The simple exceptional group $\mathsf{G}_2(q)$, q even, and two–character sets

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Abstract

When q is even the simple exceptional group $\mathsf{G}_2(q)$ is a subgroup of $\mathsf{PSp}_6(q)$. We study the action of the group $\mathsf{G}_2(q) < \mathsf{G}_2(q^2)$ on points of $\mathsf{PG}(5, q^2)$ and show that unions of $\mathsf{G}_2(q)$–orbits are two–character sets.

Keywords: exceptional finite groups of Lie type, symplectic polarity, generalized hexagon, two–character set.

1 Introduction

A *two–character set* in the projective space $\mathsf{PG}(d, q)$ is a set $\mathcal{S}$ of n points with the property that the intersection number with any hyperplane only takes two values, $n - w_1$ and $n - w_2$. Then the positive integers w_1 and w_2 are called the weights (constants) of the two–character set. Embed $\mathsf{PG}(d, q)$ as a hyperplane Π in $\mathsf{PG}(d+1, q)$. The *linear representation graph* $\Gamma_d^*(\mathcal{S})$ is the graph having as vertices the points of $\mathsf{PG}(d+1, q) \setminus \Pi$ and where two vertices are adjacent whenever the line defined by them meets $\mathcal{S}$. It follows that $\Gamma_d^*(\mathcal{S})$ has $v = q^{d+1}$ vertices and valency $k = (q-1)n$. Delsarte [3] proved that this graph is strongly regular because $\mathcal{S}$ is a two–character set. The other parameters of the graph are $\lambda = k - 1 + (k - qw_1 + 1)(k - qw_2 + 1)$ and $\mu = k + (k - qw_1)(k - qw_2)$. On the other hand, regarding the coordinates of the elements of $\mathcal{S}$ as columns of the generator matrix of a code $\mathcal{L}$ of length n and dimension $d+1$, then the two–character set property of $\mathcal{S}$ translates into the fact that the code $\mathcal{L}$ has two weights (w_1 and w_2) [1]. Such a code is said to be a *projective two–weight code*. The weights of the code are precisely the weights of the two–character set. Moreover, if $\mathsf{GF}(q_0)$ is a subfield of $\mathsf{GF}(q)$, with $q_0^r = q$, then the projective two–weight code C, which is defined over $\mathsf{GF}(q)$, determines a projective two–weight code C' of length n' and dimension kr, with weights w_1' and w_2', where $n' = ((q-1)n)/(q_0 - 1)$, $w_1' = qw_1/q_0$ and $w_2' = qw_2/q_0$.

Several authors, too many to be quoted here, studied two–character sets, sometimes involving algebra, sometimes involving geometry. Very recently, De Wispelaere and Van Maldeghem [4], [5] discovered a new infinite class of two–character sets in $\mathsf{PG}(5, q^2)$ using a very interesting idea: an anti–isomorphism between two skew planes of $\mathsf{PG}(5, q^2)$ together with a certain

Baer subplane. Its automorphism group is a non–split extension $(q+1)\cdot(\Gamma L(3,q)\times 2)$. From this construction a new distance–2 ovoid of the split Cayley hexagon $\mathsf{H}(4)$ arises. We recall that a distance−2 ovoid of a generalized hexagon of order (s,t), is a set of points of the hexagon of size s^2t^2+st+1, such that every line of the generalized hexagon is incident with exactly one element of the set. Another two–character set found by De Wispelaere and Van Maldeghem arises from certain orbits of the group $\mathsf{PSL}_2(13)$, which is a maximal subgroup of $\mathsf{G}_2(4)$, on points of $\mathsf{PG}(5,4)$.

In this paper we will construct infinite families of two–character sets of $\mathsf{PG}(5,q^2)$, q even, given by the orbits of the action of the exceptional finite simple group $\mathsf{G}_2(q)$ on points of $\mathsf{PG}(5,q^2)$. Apart from well–known instances (a Baer subgeometry and its complement in $\mathsf{PG}(5,q^2)$), the other examples we have found (and the associated strongly regular graphs) seem to be new. The automorphism groups of all our two–character sets are the direct products of the standard extensions of $\mathsf{G}_2(q)$ by all field automorphisms, with a group of order 2 (the latter a Baer involution in $\mathsf{PG}(5,q)$ corresponding to the field automorphisms of $\mathsf{GF}(q^2)$ fixing $\mathsf{GF}(q)$ pointwise).

Our construction will follow from a rather simple observation implied by an old result of Thas [8]. Let us review briefly Thas' result and prove the corollary we have in mind.

2 Baer subhexagons of generalized hexagons of order s^2

A *generalized hexagon* is a point-line incidence structure $\mathcal{H}=(\mathcal{P},\mathcal{L},\mathtt{I})$, with point set $\mathcal{P}$, line set $\mathcal{L}$ and symmetric incidence relation $\mathtt{I}$ whose incidence graph — i.e., the bipartite graph with vertex set $\mathcal{P}\cup\mathcal{L}$ and edge set given by $\mathtt{I}$ — has diameter 6 and girth 12 — the latter is the length of the smallest circuit. From now on, we will consider distances of elements of a generalized hexagon in its incidence graph. A generalized hexagon $\mathcal{H}$ is called *of order* (s,t) if all lines are incident with exactly $s+1$ points and all lines are incident with exactly $t+1$ lines. A generalized hexagon has an order as soon as two elements at distance 1 or 5 are incident with at least 3 elements. If $s=t$, then we say that the generalized hexagon has order s. A *subhexagon* $\mathcal{H}'=(\mathcal{P}',\mathcal{L}',\mathtt{I}')$ of $\mathcal{H}$ is a substructure with $\mathcal{P}'\subseteq\mathcal{P}$, $\mathcal{L}'\subseteq\mathcal{L}$ and $\mathtt{I}'$ coincides with $\mathtt{I}$ on $\mathcal{P}'\times\mathcal{L}'$.

Generalized hexagons were introduced by Tits in the appendix of [9]. More information is contained in the monograph [10].

We also use standard terminology of incidence geometry, such as *collinear points* (points at distance ≤ 2 from each other), *concurrent lines* (lines at distance ≤ 2 from one another), etc.

We have the following result.

Theorem [Thas [8]]. *Let $\mathcal{H}'$ be a subhexagon of order (s',t') of the generalized hexagon $\mathcal{H}$ of order (s,t), with $s' < s$. Then $(s't')^2 \leq st$ and if $(s't') = st$, then every point of $\mathcal{H}$ that is not collinear with any point of $\mathcal{H}'$ is at distance 3 from exactly $t'+1$ lines of $\mathcal{H}'$.*

A peculiar situation arises when we have a generalized hexagon $\mathcal{H} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order s^2 and a subhexagon $\mathcal{H}' = (\mathcal{P}', \mathcal{L}', \mathtt{I})$ of order s. Then $(ss)^2 = s^2s^2$ and the equality in the previous theorem holds. We call $\mathcal{H}'$ a *Baer subhexagons* in this case. Since here $s = t$ and $s' = t'$, the situation is very symmetric in points and lines. The distance of an element of $\mathcal{H}$ to $\mathcal{H}'$ is the distance of that element to a nearest element of $\mathcal{H}'$. The previous theorem implies that the maximal distance to $\mathcal{H}'$ is 3. We will denote the set of points (lines) of $\mathcal{H}$ at distance $0,1,2,3$ from $\mathcal{H}'$ by $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$, respectively. Note that $\mathcal{P}_0 = \mathcal{P}'$ and $\mathcal{L}_0 = \mathcal{L}'$. We have the following corollary to the above theorem.

Corollary 1. *Let $\mathcal{H}'$ be a Baer subhexagon of order s of the generalized hexagon $\mathcal{H}$. With the above notation, the number $n_{i,j}$ of elements of $\mathcal{P}_j$ and $\mathcal{L}_j$ incident with an element of $\mathcal{L}_i$ and $\mathcal{P}_i$, respectively, only depends on i and j and is given by the following matrix $N = (n_{i,j})_{0 \leq i,j \leq 3}$.*

$$N = \begin{bmatrix} s+1 & s^2-s & 0 & 0 \\ 1 & 0 & s^2 & 0 \\ 0 & 1 & 0 & s^2 \\ 0 & 0 & s+1 & s^2-s \end{bmatrix}.$$

Proof. The first row of N follows from the definition of $\mathcal{L}_1$; the second row from the definition of $\mathcal{L}_2$. Now let $p \in \mathcal{P}_2$. Then p is incident with a line L incident with a unique point x of $\mathcal{H}'$, by definition. If any other line through p would contain a point x' of $\mathcal{H}'$ or meet a line L' of $\mathcal{H}'$, then the unique path in $\mathcal{H}$ connecting x with x' or L', respectively, would also be contained in $\mathcal{H}'$, a contradiction since this path would contain p. Hence $n_{2,1} = 1$, $n_{2,2} = 0$ and $n_{2,3} = s^2$. The last row follows directly from the previous theorem. $\square$

This corollary enables one to determine the number of elements in any set $\mathcal{P}_j$ or $\mathcal{L}_j$ at distance k from any element of $\mathcal{P}_i$ or $\mathcal{L}_i$, and this number $n_{i,j}^{(k)}$ only depends on i,j,k, for $i,j \in \{0,1,2,3\}$ and $0 \leq k \leq 6$. For instance, one checks that $n_{i,j}^{(2)}$ is equal to the (i,j)-entry of N^2 if $i \neq j$, and to the (i,j)-entry of $(N-J)N$ if $i = j$ (where J is the matrix with on every position 1). It is a bit more cumbersome to find such expressions for $n_{i,j}^{(4)}$, but it is still elementary to deduce it directly from Corollary 1. In fact, one shows:

Corollary 2. *Let $\mathcal{H}'$ be a Baer subhexagon of order s of the generalized hexagon $\mathcal{H}$. With the above notation, the number $m_{i,j}$ of elements of $\mathcal{P}_j$ not opposite an element of $\mathcal{P}_i$ only depends on i and j and is given by the* *following matrix* $M = (m_{i,j})_{0 \le i,j \le 3}$.

$$\begin{bmatrix} s^4+s^3+s^2+s+1 & s^5-s & s^7-s^3 & s^8-s^7+s^6-s^5 \\ s^3+s^2+s+1 & s^5+s^4-s & s^7-s^3 & s^8-s^7+s^6-s^5 \\ s^3+s^2+s^+1 & s^5-s & s^7+s^4-s^3 & s^8-s^7+s^6-s^5 \\ s^3+s^2+s+1 & s^5-s & s^7-s^3 & s^8-s^7+s^6-s^5+s^4 \end{bmatrix}.$$

Proof. As an example let us derive $m_{3,3} = n^{(0)}_{3,3} + n^{(2)}_{3,3} + n^{(4)}_{3,3}$. Consider an arbitrary point $p \in \mathcal{P}_3$. We will say that an element of $\mathcal{P}_i \cup \mathcal{L}_i$ has type i, $i = 0,1,2,3$. To a point x of $\mathcal{H}$ at distance 4 from p corresponds a unique sequence $i_1i_2i_3i_4$ with i_j the type of the unique element of $\mathcal{H}$ at distance j from p and $4-j$ from x, $j = 1,2,3,4$. We put $i_0 = 3$, the type of p. We now count the number of points corresponding with such a sequence for which $i_4 = 3$. It is easy to see with Corollary 1 that the only sequences for which this number is nonzero are 2123, 2323, 2333, 3233, 3323 and 3333. For each such sequence A and each $j \in \{1,2,3\}$, we put

$$f^{(A)}_{j,j+1} = \begin{cases} n_{j,j+1} - 1 & \text{if } i_{j-1} = i_{j+1}, \\ n_{j,j+1} & \text{if } i_{j-1} \neq i_{j+1}. \end{cases}$$

Then the number of points $N(i_1i_2i_3i_4)$ corresponding to the sequence $A = i_1i_2i_3i_4$ equals $n_{i_0,i_1} f^{(A)}_{i_1,i_2} f^{(A)}_{i_2,i_3} f^{(A)}_{i_3,i_4}$. Hence

$$\begin{aligned} n^{(4)}_{3,3} &= N(2123) + N(2323) + N(2333) + N(3233) + N(3323) + N(3333) \\ &= s^8 - s^7 + s^6 - s^5 + s^3 - s^2. \end{aligned}$$

Similarly

$$n^{(2)}_{3,3} = n_{3,2}(n_{2,3} - 1) + n_{3,3}(n_{3,3} - 1) = s^4 - s^3 + s^2 - 1$$

and the result follows. □

3 An application to two–character sets

3.1 The group $\mathsf{G}_2(q)$ and the split Cayley hexagon

Let $\mathbb{K}$ be a field. Let $\mathsf{Q}(6,\mathbb{K})$ be a nonsingular quadric in the projective space $\mathsf{PG}(6,\mathbb{K})$ that has on it totally singular planes. Let $\mathsf{P}\Omega_7(\mathbb{K})$ be the projective commutator orthogonal group of $\mathsf{Q}(6,\mathbb{K})$. The Cartan–Dickson–Chevalley exceptional group $\mathsf{G}_2(\mathbb{K})$ is a subgroup of $\mathsf{P}\Omega_7(\mathbb{K})$. It occurs as the stabilizer in $\mathsf{P}\Omega_7(\mathbb{K})$ of a configuration $\mathcal{F}$ of points, lines and planes on $\mathsf{Q}(6,\mathbb{K})$. The geometry of $\mathcal{F}$ was explored first by J. Tits [9] and later, from a somewhat different standpoint, by R.H. Dye [6]. Both approaches involve the geometry associated with the triality of a hyperbolic quadric in $\mathsf{PG}(7,\mathbb{K})$. Tits, in particular, by identifying the stabilizer of $\mathcal{F}$ with $\mathsf{G}_2(\mathbb{K})$, obtained a lot of information on the action of $\mathsf{G}_2(\mathbb{K})$.

The group $\mathsf{G}_2(\mathbb{K})$ is also contained in the automorphism group of a classical generalized hexagon, called the *split Cayley hexagon* and denoted by $\mathsf{H}(\mathbb{K})$. The latter is a "truncation" of the configuration $\mathcal{F}$, just keeping the points and lines. So the points of $\mathsf{H}(\mathbb{K})$ are all the points of $\mathsf{Q}(6,\mathbb{K})$ and the lines of $\mathsf{H}(\mathbb{K})$ are certain lines of $\mathsf{Q}(6,\mathbb{K})$. When $\mathbb{K}$ is the finite field $\mathsf{GF}(q)$, the hexagon is denoted $\mathsf{H}(q)$ and it has order q. For more details, see [10]. The precise definition of the line set is not necessary for the purpose of the present paper; it suffices to remark that (1) it can be defined using the Plücker coordinates of the lines and a set of equations with coefficients in the prime field of $\mathsf{GF}(q)$, that (2) two points of $\mathsf{PG}(5,q)$ are conjugate with respect to ρ if and only if they are not opposite in $\mathsf{H}(q)$, and that (3) the $q+1$ lines on a point are coplanar.

When q is even, the orthogonal group $\mathsf{P\Omega}_7(q)$ is isomorphic to the symplectic group $\mathsf{PSp}_6(q)$, so that $\mathsf{G}_2(q)$ may be regarded as a subgroup of $\mathsf{PSp}_6(q)$. Then, it follows that $\mathsf{H}(q)$ has a representation in $\mathsf{PG}(5,q)$ as a *perfect symplectic hexagon*, [10, p.74]: the points of $\mathsf{H}(q)$ are all the points of $\mathsf{PG}(5,q)$ and the lines of $\mathsf{H}(q)$ are certain totally isotropic lines of $\mathsf{PG}(5,q)$ with respect to a symplectic polarity ρ.

Now we embed $\mathsf{PG}(5,q)$ in $\mathsf{PG}(5,q^2)$, thereby extending $\mathsf{H}(q)$ to $\mathsf{H}(q^2)$ and ρ extends (in a unique way, actually) to $\mathsf{PG}(5,q^2)$ in such a way that all lines of $\mathsf{H}(q)$ are totally isotropic. The points of $\mathsf{H}(q^2)$ are all points of $\mathsf{PG}(5,q^2)$. Every hyperplane H in $\mathsf{PG}(5,q^2)$ is the image of a unique point p of $\mathsf{H}(q^2)$ under ρ. Since H is the set of points conjugate to p under ρ, it is also the set of points of $\mathsf{H}(q^2)$ lying at distance at most 4 from p. The following proposition can be easily verified using the fact that a line pencil in $\mathsf{H}(q^2)$ is a planar line pencil in $\mathsf{PG}(5,q^2)$.

Proposition 3.1. *Let $\mathcal{P}_i$ be the set of points of $\mathsf{H}(q^2)$ at distance i from $\mathsf{H}(q)$, $i = 0,1,2,3$. Then $\mathcal{P}_1$ is precisely the set of points of $\mathsf{PG}(5,q^2) \setminus \mathsf{PG}(5,q)$ incident with lines of $\mathsf{H}(q)$; $\mathcal{P}_2$ is the set of points of $\mathsf{PG}(5,q^2) \setminus \mathsf{PG}(5,q)$ incident with a totally isotropic line of $\mathsf{PG}(5,q)$ with respect to ρ; finally $\mathcal{P}_3$ is the set of points of $\mathsf{PG}(5,q^2) \setminus \mathsf{PG}(5,q)$ incident with a nonisotropic line of $\mathsf{PG}(5,q)$. We also have*

$$|\mathcal{P}_0| = q^5+q^4+q^3+q^2+q+1, \qquad |\mathcal{P}_1| = q^7-q,$$

$$|\mathcal{P}_2| = q^9-q^3, \qquad |\mathcal{P}_3| = q^{10}-q^9+q^8-q^7+q^6-q^5.$$

3.2 The two–character sets

Corollary 2 and Proposition 3.1 implies the following theorem. We use the same notation as in the previous subsection.

Theorem 3.2. *Let $\mathcal{P}_i$ be the set of points of $\mathsf{H}(q^2)$ at distance i from $\mathsf{H}(q)$, $i = 0, 1, 2, 3$. Then each $\mathcal{P}_i$, $i \in \{0, 1, 2, 3\}$ is a two-character set of $\mathsf{PG}(5, q^2)$, and so is each $\mathcal{P}_i \cup \mathcal{P}_j$, $i, j \in \{0, 1, 2, 3\}$, $i \neq j$, and also each $\mathcal{P}_i \cup \mathcal{P}_j \cup \mathcal{P}_k$, $|\{0, 1, 2, 3\} \setminus \{i, j, k\}| = 1$. Omitting the well known Baer subgeometry $\mathcal{P}_0$ and its complement $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$, the corresponding weights w_1 and w_2 are as follows.*

	w_1	w_2
$\mathcal{P}_1$	$q^7 - q^5$	$q^7 - q^5 - q^4$
$\mathcal{P}_2$	$q^9 - q^7$	$q^9 - q^7 - q^4$
$\mathcal{P}_3$	$q^{10} - q^9$	$q^{10} - q^9 - q^4$
$\mathcal{P}_0 \cup \mathcal{P}_1$	q^7	$q^7 + q^4$
$\mathcal{P}_0 \cup \mathcal{P}_2$	$q^9 - q^7 + q^5$	$q^9 - q^7 + q^5 + q^4$
$\mathcal{P}_0 \cup \mathcal{P}_3$	$q^{10} - q^9 + q^5$	$q^{10} - q^9 + q^5 + q^4$
$\mathcal{P}_1 \cup \mathcal{P}_2$	$q^9 - q^7 + q^6 - q^5$	$q^9 - q^7 + q^6 - q^5 - q^4$
$\mathcal{P}_1 \cup \mathcal{P}_3$	$q^{10} - q^9 + q^7 - q^5$	$q^{10} - q^9 + q^7 - q^5 - q^4$
$\mathcal{P}_2 \cup \mathcal{P}_3$	$q^{10} - q^7$	$q^{10} - q^7 - q^4$
$\mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2$	q^9	$q^9 + q^4$
$\mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_3$	$q^{10} - q^9 + 2q^7$	$q^{10} - q^9 + 2q^7 + q^4$
$\mathcal{P}_0 \cup \mathcal{P}_2 \cup \mathcal{P}_3$	$q^{10} - q^7 + q^5$	$q^{10} - q^7 + q^5 + q^4$

For the well know examples $\mathcal{P}_0$ and $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ we refer to [1, Ex. RT1]. The strongly regular graphs associated to the two–character sets $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ all have q^6 vertices and valency k equal to $(q-1)(q^7-q)$, $(q-1)(q^9-q^3)$ and $(q-1)(q^{10}-q^9+q^8-q^7+q^6-q^5)$, respectively. It is easy to determine the parameters of the associated strongly regular graphs and codes for all two–character sets of the theorem, but we omit numbers. Checking the list in [1] and the papers [4], [5], they turn out to be most probably new.

Of course the automorphism group of all our two–character sets is the direct product of the standard extension of the group $\mathsf{G}_2(q)$ with all field automorphism of $\mathsf{GF}(q)$, with the group of order 2 generated by the involution induced by the unique field automorphism of order two in $\mathsf{GF}(q^2)$.

3.3 Definition with orbits

It has been proved in [2, Lemmas 5.2,5.4] that the group $\mathsf{G}_2(q)$ has two orbits on totally isotropic lines of $\mathsf{PG}(5, q)$ of lengths $(q^6-1)/(q-1)$ ($\mathsf{H}(q)$–lines) and $q^2(q^6-1)/(q-1)$ and acts transitively on the $q^4(q^4+q^2+1)$ nonisotropic lines of $\mathsf{PG}(5, q)$. In fact, this follows directly from the fact that $\mathsf{G}_2(q)$ acts *distance-transitively* on $\mathsf{H}(q)$, i.e., transitive on ordered pairs of points at fixed but arbitrary distance from one another. This now implies the following alternative construction of the sets $\mathcal{P}_i$, $i \in \{0, 1, 2, 3\}$.

Proposition 3.3. *The group $\mathsf{G}_2(q)$ has exactly four orbits on points of $\mathsf{PG}(5, q^2)$, and these are $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$*

Proof. Since $\mathsf{G}_2(q)$ acts distance-transitively on $\mathsf{H}(q)$, the action induced on each line L of $\mathsf{PG}(5,q)$ of the stabilizer $\mathsf{G}_2(q)_L$ contains $\mathsf{PSL}_2(q)$ in its natural action. But this is transitive on the imaginary points of L over $\mathsf{GF}(q^2)$. The proposition is now clear. $\square$

Remark. Note that usually the union of two two–character sets is not a two–character set, since the number of possibilities for the intersection numbers with hyperplanes is generically equal to four. But in our case, all possible unions of $\mathcal{P}_i$, $i = 0, 1, 2, 3$, do give rise to two–character sets. This is a rather remarkable observation. Probably this can be partly explained by the fact that the *duals* of the two–character sets $\mathcal{P}_i$, $i = 0, 1, 2, 3$, are also two–character sets in the dual space. With the *dual* of a two–character set $\mathcal{C}$ we mean the set of hyperplanes meeting $\mathcal{C}$ is the minimal number of points.

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