way

$$YSD_{ty}D_{sz}$$

together with

$$YTC_{sy}C_{tz}$$

leads to a situation where there is no acceptable block on Y and D_1.

If both the block on Y and S and the one on Y and T are determined by R_c, then we are able to complete this set of blocks on Y into a spread S_y. Nevertheless, these two blocks force us to take

$$ZSD_{tz}D_{sy}$$

$$ZTB_{ty}B_{sz}$$

as blocks on Z and this combination of blocks can never be in a unital of $\mathcal{D}$, as we shall show. Indeed, a block on S and B_{sy} is determined by one of the non-spread lines on b_{sz}

$$\boxed{\begin{array}{c} b_{sz}a_{ty}a_2 \\ b_{sz}c_{ty}c_2 \\ b_{sz}b_{tz}z \end{array}}$$

and each of these lines give a contradiction with the known blocks of S_x, S_z and S_y respectively. Hence the blocks of $\mathcal{U}$ on Y and S and T respectively are uniquely determined by R_d and R_b. Finally, considering the possible blocks on Z and these two points we find

$$ZSC_{sy}C_{tz}$$

$$ZTC_{ty}C_{sz}$$

as the only plausible combination (all other combinations allow no block on Z and D_2). These two sets can be completed into non-Hermitian spreads S_y and S_z, which are compatible with S_α, and this in a unique way.

Blocks cfr S_y				**Blocks cfr S_z**			
Y	S	D_{ty}	D_{sz}	Z	S	C_{tz}	C_{sy}
Y	T	B_{sy}	B_{tz}	Z	T	C_{ty}	C_{sz}
Y	D_2	C_{ty}	A_{tz}	Z	D_1	B_{sz}	D_{sy}
Y	D_1	C_{sy}	A_{sz}	Z	C_2	A_{sy}	D_{sz}
Y	A_1	D_{sy}	C_{sz}	Z	A_1	A_{sz}	B_{sy}
Y	A_2	A_{sy}	B_{sz}	Z	D_2	B_{ty}	D_{tz}
Y	C_2	D_{tz}	A_{ty}	Z	A_2	B_{tz}	A_{ty}
Y	C_1	B_{ty}	C_{tz}	Z	C_1	A_{tz}	D_{ty}
Y	α	X	Z	Z	α	Y	X

Note that the spread lines in previous table are denoted in order that they are forced to be so-called spread lines.

To end the proof of this theorem it suffices to take a general point p of $\mathcal{Q}$ and show that the spread S_p is fixed. Before coming to this part we claim that S_p is fixed for all $p \in x^\perp$. First, take p equal to a_1. From previous findings we already know four out of the nine spread lines of S_{a_1}, say $L_1, \ldots, L_4$. Showing that these four lines are as such that both L_1 and L_2 are not in a regulus contained in S_{a_1} with L_3 and L_4; nor is L_3 with L_4 implies the uniqueness of S_{a_1}. Indeed, if this is the case then L_1 and L_2 necessarily determine a regulus of the spread and hence S_{a_1} is fixed. Take

$$\begin{array}{rclll} L_1 & = & y & d_{sy} & c_{sz} \\ L_2 & = & z & a_{sz} & b_{sy} \\ L_3 & = & \alpha & d_{sz} & c_{ty} \\ L_4 & = & x & d_{sz} & c_{ty} \end{array}$$

as the four known lines. After some calculations we find following lines M_{ij}

$$\begin{array}{rclll} M_{13} & = & s & d_{ty} & c_{sy} \\ M_{14} & = & c_{tz} & d_{sz} & t \\ M_{23} & = & a_{tz} & d_{tz} & d_1 \\ M_{24} & = & b_{ty} & c_{ty} & d_2 \\ M_{34} & = & a_2 & y & z \\ M_{12} & = & a_2 & c_1 & c_2 \end{array}$$

as third lines in $\mathcal{R}(L_i, L_j)$. Since S and C_{sy}, T and D_{sz}, A_1 and D_{tz}, B_{ty} and D_2 and finally Y and Z are already in blocks of S_z, S_x, S_x, S_z and S_α respectively, we find on the one hand that $A_1A_2C_1C_2$ is a block of the unital and on the other hand that S_{a_1} is fixed.

For p equal to a_2 we immediately find a regulus of S_{a_2}, namely

$$\begin{array}{ccc} \alpha & d_{sy} & c_{tz} \\ x & b_{sy} & a_{tz} \\ a_1 & c_1 & c_2 \end{array}$$

and hence also S_{a_2} is fixed.

For $p \in \{s, t, d_1, d_2\}$ we know that STD_1D_2 determines a first line of the spread S_p. Apart from this line we have six other, two by two distinct, lines (corresponding to S_α, S_x, S_y, S_z, S_{a_1} and S_{a_2}) and obviously seven out of nine lines of the spread completely determine the spread.

If p equals c_1 or c_2 we obtain at least seven distinct lines of S_p whens considering all previous constructed spreads. Hence S_p is fixed.

Finally, consider p any point of $\mathcal{Q}$ which is non-collinear to x. Then S_p is determined by the unique elements of S_α, S_x and $S_{M_i^p}$ (with M_i, $i \in \{1, \ldots, 5\}$, a line on x and M_i^p the projection of p onto M_i) it belongs to. On can easily see that we thus establish a line set which uniquely determines all lines of S_p and we are done.

$\square$

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