Generalized hexagons and Singer geometries

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Abstract

In this paper, we consider a set $\mathcal{L}$ of lines of $\mathsf{PG}(5,q)$ with the properties that (1) every plane contains 0, 1 or $q+1$ elements of $\mathcal{L}$, (2) every solid contains no more than q^2+q+1 and no less than $q+1$ elements of $\mathcal{L}$, and (3) every point of $\mathsf{PG}(5,q)$ is on $q+1$ members of $\mathcal{L}$, and we show that, whenever (4) $q \neq 2$ (respectively, $q=2$) and the lines of $\mathcal{L}$ through some point are contained in a solid (respectively, a plane), then $\mathcal{L}$ is necessarily the set of lines of a regularly embedded split Cayley generalized hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, with q even. We present examples of such sets $\mathcal{L}$ not satisfying (4) based on a Singer cycle in $\mathsf{PG}(5,q)$, for all q.

1 Introduction

Recognizing specific geometric structures by certain properties — preferably as weak as possible — is very important in (finite) geometry, since various configurations can turn up unexpectedly in completely different contexts, disguised in an unusual definition. Data that are available for structures in projective spaces are in many situations the intersection numbers with respect to subspaces. A typical example are the conics in Desarguesian planes $\mathsf{PG}(2,q)$, with q odd, which are characterized as sets of $q+1$ points meeting lines in 0, 1 or 2 points. Other, more involved, examples are the characterizations of quadrics, Hermitian and Veronesean varieties by intersection numbers.

In the present paper, our aim is to characterize the standard embedding in $\mathsf{PG}(5,q)$ of the split Cayley hexagon $\mathsf{H}(q)$, q even, by intersection numbers with subspaces. Of course, since the points of $\mathsf{H}(q)$ are all the points of $\mathsf{PG}(5,q)$, such a characterization is impossible

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if we (only) consider intersections of the point set of the hexagon with subspaces. That is why we consider the intersections of subspaces with the **line set** of $\mathsf{H}(q)$, i.e., we count the number of lines of a given embedded $\mathsf{H}(q)$ in a given subspace and then assume that we have a set of lines with these intersection numbers (in practice, we take weaker assumptions). It is also natural to require that we deal with a tactical configuration, i.e., we assume that each point of the projective space is incident with exactly $q+1$ lines of our set. A similar characterization for the standard embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(6,q)$ has been proved in [7].

Before getting down to precise statements, let us recall the definition and construction of the split Cayley hexagons $\mathsf{H}(q)$.

A *point-line geometry* is a triple $(\mathcal{P},\mathcal{L},\mathtt{I})$ consisting of a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines, and a symmetric incidence relation $\mathtt{I}$ saying precisely which points are incident with which lines (and conversely). The *incidence graph* of a point-line geometry $(\mathcal{P},\mathcal{L},\mathtt{I})$ is the graph with vertex set $\mathcal{P}\cup\mathcal{L}$ and adjacency relation $\mathtt{I}$. A *generalized hexagon* is a point-line geometry for which the incidence graph has diameter 6 and girth 12, i.e., the maximal distance between two vertices is 6, and the length of a shortest circuit is 12. Whenever each vertex of the incidence graph of a generalized hexagon has valency at least 3, this (bipartite) graph is bi-valent. If the valency of the vertices belonging to $\mathcal{P}$ and $\mathcal{L}$ is equal to $s+1$ and $t+1$, respectively, then we say that the generalized hexagon has order (s,t). Distances between elements of a point-line geometry are always measured in the incidence graph. Elements at distance 6 from each other in a generalized hexagon are called *opposite*.

Let q be any prime power. Up to isomorphism, the *split Cayley hexagon* $\mathsf{H}(q)$ is defined as follows (see Tits [8]). Let $\mathsf{Q}(6,q)$ be the parabolic quadric in $\mathsf{PG}(6,q)$ defined by the equation $X_0X_4+X_1X_5+X_2X_6=X_3^2$. Then the points of $\mathsf{H}(q)$ are the points of $\mathsf{Q}(6,q)$, the lines of $\mathsf{H}(q)$ are the lines of $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01},p_{02},\ldots,p_{56})$ satisfy the six relations $p_{12}=p_{34}, p_{56}=p_{03}, p_{45}=p_{23}, p_{01}=p_{36}, p_{02}=-p_{35}$ and $p_{46}=-p_{13}$. Incidence is inherited from $\mathsf{PG}(6,q)$. For more details, properties and information about $\mathsf{H}(q)$ we refer to [9].

When q is even, then the point with coordinates $(0,0,0,1,0,0,0)$ has the property that each line of $\mathsf{PG}(6,q)$ though that point meets $\mathsf{Q}(6,q)$ in exactly one point. Projection of $\mathsf{H}(q)$ from that point onto any hyperplane not containing $(0,0,0,1,0,0,0)$ yields a representation of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$. It is exactly this representation that we will characterize in the present paper. We call it an *embedding* of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$. (Abstractly, an *embedding* of a point-line geometry $(\mathcal{P},\mathcal{L},\mathtt{I})$ in $\mathsf{PG}(n,q)$, for some n, is an injective mapping of $\mathcal{P}$ in the point set of $\mathsf{PG}(n,q)$ inducing an injective mapping from $\mathcal{L}$ into the line set of $\mathsf{PG}(n,q)$

and such that the image of $\mathcal{P}$ generates $\mathsf{PG}(n,q)$.) It has the following properties, see [9] and [6].

First of all, it is *polarized*. This means that for every point x of $\mathsf{H}(q)$, the set of points not opposite x is contained in a hyperplane of $\mathsf{PG}(5,q)$. Also, the set of lines of $\mathsf{H}(q)$ incident with x is contained in a plane of $\mathsf{PG}(5,q)$, and we say that the embedding is *flat*. Flat and polarized are abbreviated by the notion *regular embedding*. By [5], up to projectivity, the example above is the only flatly embedded generalized hexagon of order (q,q) in $\mathsf{PG}(5,q)$.

A line L of $\mathsf{PG}(5,q)$ that does not belong to $\mathsf{H}(q)$ contains $q+1$ points at mutual distance (in the hexagon) either 4, or 6. In the former case, we call the line *ideal*. In this case, there is a unique point x of $\mathsf{H}(q)$ collinear to all points of L, and we call x the *focus* of the ideal line L. If all points of L are mutually opposite, then we say that the points of L form a *point regulus*.

We are now ready to state our main results.

2 Main Results

Let $\mathcal{L}$ be a set of lines of $\mathsf{PG}(5,q)$. Then $\mathcal{L}$ is called a *pseudo-hexagon* if it satisfies the properties (Pt), (Pl) and (Sd).

(Pt) Every point of $\mathsf{PG}(5,q)$ is incident with exactly $q+1$ elements of $\mathcal{L}$.

(Pl) Every plane of $\mathsf{PG}(5,q)$ is incident with either 0, 1 or $q+1$ elements of $\mathcal{L}$.

(Sd) We either have that every solid of $\mathsf{PG}(5,q)$ is incident with no more than q^2+q+1 and no less than $q+1$ elements of $\mathcal{L}$, or no solid of $\mathsf{PG}(5,q)$ is incident with strictly less than q^2+q+1 and strictly more than $q+1$ elements of $\mathcal{L}$.

(Sd′) Every solid of $\mathsf{PG}(5,q)$ is incident with either q^2+q+1 or $q+1$ elements of $\mathcal{L}$.

(Hp) Every hyperplane of $\mathsf{PG}(5,q)$ is incident with exactly q^3+q^2+q+1 members of $\mathcal{L}$.

(To) The set $\mathcal{L}$ contains $q^5+q^4+q^3+q^2+q+1$ lines.

A pseudo-hexagon $\mathcal{L}$ with the additional property that for some point x, the members of $\mathcal{L}$ through x are contained in a plane will be called *flat* (and the point x will also be called *flat*).

Our Main Result reads as follows.

Main Result. *If $\mathcal{L}$ is a pseudo-hexagon, then it also satisfies* (Sd′), (Hp) *and* (To). *If moreover $q \neq 2$ and for some point x the members of $\mathcal{L}$ through x are contained in a solid, then $\mathcal{L}$ is flat, all points of $\mathsf{PG}(5,q)$ are flat and $\mathcal{L}$ is the line set of a regularly embedded split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, with q even. If $q = 2$ and some point x is flat, then we have the same conclusion. Conversely, the line set of every regularly embedded split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, q even, is a pseudo-hexagon.*

We will also provide a class of examples of pseudo-hexagons that are not flat. For $q = 2$, there is a complete classification.

Theorem. *Up to isomorphism, there are exactly three pseudo-hexagons in $\mathsf{PG}(5,2)$.*

3 Proof of the Main Result

First let $\mathcal{L}$ be the line set of a regularly embedded split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$. Since all points of $\mathsf{PG}(5,q)$ are points of $\mathsf{H}(q)$, and since $\mathsf{H}(q)$ has order q, Condition (Pt) follows. Since the embedding is flat, Condition (Pl) is trivial.

Recall that, because q is even, every element of $\mathcal{L}$ is an absolute line with respect to some symplectic polarity ρ. Moreover, two points x, y are conjugate under ρ (meaning x is incident with y^ρ) if and only if x and y are at distance at most 4 in the incidence graph of $\mathsf{H}(q)$. Every solid is the image under ρ of a line L of $\mathsf{PG}(5,q)$, and there are three possibilities.

Either L is a line of $\mathsf{H}(q)$

Then L^ρ consists of all points at distance ≤ 3 from L in $\mathsf{H}(q)$. There are $q^2 + q + 1$ lines of $\mathsf{H}(q)$ in that set of points.

Or it is any other absolute line with respect to ρ

Then L is an ideal line with focus, say, x and L^ρ contains all lines of $\mathsf{H}(q)$ through x, which lie in a plane π. Every other line of $\mathsf{H}(q)$ in L^ρ must meet π in some point $u \neq x$, but then it is easily seen that u is the only point of that line at distance ≤ 4 from all points of L, which yields a contradiction. Hence L^ρ contains $q + 1$ lines of $\mathsf{H}(q)$.

Or it is a non absolute line with respect to ρ

In this case, the points of L form a point regulus of $\mathsf{H}(q)$ and the set of all points of $\mathsf{H}(q)$ at distance ≤ 4 from each point of L consists of all points on the elements of a line regulus. Hence here, too, we have $q+1$ lines of $\mathsf{H}(q)$ in L^ρ.

This shows (Sd$'$).

Next, Condition (Hp) follows from the fact that every hyperplane of $\mathsf{PG}(5,q)$ is the image under ρ of a point of x of $\mathsf{H}(q)$, and the lines in x^ρ are those at distance ≤ 3 from x. There are q^3+q^2+q+1 such lines.

Finally, Condition (To) is immediate.

For the remainder of this section, we assume now that a set of lines $\mathcal{L}$ satisfies (Pt), (Pl) and (Sd).

We prove a sequence of lemmas culminating in a complete proof of our Main Result.

Lemma 1 *The set $\mathcal{L}$ satisfies* (To).

Proof. If N is the total number of lines, then, since there are precisely $q+1$ lines through each point, and $q+1$ points on each line, there are as many lines in $\mathcal{L}$ as there are points in $\mathsf{PG}(5,q)$. This proves the lemma. $\square$

Lemma 2 *The set $\mathcal{L}$ satisfies* (Hp).

Proof. Let I be the index set $\{1,2,\ldots,q^5+q^4+q^3+q^2+q+1\}$. Let $\{H_i \mid i\in I\}$ be the set of hyperplanes of $\mathsf{PG}(5,q)$. Let t_i be the number of lines of $\mathcal{L}$ contained in H_i, $i\in I$. Counting pairs (L,H_i), with $L\in\mathcal{L}$, $i\in I$ and L in H_i, in two different ways, we obtain

$$\sum_{i\in I} t_i = (q^5+q^4+q^3+q^2+q+1)(q^3+q^2+q+1).$$

Counting the ordered triples (L,L',H_i), with $L,L'\in\mathcal{L}$, $L\neq L'$, $i\in I$ and both L,L' in H_i, in two different ways, we obtain

$$\begin{aligned}\sum_{i\in I} t_i(t_i-1) &= (q^5+q^4+q^3+q^2+q+1)(q^2+q)(q^2+q+1)+\\ &\quad +(q^5+q^4+q^3+q^2+q+1)(q^5+q^4+q^3)(q+1).\end{aligned}$$

Then one calculates easily that

$$\sum_{i\in I} t_i^2 = (q^5+q^4+q^3+q^2+q+1)(q+1)^2(q^2+1)^2$$

and hence $|I|\sum_{i\in I} t_i^2 - (\sum_{i\in I} t_i)^2 = 0$. Consequently the variance (in the probabilistic sense viewing the t_i as test cases of a certain variable) of the t_i's is zero and each t_i is equal to the average $(\sum_{i\in I} t_i)/|I|$, which is q^3+q^2+q+1. $\square$

Lemma 3 *The set $\mathcal{L}$ satisfies* (Sd$'$).

Proof. Let I be the index set $\{1,2,\ldots,(q^4+q^3+q^2+q+1)(q^4+q^2+1)\}$. Let $\{S_i \mid i\in I\}$ be the set of solids of $\mathsf{PG}(5,q)$. Let t_i be the number of lines of $\mathcal{L}$ contained in S_i, $i\in I$. Counting pairs (L,S_i), with $L\in\mathcal{L}$, $i\in I$ and L in S_i, in two different ways, we obtain

$$\sum_{i\in I} t_i = (q^5+q^4+q^3+q^2+q+1)(q^2+1)(q^2+q+1).$$

Counting the ordered triples (L,L',S_i), with $L,L'\in\mathcal{L}$, $L\neq L'$, $i\in I$ and both L,L' in S_i, in two different ways, we obtain

$$\begin{aligned}\sum_{i\in I} t_i(t_i-1) &= (q^5+q^4+q^3+q^2+q+1)(q^2+q)(q^2+q+1)+\\ &\quad +(q^5+q^4+q^3+q^2+q+1)(q^5+q^4+q^3).\end{aligned}$$

Then one calculates easily that

$$\sum_{i\in I} t_i^2 = (q^5+q^4+q^3+q^2+q+1)(q^3+2q^2+q+1)(q^2+q+1)$$

and hence

$$\begin{aligned}&\sum_{i\in I}(t_i-(q+1))(t_i-(q^2+q+1)) =\\ &\quad = \sum_{i\in I} t_i^2 - (q^2+2q+2)\sum_{i\in I} t_i + |I|(q+1)(q^2+q+1)\\ &\quad = (q^5+q^4+q^3+q^2+q+1)(q^2+q+1)[q^3+2q^2+q+1\\ &\qquad - q^4-2q^3-3q^2-2q-2+q^4+q^3+q^2+q+1]\\ &\quad = 0.\end{aligned}$$

By (Sd), since either $q+1 \leq t_i \leq q^2+q+1$ or t_i does not lie between $q+1$ and q^2+q+1, for all $i \in I$, no term of the left hand side is either strictly positive or strictly negative, so each term must be zero. But this means that $t_i \in \{q+1, q^2+q+1\}$. $\square$

Before proving some structural properties of $\mathcal{L}$, we count various types of subspaces containing various numbers of lines of $\mathcal{L}$.

Lemma 4 *There are $q^2(q^4+q^2+1)(q^2+q+1)$ solids containing exactly $q+1$ lines of $\mathcal{L}$, and there are $q^5+q^4+q^3+q^2+q+1$ solids containing q^2+q+1 members of $\mathcal{L}$. Every member of $\mathcal{L}$ is contained in exactly q^2+q+1 solids containing q^2+q+1 members of $\mathcal{L}$.*

Proof. Let a and b be the numbers of solids containing exactly $q+1$ and q^2+q+1 members of $\mathcal{L}$, respectively. Then, counting the pairs (L, S), with $L \in \mathcal{L}$ and S a solid containing L, we obtain

$$\left\{ \begin{array}{rcl} a(q+1)+b(q^2+q+1) & = & (q^5+q^4+q^3+q^2+q+1)(q^2+q+1)(q^2+1) \\ a+b & = & (q^4+q^3+q^2+q+1)(q^4+q^2+1). \end{array} \right.$$

The first assertion of the lemma easily follows.

Now let $L \in \mathcal{L}$ arbitrarily. Let c and d be the number of solids through L containing exactly $q+1$ members of $\mathcal{L}$ and q^2+q+1 members of $\mathcal{L}$, respectively. Then counting the pairs (M, S), with $M \in \mathcal{L} \setminus \{L\}$ and S a solid containing L and M, we obtain

$$\left\{ \begin{array}{rcl} cq+d(q^2+q) & = & (q^2+q)(q^2+q+1)+(q^5+q^4+q^3) \\ c+d & = & (q^2+q+1)(q^2+1). \end{array} \right.$$

Now also the second assertion of the lemma follows. $\square$

Lemma 5 *There are exactly $q^5+q^4+q^3+q^2+q+1$ planes of $\mathsf{PG}(5,q)$ containing $q+1$ members of $\mathcal{L}$, and through every line of $\mathcal{L}$, there are exactly $q+1$ such planes.*

Proof. Fix $L \in \mathcal{L}$. Then there are q^2+q lines of $\mathcal{L}$ concurrent with, but distinct from L. Counting in two ways the pairs (M, π), with $L \neq M \in \mathcal{L}$ concurrent with L and π a plane containing L and q other members of $\mathcal{L}$, we easily obtain that there are exactly $q+1$ such planes π. Now we count in two ways all the pairs (M, π), with $M \in \mathcal{L}$, and π a plane containing M and q other lines of $\mathcal{L}$. Since there are equally many lines M in such planes π as there are planes π through such lines M, there are also equally many such planes as such lines. $\square$

Lemma 6 *For every plane π of $\mathsf{PG}(5,q)$ containing $q+1$ members of $\mathcal{L}$, there are exactly $q+1$ solids incident with π and containing q^2+q+1 lines of $\mathcal{L}$.*

Proof. Fix a plane π, put $I = \{1, 2, \ldots, q^2+q+1\}$ and let $\{p_i \mid i \in I\}$ be the set of points of π. Let t_i be the number of lines of $\mathcal{L}$ incident with p_i and not contained in π. Let N be the number of solids incident with π and containing q^2+q+1 lines of $\mathcal{L}$. Counting in two ways the pairs (M, S), with $M \in \mathcal{L}$ meeting π in exactly one point and S a solid containing π and M (and hence containing q^2+q+1 members of $\mathcal{L}$), we obtain $\sum_{i\in I} t_i = Nq^2$.

Now we count the incident pairs (p_i, M), with $i \in I$ and $M \in \mathcal{L}$ in π. We obtain $\sum_{i\in I}(q+1-t_i) = (q+1)^2$. Combined with the previous paragraph this implies $N = q+1$. □

Lemma 4 implies that there exists a solid S containing exactly q^2+q+1 lines of $\mathcal{L}$. We fix S and we denote by $\mathcal{W}$ the set of lines of $\mathcal{L}$ contained in S. Property (Pl) implies that there are at most three types of planes in S. Planes *of Type* I contain $q+1$ lines of $\mathcal{W}$, planes *of Type* II contain only one line of $\mathcal{W}$, and planes *of Type* III do not contain any line of $\mathcal{W}$.

Lemma 7 *The solid S contains exactly $q+1$ planes of Type* I*, and no planes of Type* III.

Proof. Let N_X be the number of planes of Type X in S, with $X =$ I,II,III. Counting the pairs (L, π), with $L \in \mathcal{W}$ and π a plane of S containing L, we obtain

$$\begin{cases} N_{\mathrm{II}} + N_{\mathrm{I}}(q+1) &= (q^2+q+1)(q+1) \\ N_{\mathrm{III}} + N_{\mathrm{II}} + N_{\mathrm{I}} &= q^3+q^2+q+1, \end{cases}$$

from which we deduce that $N_{\mathrm{I}}q - N_{\mathrm{III}} = q^2+q$. Hence $N_{\mathrm{I}} \geq q+1$ and if equality holds, then $N_{\mathrm{III}} = 0$.

We now count the number of pairs (π, T), with π a plane of $\mathsf{PG}(5,q)$ containing $q+1$ lines of $\mathcal{L}$, and T a solid of $\mathsf{PG}(5,q)$ containing π and containing q^2+q+1 members of $\mathcal{L}$, in two ways. If $\overline{N}$ is the average number of planes with $q+1$ lines of $\mathcal{L}$ contained in solids with q^2+q+1 lines of $\mathcal{L}$, then by Lemmas 4, 5 and 6, we have

$$\overline{N} = \frac{(q^5+q^4+q^3+q^2+q+1)(q+1)}{q^5+q^4+q^3+q^2+q+1} = q+1.$$

But varying S in the first part of our proof, we now see that $N_{\mathrm{I}} = q+1$. □

Lemma 8 *For every point p in S, there are equally many lines of $\mathcal{W}$ incident with p as there are planes of Type* I *in S incident with p.*

Proof. Let n be the number of lines of $\mathcal{W}$ through p, and let m be the number of planes of Type I in S incident with p. Counting in two ways the number of incident pairs (L, π), with $L \in \mathcal{W}$ and π a plane of S with $p \in \pi$, we obtain

$$n(q+1) + (q^2 + q + 1 - n) = m(q+1) + (q^2 + q + 1 - m),$$

which proves $n = m$. $\square$

We now prove a structural property of $\mathcal{L}$.

Lemma 9 *If the point x of S is incident with at least three lines of $\mathcal{W}$ not contained in a common plane, then the incidence structure Ω with as points the element of $\mathcal{W}$ through x, as lines the planes of Type* I *of S through x, and as incidence the natural one, is a (possibly degenerate) projective plane. Hence each plane of Type* I *in S through x contains at least two members of $\mathcal{W}$ through x, and any two planes of Type* I *in S through x intersect in a line belonging to $\mathcal{W}$.*

Proof. Let n be the number of planes of Type I through x, which is also equal to the number of lines of $\mathcal{W}$ through x, by Lemma 8. Let Ω be the linear space with point set the members of $\mathcal{W}$ incident with x, and line set the planes of Type I in S containing at least two members of $\mathcal{W}$ through x. If m is the number of planes of Type I in S through x containing at least two lines of $\mathcal{W}$ that are also incident with x (and so $m \leq n$), then by the main result of [1], we have $n \leq m$. We conclude that $n = m$. Hence every plane of Type I in S containing x contains at least two lines of $\mathcal{W}$ through x. Moreover Ω is a projective plane, possibly degenerate; see [2] for the definition of a degenerate projective plane. Hence also any two planes of Type I of S through x intersect in a line of $\mathcal{W}$. $\square$

The next lemmas concern hyperplanes. So we fix a hyperplane H of $\mathsf{PG}(5, q)$, and we prove some results similar to the previous ones for the solid S. Also, from now on, we do not fix S anymore.

Recall that H contains exactly $q^3 + q^2 + q + 1$ members of $\mathcal{L}$. There are three types of solids. A solid of *Type* I is a solid containing $q^2 + q + 1$ members of $\mathcal{L}$. A solid of *Type* IIa is a solid containing $q + 1$ members of $\mathcal{L}$, all contained in one plane. A solid of *Type* IIb is a solid containing $q + 1$ mutually non-intersecting members of $\mathcal{L}$.

Lemma 10 *A plane of Type* II *is contained in exactly one solid of Type* I*.*

Proof. Let π be a plane of Type II. Let L be the unique member of $\mathcal{L}$ contained in π. Let N_{I} be the number of solids of Type I containing π, and let N_{II} be the number of solids of Types IIa and IIb containing π. Then $N_{\mathrm{I}} + N_{\mathrm{II}} = q^2 + q + 1$. We count in two different ways the pairs (S, M), where S is a solid of $\mathsf{PG}(5, q)$ containing π, M is a member of $\mathcal{L}$ intersecting π in a unique point, and M is contained in S. Noting that exactly $q^2(q+1)$ members of $\mathcal{L}$ meet π in a point off L, and $q(q+1)$ meet π in a unique point of L, we obtain

$$N_{\mathrm{I}}(q^2 + q) + N_{\mathrm{II}}q = (q^2 + q)(q + 1),$$

from which easily follows that $N_{\mathrm{I}} = 1$. □

We now want to count the number of various types of solids and planes in H. In order to do so, we need a lemma about the intersection of two solids of Type I.

Lemma 11 *If a plane π of $\mathsf{PG}(5, q)$ is contained in two different solids of Type* I*, then it is itself of Type* I*.*

Proof. This follows directly from Lemmas 7 and 10. □

The proof of the next lemma is suspiciously analogous to the proof of Lemma 8. This will be explained by Lemma 16 below.

Lemma 12 *Consider a member L of $\mathcal{L}$ contained in H. The number of planes of Type* I *through L in H is equal to the number of solids of Type* I *through L in H.* □

Proof. Let n be the number of solids of Type I in H containing L, and let m be the number of planes of Type I in H containing L. Counting in two ways the number of incident pairs (S, π), with S a solid of Type I containing L and π a plane in H through L, we obtain, using Lemmas 4 (last assertion), 6 and 10,

$$n(q + 1) + (q^2 + q + 1 - n) = m(q + 1) + (q^2 + q + 1 - m),$$

which proves $n = m$. □

We now count the number of planes and solids in H of various types.

Lemma 13 *The hyperplane H contains exactly $q + 1$ solids of Type* I*, it contains exactly $q^3 + q^2$ solids of type* IIa *and exactly q^4 solids of Type* IIb*. Also, H contains exactly $q^2 + q + 1$ planes of Type* I*, it contains exactly $q^2(q^2 + q + 1)(q + 1)$ planes of Type* II*, and exactly q^6 planes of Type* III*.*

Proof. Let N_X be the number of solids of Type X in H, $X \in \{\mathrm{I}, \mathrm{IIa}, \mathrm{IIb}\}$. Then a double count of pairs (L, S), where L is a line of H in $\mathcal{L}$ and S is a solid in H containing L, reveals

$$N_{\mathrm{I}}(q^2+q+1) + (q^4+q^3+q^2+q+1-N_{\mathrm{I}})(q+1) = (q^3+q^2+q+1)(q^2+q+1),$$

which proves the first assertion of the lemma.

Now let n_0 be the number of incident pairs (L, π), with $L \in \mathcal{L}$ in H and π a plane of Type I in H, and let n_1 be the number of incident pairs (L, S), with $L \in \mathcal{L}$ in H and S a solid of Type I in H. If N'_{I} is the number of planes of Type I in H, then we easily obtain $n_0 = N'_{\mathrm{I}}(q+1)$ and $n_1 = (q+1)(q^2+q+1)$. But first counting the lines L, we deduce from Lemma 12 that $n_0 = n_1$. Hence $N'_{\mathrm{I}} = q^2+q+1$.

Next we count in two ways the pairs (L, π), where L is a member of $\mathcal{L}$ contained in H, and π is a plane in H containing L. If N'_Y is the number of planes of Type Y, $Y \in \{\mathrm{II}, \mathrm{III}\}$, contained in H, then we obtain

$$(q^3+q^2+q+1)(q^2+q+1) = N'_{\mathrm{I}}(q+1) + N'_{\mathrm{II}},$$

implying $N'_{\mathrm{II}} = q^2(q^2+q+1)(q+1)$. It follows now easily that there are $N'_{\mathrm{III}} = q^6$ planes of Type III in H.

Now we count in two ways the pairs (π, S), where π is a plane of Type I in H, and S is a solid (necessarily of Type I or IIa) in H containing π. We obtain $(q^2+q+1)(q+1) = N_{\mathrm{I}}(q+1) + N_{\mathrm{IIa}}$, which implies $N_{\mathrm{IIa}} = q^3+q^2$. It now also easily follows that there are $N_{\mathrm{IIb}} = q^4$ solids of Type IIb in H.

The lemma is completely proved. $\square$

We now prove a lemma analogously to Lemma 8

Lemma 14 *For every point x in H, there are equally many lines of $\mathcal{L}$ in H incident with x as there are solids of Type* I *in H incident with x.*

Proof. Let n be the number of lines of $\mathcal{L}$ in H through x, and let m be the number of solids of Type I in H incident with x. Counting in two ways the number of incident pairs (L, S), with $L \in \mathcal{L}$ in H, and S a solid of H through x, we obtain

$$n(q^2+q+1)+(q^3+q^2+q+1-n)(q+1) = m(q^2+q+1)+(q^3+q^2+q+1-m)(q+1),$$

which proves $n = m$. $\square$

Lemma 15 *Every line of* $\mathsf{PG}(5,q)$ *not belonging to* $\mathcal{L}$ *is incident with exactly* $q+1$ *solids of Type* I.

Proof. Let L be a line not belonging to $\mathcal{L}$. Let N_{I} be the number of solids of Type I containing L, and let N_{II} be the number of solids of Types IIa and IIb containing L. Then $N_{\mathrm{I}} + N_{\mathrm{II}} = (q^2+q+1)(q^2+1)$. We count in two different ways the pairs (S, M), where S is a solid of $\mathsf{PG}(5,q)$ containing L, and M is a member of $\mathcal{L}$ contained in S. Noting that exactly $(q+1)^2$ members of $\mathcal{L}$ meet L in a point, we obtain

$$N_{\mathrm{I}}(q^2+q+1) + N_{\mathrm{II}}(q+1) = (q+1)^2(q^2+q+1) + (q^5+q^4+q^3-q),$$

from which easily follows that $N_{\mathrm{I}} = q+1$. $\square$

We now prove that the notion of a pseudo-hexagon of $\mathsf{PG}(5,q)$ is a self-dual one, in a sense made precise in the following lemma.

Lemma 16 *The set* $\mathcal{L}'$ *of solids of Type* I *is a pseudo-hexagon in the dual of* $\mathsf{PG}(5,q)$. *Furthermore, a plane of Type* X *with respect to* $\mathcal{L}'$ *has the same Type* X *with respect to* $\mathcal{L}$, $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. *Also, a line of* $\mathsf{PG}(5,q)$ *not belonging to* $\mathcal{L}$ *is a solid of Type* IIa *in the dual of* $\mathsf{PG}(5,q)$ *with respect to* $\mathcal{L}'$ *if and only if it is contained in a plane of Type* I.

Proof. The first assertion follows from Lemma 4 (last assertion), Lemma 6, Lemma 7, Lemma 10, Lemma 13 and Lemma 15.

The second assertion follows from Lemma 6, Lemma 7 and Lemma 10. The last assertion follows from the defining property of solids of Type IIa and by Lemmas 6 and 15. $\square$

As a result of the previous lemma, we can dualize all lemmas proved sofar. In particular, two planes of Type I that meet in a line, meet in a member of $\mathcal{L}$. Also, the dual of Lemma 8 is Lemma 12, explaining why the proofs of these lemmas are so similar.

As a further application we can now show our First Main Result in case there is one plane of Type I in which all lines are concurrent. For the ease of phrasing, we use the terminology introduced in Section 2 and which we repeat here for the convenience of the reader. A point x of $\mathsf{PG}(5,q)$ will be called *flat* if the $q+1$ lines of $\mathcal{L}$ incident with x are contained in a common plane.

Lemma 17 *If at least one point of* $\mathsf{PG}(5,q)$ *is flat, then all points are flat.*

Proof. We assume dually that a hyperplane H is dually flat, which means that all solids of Type I in H contain a common plane π of Type I. A simple count of the number of members of $\mathcal{L}$ in H that are contained in the union of these $q+1$ solids of Type I reveals that all members of $\mathcal{L}$ in H are contained in these solids; similarly all planes of Type I in H are contained in the union of these $q+1$ solids. Since every point y of H outside π is contained in a unique solid S of Type I in H, it is contained in a unique member L of $\mathcal{L}$ in H by Lemma 14, which must then be contained in S by the foregoing. Similarly, now using Lemma 12, we see that L is contained in a unique plane of Type I in H, which must be contained in S; by Lemma 8 this plane is the unique plane π_L of Type I in S (and H) containing y. Hence the q planes of Type I in S distinct from π pairwise meet at a line of π, and hence they necessarily all share a common line L_∞ contained in π. Since also the lines of $\mathcal{L}$ in these planes do not meet outside π, all lines of π_L are concurrent in a point $x_\infty \in L_\infty$, and there are q such points x_∞ on L_∞, leaving a unique point $z \in L_\infty$ which is not incident with any member of $\mathcal{L}$ that lies in S, but not in π. Since x_∞ is incident with $q+1$ members of $\mathcal{L}$, only one (L_∞) of which is contained in π, the line L_∞ is the unique member of $\mathcal{L}$ in π through x_∞. Hence all other lines of $\mathcal{L}$ in π must be incident with the point z of L_∞. So z is flat, and also all points of π are flat.

The previous paragraph shows that, if we again dualize, if some point x is flat, and if we denote by π_x the plane containing all members of $\mathcal{L}$ through x, then all solids of Type I through π_x are contained in a hyperplane H, which in turn implies that all points of π_x are flat. If we now apply this to all points of π_x, then we see that all points of H are flat. Since every member of $\mathcal{L}$ meets H in at least one point, this now easily implies the lemma. $\square$

Lemma 18 *Assume that at least one point of* $\mathsf{PG}(5,q)$ *is flat. The incidence graph of the point-line geometry* $\mathcal{H}$ *with point set the set of points of* $\mathsf{PG}(5,q)$ *and line set* $\mathcal{L}$ *(with natural incidence relation) has girth at least* 12.

Proof. Suppose by way of contradiction that the incidence graph contains a cycle of length at most 11. Since the incidence graph is bipartite, and since two points are contained in at most one line, this implies that the geometry $\mathcal{H}$ contain a triangle, quadrangle, or pentagon ξ. Then ξ is contained in a hyperplane. Since all points are flat, the proof of the previous lemma shows that all hyperplanes are dually flat. This implies, as in the proof of Lemma 17, that the only points of H incident with at least two members of $\mathcal{L}$ in H are contained in a unique plane π of Type I. Hence ξ is contained in π, clearly a contradiction, as all lines of $\mathcal{L}$ in π are incident with a common point. $\square$

End of the proof of the Main Result in case of at least one flat point. The point-line geometry $\mathcal{H}$ with point set the set of points of $\mathsf{PG}(5,q)$ and line set $\mathcal{L}$ has order q and has girth at least 6. An easy count reveals that every element is at distance at most 6 from any other element. Hence $\mathcal{H}$ is a generalized hexagon and the result follows from Lemma 17 (which means that the embedding of $\mathcal{H}$ in $\mathsf{PG}(5,q)$ is flat) and the main result of [6].

Our next aim is to show that, if $q>2$, and if for some point x of $\mathsf{PG}(5,q)$, the lines of $\mathcal{L}$ though x are contained in a solid, then at least one point of $\mathsf{PG}(5,q)$ is flat. Hence, from now on, we assume that $q>2$, that no point of $\mathsf{PG}(5,q)$ is flat, and that we have a point x_0 of $\mathsf{PG}(5,q)$ with the property that the lines of $\mathcal{L}$ through x_0 are contained in (and hence span) a solid S_0. For reference we call this Assumption $(*)$. We seek a contradiction.

Lemma 19 *Under Assumption $(*)$, there are q members of $\mathcal{L}$ through x_0 contained in a plane π_0 of S_0. Also, S_0 is of Type* I *and in S_0, there is a unique member L_0 of $\mathcal{L}$ (which is incident with x_0) incident with q planes of Type I contained in S_0, and there is a hyperbolic quadric Q in S_0 containing L_0 and a line $M_0 \in \mathcal{L}$ in π_0 not meeting L_0 such that the members of $\mathcal{L}$ in S_0 are the following.*

- (i) *The lines L_0 and M_0.*
- (ii) *For each point x of L_0 different from x_0, the $q-1$ lines through x not on Q, but contained in the tangent plane at x to Q.*
- (iii) *The lines through x_0 meeting M_0 but not contained in Q.*
- (iv) *The $q-1$ lines on Q skew to both L_0 and M_0.*

Proof. Since we assume that x_0 is not flat, S_0 is of Type I, and we can apply Lemma 9 to obtain a (possibly degenerate) projective plane Ω with point set the lines of $\mathcal{L}$ through x_0, and with line set the set of planes of type I containing x_0 in S_0. If Ω were nondegenerate, then it would have order p^h, for some positive integer h, with $q=p^n$, p a prime number. Indeed, Ω is a subplane of the projective plane defined by all lines and planes of S_0 through x_0, and as such it is Desarguesian. It now follows that $p^n=p^{2h}+p^h$, a contradiction. So Ω is degenerate and q members of $\mathcal{L}$ through x_0 are contained in a plane π_0 of Type I. Let $L_1,L_2,\ldots,L_q$ be the lines of $\mathcal{L}$ through x_0 contained in π_0. Let $\{x_0,x_1,x_2,\ldots,x_q\}$ be the set of points incident with L_0. It is now clear that the $q+1$ planes of Type I in S_0 are π_0 and the planes $\pi_i:=\langle L_0,L_i\rangle$, $i=1,2,\ldots,q$. It follows that all lines of $\mathcal{L}$ through x_i, with $i\in\{1,2,\ldots,q\}$, and contained in S_0, are contained in one of the planes π_j. By

Lemma 8, there are q lines of $\mathcal{L}$ in S_0 through x_i and so we may choose indices in such a way that the lines in S_0 through x_i are contained in π_i, for all $i \in \{1, 2, \ldots, q\}$.

Now let M_0 be the member of $\mathcal{L}$ in π_0 not through x_0. Let K_i be the line through x_i in π_i not belonging to $\mathcal{L}$, $i = 1, 2, \ldots, q$. Then M_0 meets K_i, since otherwise M_0 meets a line K of $\mathcal{L}$ in π_i, and then $\langle M_0, K\rangle$ would be a plane of Type I, contradicting the facts that it cannot be equal to π_0 (since it contains x_i and $i \neq 0$) and that it cannot be equal to any other π_j, since M_0 is not contained in π_j, $j \in \{1, 2, \ldots, q\}$. Evidently, also L_0 meets K_i.

One can count that there are $q^2 + 2$ lines of $\mathcal{L}$ belonging to a plane of Type I contained in S_0. Hence there remain $q - 1$ members of $\mathcal{L}$ not meeting any of these $q^2 + 2$ previous lines and contained in S_0. Noting that such a line must meet π_i in a point of K_i, we see that these $q - 1$ lines, together with L_0 and M_0, form a regulus; the opposite regulus contains the lines K_j, $j \in \{1, 2, \ldots, q\}$.

The lemma is now clear. $\square$

A solid of Type I structured as in the previous lemma will be called a *hyperbolic* solid, and the point x_0 its *origin*, the line L_0 its *ridge* and the plane π_0 its *ceiling*.

A plane π of type I contained in a hyperplane H will be called *isolated in H* if it is not contained in any solid of Type I contained in H. A member of $\mathcal{L}$ in H will be called *isolated in H* if it is not contained in any plane of Type I contained in H; by Lemma 12, this is equivalent to being not contained in any solid of Type I contained in H.

Lemma 20 *Every plane of type* I *is isolated in some hyperplane.*

Proof. Let π be a plane of Type I. Suppose by way of contradiction that π is not isolated in any hyperplane it belongs to. We know that there are $q+1$ solids of Type I through π. Since every hyperplane through π must contain at least one of these, all these solids lie in one hyperplane H containing π. It follows that in the dual of $\mathsf{PG}(5, q)$, the hyperplane H is a flat point with respect to $\mathcal{L}'$. The proof of Lemma 17 now implies that all points of π are flat, contradicting our assumptions. $\square$

Lemma 21 *Let π be a plane of Type* I *isolated in the hyperplane H. Then every member of $\mathcal{L}$ in π is contained in a unique solid of Type* I *belonging to H. Also, every point x of π that is incident with a member of $\mathcal{L}$ is contained in a unique plane of Type* I *distinct from π and contained in H.*

Proof. We count the number of solids of Type I in H through a line $L \in \mathcal{L}$ of π. Such a solid cannot contain π. Note that no solid through L not containing π is of Type IIa, since otherwise there would be some member $L' \in \mathcal{L}$ meeting L and not contained in π, which would imply that the solid of Type I spanned by π and L' is contained in H, and contains π, contradicting the isolation of π. Suppose that there are N solids of Type I in H through L and not containing π. Then there are $q^2 - N$ solids of Type IIb in H through L and not containing π. A double count of the incident pairs (M, S), with $M \in \mathcal{L}$ in H but not contained in π, and S a solid in H through L but not containing π reveals that $N(q^2+q) + (q^2 - N)q = q^3 + q^2$, implying $N = 1$. Since π contains exactly $q+1$ lines of $\mathcal{L}$, every solid of Type I in H is incident with a unique line of $\mathcal{L}$ in π.

Suppose now, by way of contradiction, that some point x incident with a member L of $\mathcal{L}$ in π is incident with two planes π_1 and π_2 of Type I in H, and both different from π. Since $\langle \pi_1, L\rangle$ and $\langle \pi_2, L\rangle$ are solids of Type I, they have to coincide since they both contain L. Then x is incident in $\langle \pi_1, L\rangle$ with at least two planes of Type I, hence by Lemma 8, x must be incident with at least two members of $\mathcal{L}$ lying in $\langle \pi_1, L\rangle$, contradicting the isolation of π. Now let S be a solid through some line L of $\mathcal{L}$ in π. Then S contains precisely $q+1$ planes of Type I meeting L in different points by the foregoing. Hence every point of L is contained in at least one such plane, and the lemma is proved. $\square$

Lemma 22 *Let π be a plane of Type* I *isolated in the hyperplane H. Suppose that L_1, L_2, L_3 are three members of $\mathcal{L}$ in π not incident with a common point. Let π_i, $\pi_i \neq \pi$, be the unique plane of Type* I *in H containing the intersection point $L_j \cap L_k$, for $\{i,j,k\} = \{1,2,3\}$. Then $\pi_1 \cap \pi_2 \cap \pi_3$ is a line belonging to $\mathcal{L}$.*

Proof. Let S_i be the unique solid of Type I in H containing L_i. Since these solids have no common point in π, they have a line L in common. Since $S_i = \langle L_i, \pi_j\rangle = \langle L_i, \pi_k\rangle$, for $\{i,j,k\} = \{1,2,3\}$, we deduce that $\pi_i = \langle L_j \cap L_k, L\rangle$, with $\{i,j,k\} = \{1,2,3\}$, and so L is the intersection of the planes π_1, π_2, π_3. The assertion now follows from the fact that two planes of Type I that meet in a line always meet in a line of $\mathcal{L}$. $\square$

We now return to Assumption $(*)$. By Lemma 21, there is a hyperplane H_0 in which π_0 is isolated.

Lemma 23 *Assumption $(*)$ leads to a contradiction.*

Proof. In π_0, we denote the lines of $\mathcal{L}$ through x_0 by $L_1, L_2, \ldots, L_q$. The unique member of $\mathcal{L}$ in π_0 not through x_0 will be denoted by L_0. The unique line in π_0 through x_0 not belonging to $\mathcal{L}$ will be denoted by L. Put $x_i = L_i \cap L_0$, $i = 1, 2, \ldots, q$. Let π be the unique plane of Type I in H_0 through x_0 and different from π_0. Let π_i be the unique plane of Type I in H_0 through x_i and different from π_0, $i = 1, 2, \ldots, q$. Let x be the intersection of L with L_0. By Lemma 22, the line $M := \pi \cap \pi_1$ is also contained in π_i, for all $i \in \{2, 3, \ldots, q\}$. The $q+1$ solids of Type I in H_0 are $S := \langle L_0, M \rangle$ and $S_i := \langle L_i, M \rangle$, for $i = 1, 2, \ldots, q$. If the plane $\langle x, M \rangle$ were of Type I, then in S, there would be $q + 1$ planes of type I through each point of M, hence for each such point, all lines of $\mathcal{L}$ through it would be contained in S (see Lemma 8), and it is easy to see that in this case all points on M would be flat, a contradiction. Hence, in H_0 (and S), the plane π_x, $\pi_x \neq \pi_0$, of Type I on x meets M in a unique point y. The q lines in S on any other point of M all must be contained in one plane π_i, for some $i \in \{1, 2, \ldots, q\}$ (as otherwise a plane generated by two such lines would be of Type I and not belong to $\{\pi_x\} \cup \{\pi_i \mid i \in \{1, 2, \ldots, q\}\}$). Hence for each plane π_i, $i \in \{1, 2, \ldots, q\}$, there is a point $y_i \in M$, $y_i \neq y$, incident with q members of $\mathcal{L}$ incident with π_i. It follows that y is incident with $q+1$ members of $\mathcal{L}$ lying in S, and as $\pi_i \cap \pi_x$ with $i \in \{1, 2, \ldots, q\}$ belongs to $\mathcal{L}$, q of them lie in π_x (and these are all lines through y in π_x not incident with x). Hence S is a hyperbolic solid with origin y, ridge M and ceiling π_x. From the structure of hyperbolic solids explained in Lemma 19, we deduce that the set of points of S not incident with a member of $\mathcal{L}$ contained in S are the $q(q-1)$ points of the plane $\beta := \langle x, M \rangle$ not on the lines M and xy. Also, the lines x_1y_1, x_2y_2, $\ldots$, x_qy_q and xy are a set of generators of a hyperbolic quadric Q^+ in S.

Fix $i \in \{1, 2, \ldots, q\}$. The point y_i is incident with $q+1$ solids of Type I contained in H_0. By Lemma 14, all lines of $\mathcal{L}$ on y_i are contained H_0. Since q of these lines are contained in the plane π_i, H_0 contains a hyperbolic solid through π_i with origin y_i. Since there are only two solids of Type I through π_i in H_0, this solid must be S_i (otherwise S would have two origins, which is impossible). Since $\pi \subset S_i$, it follows that there is a point z in π incident with q members of $\mathcal{L}$ contained in π. Since neither yz nor x_0z belong to $\mathcal{L}$ (noting that all members of $\mathcal{L}$ through y are either in π_x, which meets π in the unique point y, or equal M), we see that z belongs to the line x_0y. Moreover, the line y_iz is the ridge of S_i. So it follows that the only points of S_i not incident with any member of $\mathcal{L}$ contained in S_i, are the $q(q-1)$ points of the plane $\beta_i := \langle x_i, y_i, z \rangle$ not on the lines x_iy_i and y_iz. Note that, for $i \neq j$, the planes β_i and β_j have only one point in common. This is the point z.

We now count the number of isolated members of $\mathcal{L}$ in H_0. There are $q + 1$ solids of Type I in H_0, hence counting with multiplicities, we obtain $(q+1)(q^2+q+1)$ lines of $\mathcal{L}$ contained in solids of Type I contained in H_0. This way, we counted every such line once, except for the lines in π different from M, which we counted q times, the lines in each π_i different from M, which we counted twice, $i \in \{1, 2, \ldots, q\}$, and M, which we counted $q+1$ times. So we finally obtain $(q+1)(q^2+q+1) - (q-1)q - q^2 - q = q^3 + 2q + 1$

lines of $\mathcal{L}$ in solids of Type I contained in H_0. This means that there are precisely $q^2 - q$ isolated lines. Each of these lines meets any solid of Type I in a point not incident with any member of $\mathcal{L}$ contained in that solid. Hence we have $q^2 - q$ lines in H_0 meeting β and each β_i, $i = 1, 2, \ldots, q$, in distinct points.

We now project from the point z onto the solid S the isolated lines in H_0 and the planes β_i, $i \in \{1, 2, \ldots, q\}$. The plane β_i is projected onto the line $x_i y_i$. Each isolated line R is projected onto a line R' meeting all of $x_i y_i$, $i \in \{1, 2, \ldots, q\}$. If $q > 2$, this implies that R' is contained in Q^+ and meets the line xy in a unique point. Hence the unique point of R in $\beta \subset S$ is a point of the line xy, a contradiction as $q^2 - q > q - 1$.

The lemma is proved. $\square$

4 Non-flat pseudo-hexagons

We now present a class of non-flat pseudo-hexagons.

Let G be a standard cyclic Singer group of $\mathsf{PG}(5, q)$, i.e., G is a cyclic subgroup of order $q^5 + q^4 + q^3 + q^2 + q + 1$ of $\mathsf{PGL}(6, q)$ acting sharply transitively on the point set of $\mathsf{PG}(5, q)$. Let $\mathcal{L}$ be an orbit of lines under the action of G of length $q^5 + q^4 + q^3 + q^2 + q + 1$ with the property that no nontrivial element of G fixes any plane containing a member of $\mathcal{L}$. This condition means the following. The unique subgroup H of G of order $q^2 + q + 1$ lies in a subgroup isomorphic to $\mathsf{PGL}(3, q)$ and hence fixes the planes of a (regular) spread of $\mathsf{PG}(5, q)$ (since it fixes at least one plane — that associated with $\mathsf{PGL}(3, q)$— by permuting its point set transitively, and since all point-orbits are isomorphic by the action of G), and the last condition just says that no member of $\mathcal{L}$ is contained in a plane of this spread. By the way, the point set of any plane of this spread can be obtained as point orbit under the action of H. Also, a line L of $\mathsf{PG}(5, q)$ has an orbit of length $q^5 + q^4 + q^3 + q^2 + q + 1$ under G if and only it is not fixed under the subgroup of order $q + 1$ of G. These claims are easy to verify and show that sets $\mathcal{L}$ with the required properties exist in abundance.

We show that $\mathcal{L}$ is a non-flat pseudo-hexagon. We first fix some notation. Throughout, we let g be a fixed generator of G and we choose arbitrarily a point p_0 in $\mathsf{PG}(5, q)$. For each integer i, we denote by p_i the point $p_0^{g^i}$. Let L_0 be an arbitrary member of $\mathcal{L}$ through p_0 and denote by L_i the line $L_0^{g^i}$, for all integers i. In the sequel, when dealing with subscripts of p and L, we will always assume that integers are taken modulo $q^5 + q^4 + q^3 + q^2 + q + 1$, and we will not repeat this each time.

$\mathcal{L}$ satisfies (Pt)

By the action of the group G, we only need to verify Property (Pt) for p_0.

Clearly, a line L_i is incident with p_0 if and only if p_{-i} is incident with L_0. Since L_0 contains $q+1$ points, the assertion follows.

$\mathcal{L}$ satisfies (Pl)

Suppose the plane π contains two members of $\mathcal{L}$, which we may, without loss of generality, assume to be L_0 and L_i, with p_0 incident with L_i (possibly we have to make another choice for L_0 though p_0 to obtain this, or, equivalently, we must choose on L_0 another point for p_0). Note that this implies that p_i is on L_i and p_{-i} is on L_0. Consider any point p_k on L_0 and apply g^{i+k}. The latter maps p_{-i} to p_k and p_0 onto $p_{i+k} = p_k^{g_i} \in L_i$. Hence L_{i+k} is contained in π. It follows that we already have $q+1$ members of $\mathcal{L}$ contained in π.

Suppose there are at least $q+2$ members of $\mathcal{L}$ contained in π. Then there exist three members L_u, L_v, L_w of $\mathcal{L}$ in π such that p_u, p_v, p_w are not collinear. Since $p_{-i} \in L_0$, we have $p_{j-i} = p_{-i}^{g^j} \in L_j$, for all $j \in \{u, v, w\}$. Hence the image of $\{p_u, p_v, p_w\}$ under g^{-i} belongs to π and generates it. Consequently g^{-i} fixes π, contrary to our assumptions. Note that this argument is independent of whether L_0 belongs to π or not.

Note that the previous arguments imply that the lines of $\mathcal{L}$ contained in the plane π form a dual oval. Hence p_0 cannot be a flat point.

$\mathcal{L}$ satisfies (Sd)

First we show that every solid S containing at least two members, say L_0 and L_j, of $\mathcal{L}$ contains at least $q+1$ members of $\mathcal{L}$.

If L_0 and L_j are concurrent, then the assertion follows from Property (Pl). Suppose now that L_0 and L_j are not concurrent. Let p_n be any point of L_0. Then $p_{n+j} \in L_j$. Let p_z be any point of the line p_0p_j. Then p_{z+n} belongs to p_np_{j+n} and hence the line p_zp_{z+n} is contained in S. But L_z contains $p_0^{g^z} = p_z$ and $p_n^{g^z} = p_{n+z}$, hence L_z is contained in S. There are $q+1$ choices for p_z, hence the assertion.

Note that the $q+1$ lines thus constructed form a regulus of a nonsingular hyperbolic quadric in S (as they all contain a point of p_np_{n+j}, for all p_n on L_0).

Suppose now the solid S contains at least q^2+q+2 members of $\mathcal{L}$. Then there exist four members L_u, L_v, L_w, L_t of $\mathcal{L}$ in S with p_u, p_v, p_w, p_t not coplanar. A similar argument as above shows that in this case for every integer i, with $p_{-i} \in L_0$, the map g^{-i} fixes S. Hence $\langle g^{-i} \rangle$ acts freely on the set of q^3+q^2+q+1 points of S, implying that the order of g^{-i} divides both of $q^5+q^4+q^3+q^2+q+1$ and q^3+q^2+q+1; hence it divides $q+1$ and consequently g^{-i} fixes a spread of lines of $\mathsf{PG}(5,q)$. This spread obviously consist of the lines $p_n p_{n-i}$. But these lines belong to $\mathcal{L}$, hence the orbit L_0 under G contains only q^4+q^2+1 lines, contradicting our assumptions.

We next claim that there are no solids containing r members of $\mathcal{L}$, with $q+1 < r < q^2+q+1$. Indeed, let S be a solid containing r members of $\mathcal{L}$, with $q+1 < r < q^2+q+1$. We define a linear (block) space $\mathcal{S}$ as follows. The points are the members of $\mathcal{L}$ in S, the blocks are the planes in S containing exactly $q+1$ members of $\mathcal{L}$, together with the reguli in S constructed with two skew members of $\mathcal{L}$ in S as above. It is easy to see that this defines indeed a linear space, and every block has exactly $q+1$ elements. Since $r > q+1$, we can find a block B in S and a member L of $\mathcal{L}$ in S not contained in B. Considering the blocks containing L and an element of B, we obtain $(q+1)(q-1)$ additional points of $\mathcal{S}$. Hence $\mathcal{S}$ contains at least $(q^2-1)+(q+1)+1 = q^2+q+1$ points, implying our claim.

This proves Condition (Sd).

The geometry with point set $\mathsf{PG}(5,q)$, line set $\mathcal{L}$ and natural incidence relation will be called a *Singer geometry*. Obviously, one can generalize this to other dimensions. These Singer geometries seem to have rather special properties. In particular, the line sets have few intersection numbers with respect to subspaces.

5 Pseudo-hexagons and Singer geometries in $\mathsf{PG}(5,2)$

We now take a closer look at the case $q=2$. It is more convenient to start with a given geometry rather than with $\mathsf{PG}(5,2)$ and then construct a Singer cycle.

So we consider the geometry $\mathcal{S}$ with point set the integers modulo 63 and line set $\mathcal{L}$ the triples $\{z, z+1, z+6\}$, for any z modulo 63. We let the points $0,1,2,3,4,5$ be a basis of a 6-dimensional vector space V over $\mathsf{GF}(2)$, and we denote the basis vectors $e_0, e_1, e_2, e_3, e_4, e_5$, respectively, and define the vectors e_i, $6 \le i \le 62$, by the rule $e_i = e_{i-5} + e_{i-6}$ (so that the vectors e_z, e_{z+1} and e_{z+6} lie in a common plane, and hence as points of the projective space $\mathsf{PG}(V)$, they form a line. It then follows that the relation $e_i = e_{i-5} + e_{i-6}$ also holds for $0 \le i \le 5$, modulo 63. Hence we have embedded $\mathcal{S}$ into $\mathsf{PG}(V) = \mathsf{PG}(5,2)$, and the map $e_i \mapsto e_{i+1}$ defines a Singer cycle G acting sharply transitively on both the point set and the line set of $\mathcal{S}$ (identifying the point set of $\mathcal{S}$ with the point set of $\mathsf{PG}(5,2)$ in the obvious way). If we denote a vector by its nonzero coordinates, then we obtain the following table.

$e_0 = 0$	$e_1 = 1$	$e_2 = 2$	$e_3 = 3$	$e_4 = 4$	$e_5 = 5$	$e_6 = 01$
$e_7 = 12$	$e_8 = 23$	$e_9 = 34$	$e_{10} = 45$	$e_{11} = 015$	$e_{12} = 02$	$e_{13} = 13$
$e_{14} = 24$	$e_{15} = 35$	$e_{16} = 014$	$e_{17} = 125$	$e_{18} = 0123$	$e_{19} = 1234$	$e_{20} = 2345$
$e_{21} = 01345$	$e_{22} = 0245$	$e_{23} = 035$	$e_{24} = 04$	$e_{25} = 15$	$e_{26} = 012$	$e_{27} = 123$
$e_{28} = 234$	$e_{29} = 345$	$e_{30} = 0145$	$e_{31} = 025$	$e_{32} = 03$	$e_{33} = 14$	$e_{34} = 25$
$e_{35} = 013$	$e_{36} = 124$	$e_{37} = 235$	$e_{38} = 0134$	$e_{39} = 1245$	$e_{40} = 01235$	$e_{41} = 0234$
$e_{42} = 1345$	$e_{43} = 01245$	$e_{44} = 0235$	$e_{45} = 034$	$e_{46} = 145$	$e_{47} = 0125$	$e_{48} = 023$
$e_{49} = 134$	$e_{50} = 245$	$e_{51} = 0135$	$e_{52} = 024$	$e_{53} = 135$	$e_{54} = 0124$	$e_{55} = 1235$
$e_{56} = 01234$	$e_{57} = 12345$	$e_{58} = 012345$	$e_{59} = 02345$	$e_{60} = 0345$	$e_{61} = 045$	$e_{62} = 05$

So $\mathcal{L}$ is a pseudo-hexagon as soon as we verify that no member of $\mathcal{L}$ is contained in a plane belonging to the plane spread S defined by G. But these planes are $\{z, z+9, z+18, z+27, z+36, z+45, z+54\}$, for $0 \leq z \leq 8$, and so none of them contains $\{0, 1, 6\}$.

Similarly, the set $\mathcal{L}'$ of lines $\{z, z+7, z+26\}$, z mod 63, is a pseudo-hexagon. We now show that $\mathcal{L}$ and $\mathcal{L}'$ are not projectively equivalent.

For any pseudo-hexagon arising from a Singer group in $\mathsf{PG}(5,2)$, we consider the following transformation ∂, which we call *derivative*. With each plane π of Type I, we let correspond the dual nucleus of the dual conic consisting of the lines of the pseudo-hexagon in π. This way we obtain a set of 63 lines invariant under the Singer group. It is not clear that it is again a pseudo-hexagon, but applied to $\mathcal{L}$ above, we do obtain a pseudo-hexagon $\partial\mathcal{L}$ consisting of the lines $\{z, z+2, z+12\}$, as one easily checks. In fact, $\mathcal{L}$ is isomorphic to $\partial\mathcal{L}$ using the isomorphism $i \mapsto 2i$ (modulo 63). Doing this again, we obtain $\partial^2\mathcal{L}$, consisting of the lines $\{z, z+4, z+24\}$. We now see that $\partial^6\mathcal{L} = \mathcal{L}$, and that $\partial^i\mathcal{L} \neq \mathcal{L}$, for all $i \in \{1, 2, 3, 4, 5\}$.

Playing the same game with $\mathcal{L}'$, we easily calculate that ∂ again corresponds to the isomorphism $i \mapsto 2i$ (modulo 63), but this time $\partial^3\mathcal{L}' = \mathcal{L}'$. Hence $\mathcal{L}$ and $\mathcal{L}'$ cannot be projectively equivalent.

So we already have at least three non-isomorphic pseudo-hexagons in $\mathsf{PG}(5,2)$.

Remarks.

1. In fact, all lines of $\mathsf{PG}(5,2)$ are given by the lines inside the planes of S, the lines of the line spread $\{\{z, z+21, z+42\} \mid z \text{ mod } 63\}$, and the lines of the pseudo-hexagons $\mathcal{L}, \partial\mathcal{L}, \partial^2\mathcal{L}, \partial^3\mathcal{L}, \partial^4\mathcal{L}, \partial^5\mathcal{L}, \mathcal{L}', \partial\mathcal{L}'$ and $\partial^2\mathcal{L}'$.

2. The Singer geometries Γ and Γ' defined by $\mathcal{L}$ and $\mathcal{L}'$, respectively, are not isomorphic. This follows easily from the observation that the functor ∂ can be extended to Γ and Γ' and defined abstractly as follows. The points of $\partial\Gamma$ are the points of Γ, and the lines are the triples of points $\{x, y, z\}$ such that there exist a triangle $\{a, b, c\}$ in Γ with x on bc, y on ca and z on ab, where $\{a, b, c\} \cap \{x, y, z\} = \emptyset$. Similarly for $\partial\Gamma'$.

We now want to classify pseudo-hexagons in $\mathsf{PG}(5, 2)$. For this, we invoke the theory of bislim geometries developed in [11]. As in the above Remark 2, we consider, for a given pseudo-hexagon $\mathcal{L}$ in $\mathsf{PG}(5, 2)$, the geometry $\Gamma(\mathcal{L})$ with point set the set of points of $\mathsf{PG}(5, 2)$ and line set $\mathcal{L}$, with the obvious and natural incidence relation. Then $\Gamma(\mathcal{L})$ is a so-called *bislim geometry*, i.e., a geometry with 3 points on each line and 3 lines through each point.

Following [11], we define the *local structure* of $\Gamma(\mathcal{L})$ in a point x as the geometry induced on the seven points collinear with or equal to x, ignoring lines of size one. We call $\Gamma(\mathcal{L})$ *of honeycomb type* if for each point x, the local structure in x has exactly 3 lines of size 3 and exactly 3 lines of size 2, the latter pairwise disjoint. We have the following easy but useful lemma.

Lemma 24 *If Γ is a bislim geometry with the property that every pair of concurrent lines is contained in exactly one triangle, then Γ is of honeycomb type.*

Proof. The condition obviously implies that each local structure only contains three lines of size 3. Now let x be the intersection point of two distinct intersecting lines L and M in Γ and let K be the third line of Γ through x, $L \neq K \neq M$. Let $P \in \mathcal{L}$ be such that L, K, P form a triangle, and let $N \in \mathcal{L}$ be such that L, M, N form a triangle. Then it suffices to show that the intersection y of L and N is different from the intersection z of L and P. If not, then there is a point u on L distinct from both x and $y = z$. Let Q be any line through u distinct from L. Then our assumptions imply that Q does meet any of M, K, N, P (otherwise we can form two triangles with a certain pair of intersecting lines). But then the third side of the triangle containing the sides L and Q must be a fourth line through either x or y, contradicting the fact that Γ is bislim. $\square$

Now, if a pseudo-hexagon $\mathcal{L}^*$ in $\mathsf{PG}(5, 2)$ does not define a generalized hexagon, then for every point x of $\mathsf{PG}(5, 2)$, the three lines of $\mathcal{L}^*$ through x are not contained in a plane, see the Main Result. Hence every pair $\{L.M\}$ of concurrent lines of $\Gamma(\mathcal{L}^*)$ are contained in a unique triangle and we can apply the above lemma. But then the arguments on page 91 up to 93 in [10] imply that $\Gamma(\mathcal{L}^*)$ is a quotient of the so-called *honeycomb geometry*, which is the unique bislim geometry with incidence graph the graph defined in the obvious way by a honeycomb, i.e., a regular tiling of the Euclidean real plane in hexagons. All these quotients are determined in [11]. From the main result in [10], we easily deduce that there are 17 pairwise non-isomorphic bislim geometries with 63 points which are quotients of the honeycomb geometry. We will now quickly define these, using their names given in [10].

For positive integers a, c, d, define the following geometry $\mathcal{M}_{(a,0),(b,c)}$. The points are the equivalence classes of ordered pairs (i, j) of integers with respect to the equivalence relation E defined as $(i, j)E(i', j')$ if $(i - i', j - j') = (ka + \ell c, \ell d)$, for some integers k and ℓ. The lines are the 3-sets $\{(i, j)/E, (i+1, j)/E, (i, j+1)/E\}$. This geometry has ad points.

For positive integers r, s, define the following geometry $\mathcal{M}^{**}_{(r),(s,0)}$. The points are the equivalence classes of ordered pairs (i, j) of integers with respect to the equivalence relation E^{**} defined as $(i, j)E^{**}(i', j')$ if either $(i - i', j - j') = (-k(2r + 1) + \ell s, 2k(2r + 1))$, for some integers k and ℓ, or $(i+i'+r+j, j'-j-2r-1) = (-k(2r+1)+\ell s, 2k(2r+1))$. The lines are the 3-sets $\{(i, j)/E^{**}, (i+1, j)/E^{**}, (i, j+1)/E^{**}\}$. This geometry has $(2r+1)s$ points.

One must remark that the positive integers have to be large enough in order to really obtain a bislim geometry which is of honeycomb type. In particular, $r \geq 2$ and $s \geq 4$, and for every integer linear combination of $(a, 0)$ and (c, d), say (n, m), we must have $n^2 + nm + m^2 \geq 12$; see [11] for more details.

So we see that we obtain a geometry with 63 points if we choose a, d in the above such that $ad = 63$ and r, s such that $(2r + 1)s = 63$. According to Lemma 6.2 and Proposition 6.3 of [11], the following is a complete list of pairwise non-isomorphic bislim geometries of honeycomb type with 63 points:

$$\begin{gathered}
\mathcal{M}_{(7,0),(0,9)}, \mathcal{M}_{(7,0),(1,9)}, \mathcal{M}_{(7,0),(2,9)}, \mathcal{M}_{(7,0),(6,9)}, \\
\mathcal{M}_{(9,0),(1,7)}, \mathcal{M}_{(9,0),(3,7)}, \mathcal{M}_{(9,0),(4,7)}, \mathcal{M}_{(9,0),(5,7)}, \\
\mathcal{M}_{(21,0),(1,3)}, \mathcal{M}_{(21,0),(2,3)}, \mathcal{M}_{(21,0),(3,3)}, \mathcal{M}_{(21,0),(5,3)}, \mathcal{M}_{(21,0),(6,3)}, \mathcal{M}_{(21,0),(8,3)}, \mathcal{M}_{(21,0),(19,3)}, \\
\mathcal{M}^{**}_{(4),(7,0)}, \mathcal{M}^{**}_{(3),(9,0)}.
\end{gathered}$$

In order to see whether each of these geometries defines a pseudo-hexagon, one has to see whether there exists some embedding in $\mathsf{PG}(5, 2)$. According to [3], this can be done as follows. For each geometry Γ above, define an abelian additive 2-group G with 63 generators a_x, one for each point x of Γ, and with additional relations $a_x + a_y + a_z = 0$, for every triple of points x, y, x that form a line in Γ. If this group G turns out to have less than 64 elements (and hence at least two generators have to coincide), then there is no embedding, and hence no pseudo-hexagon related with Γ. One can do this exercise by hand, and after some tedious computations, one can prove that G is trivial for all cases except for $\mathcal{M}_{(9,0),(1,7)}$ and $\mathcal{M}_{(21,0),(8,3)}$. Hence there are at most two non-isomorphic pseudo-hexagons in $\mathsf{PG}(5, 2)$. Since we already constructed two, we have exactly two of them. One can also calculate that $\mathcal{M}_{(21,0),(8,3)}$ corresponds with the pseudo-hexagon $\mathcal{L}$ above, and $\mathcal{M}_{(9,0),(1,7)}$ corresponds with $\mathcal{L}'$. This completes (a sketch of) the proof of the last theorem of the introduction.

6 A conjecture and an open problem

The following conjecture is based on research in progress and will hopefully soon be concluded. The open problem that we will state could also be formulated as a conjecture, but we do not have reasonable evidence to be convinced of either answer.

Conjecture. *If $\mathcal{L}$ is a pseudo-hexagon in $\mathsf{PG}(5, q)$, $q > 3$, with the property that for some point x, the lines of $\mathcal{L}$ through x do not span $\mathsf{PG}(5, q)$, then $\mathcal{L}$ is the line set of a regularly embedded split Cayley generalized hexagon.*

Open Problem. Are the Singer geometries the only examples of pseudo-hexagons that contain triangles?

A positive answer to this question would imply a beautiful characterization of these Singer geometries, that would be worthwhile to generalize to other dimensions. For the moment, however, a solution to this problem seems out of reach (at least, to us).

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