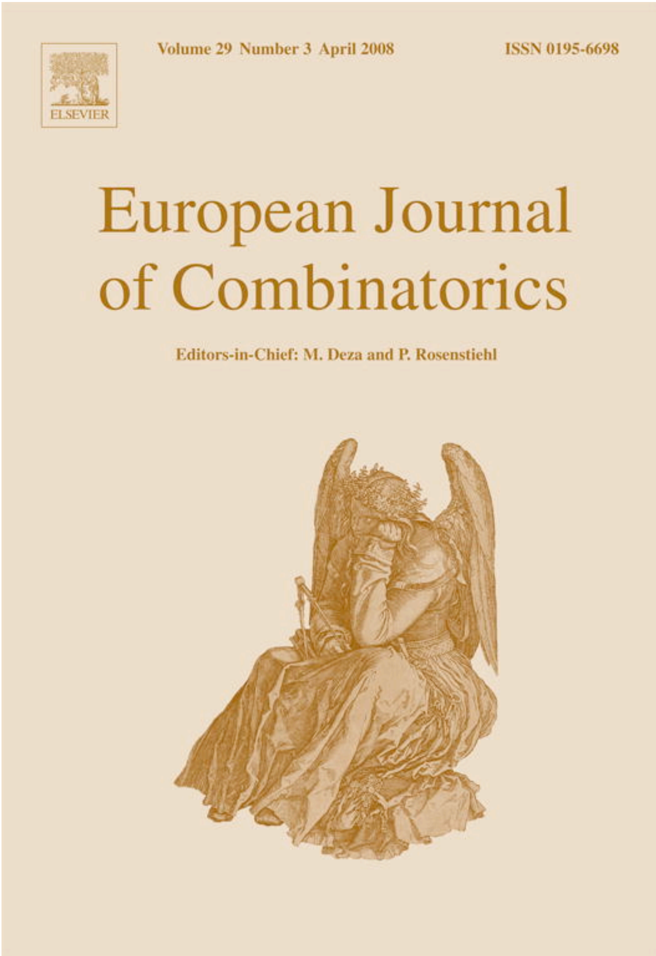
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The automorphism group of a class of strongly regular graphs related to $Q(6, q)$

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Abstract

In [A. Devillers, H. Van Maldeghem, Partial linear spaces built on hexagons, European J. Combin. 28 (2007) 901–915], Devillers and Van Maldeghem determined the automorphism group of four classes of geometries that have as collinearity graph the graph $\Gamma(q)$ of all elliptic hyperplanes of a given parabolic quadric $Q(6, q)$ in PG$(6, q)$ (adjacency is given by intersecting in a tangent 4-space). In their introduction they mention that at the time they were not able to determine the full automorphism group of $\Gamma(q)$, but that their results might be useful for proving that it is isomorphic to $P\Gamma O(7, q)$. In this note we use one of their results to prove that this is indeed the case.

1. Introduction and statement of the main theorem

Let $Q(6, q)$ be any given non-degenerate parabolic polar space in PG$(6, q)$. Define the following graph $\Gamma(q)$: the vertices of $\Gamma(q)$ are all non-degenerate elliptic quadrics $Q^-(5, q) \subset Q(6, q)$ and two vertices are adjacent provided the corresponding elliptic quadrics intersect in a tangent 4-space, that is, a cone $pQ^-(3, q)$. In [3, Theorem 3] it is shown that $\Gamma(q)$ is strongly regular (see also Thas [4]). The aim of this note is to determine the full automorphism group of $\Gamma(q)$.

In Devillers and Van Maldeghem [3] the following geometry $\Gamma_1(q)$, $q > 2$ was introduced: the points of $\Gamma_1(q)$ are all non-degenerate elliptic quadrics $Q^-(5, q) \subset Q(6, q)$ and the blocks

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are all sets of q elliptic quadrics mutually intersecting in a fixed tangent 4-space. Note that $\Gamma(q)$ is the collinearity graph of $\Gamma_1(q)$. The following result is Theorem 6 of their paper. It is also implicitly contained in Cuypers [1] as we will see at the end of this section.

Theorem 1.1 *([3]). The full collineation group of $\Gamma_1(q)$, $q > 2$, is isomorphic to $P\Gamma O(7, q)$.*

It is this theorem that will turn out to be useful for proving the result conjectured in the introduction of [3], that is, the main theorem of this article.

Main Theorem 1.2. *The full automorphism group of $\Gamma(q)$, $q > 2$, is isomorphic to $P\Gamma O(7, q)$.*

We will prove this theorem by showing that each automorphism of $\Gamma(q)$ induces an automorphism of the geometry $\Gamma_1(q)$. In order to do so we will use another geometry, $N^-O(7, q)$ (see Cuypers [1]), which is a rank 3 geometry having $\Gamma_1(q)$ as its point–line system and having $\Gamma(q)$ as its point graph (in fact we really only use this geometry if $q = 4$, but as was pointed out to the authors by a referee this geometry allows one to shorten the original argument for $q = 4$ by several pages).

We will prove that it is possible to recognizes triples of points of $N^-O(7, q)$, $q \neq 2$, that are on a plane of $N^-O(7, q)$ by counting the number of 4-cliques that contain a certain 3-clique. This allows one to recover the planes of $N^-O(7, q)$ from the graph $\Gamma(q)$. As it is possible to recover the lines of $N^-O(7, q)$ using only the points and planes of $N^-O(7, q)$ Theorem 1.1 then implies our main theorem. Before defining the geometry $N^-O(7, q)$ we will have a look at the following alternative description of the graph $\Gamma(q)$.

Embed $Q(6, q)$ in a non-degenerate elliptic quadric $Q^-(7, q)$. Then it is well known that there is a unique involutory automorphism σ of $Q^-(7, q)$ fixing $Q(6, q)$ pointwise and having no fixed points in $Q^-(7, q) \setminus Q(6, q)$. The vertices of $\Gamma(q)$ are the pairs $\{x, x^\sigma\}$, $x \in Q^-(7, q) \setminus Q(6, q)$, and two vertices $\{x, x^\sigma\}$, $\{y, y^\sigma\}$ are adjacent if and only if one of the points x and x^σ, say x, is collinear (in $Q^-(7, q)$) with one of the points y and y^σ, say y. It is easily seen that this is indeed an alternative description of the graph $\Gamma(q)$. In this description, the vertex $\{x, x^\sigma\}$ corresponds to the unique elliptic quadric $Q_x = \{x, x^\sigma\}^\perp$, where $A^\perp$ denotes the set of points collinear with all points in the point set A; the tangent 4-space corresponding to $\{x, x^\sigma\}$ and $\{y, y^\sigma\}$ is given by $Q(6, q) \cap \{x, y, x^\sigma, y^\sigma\}^\perp$ and the corresponding cone $pQ^-(3, q)$ has vertex $p = xy \cap x^\sigma y^\sigma = xy \cap Q(6, q)$.

We can now define the geometry $N^-O(7, q)$ (see Cuypers [1]). The points of $N^-O(7, q)$ are the pairs $\{x, x^\sigma\}$, $x \in Q^-(7, q) \setminus Q(6, q)$. The lines of $N^-O(7, q)$ are the pairs of lines $\{L, L^\sigma\}$, with L a line of $Q^-(7, q)$ intersecting $Q(6, q)$ exactly in a point. The planes of $N^-O(7, q)$ are the pairs of planes $\{\pi, \pi^\sigma\}$, with π a plane of $Q^-(7, q)$ intersecting $Q(6, q)$ exactly in a line. The incidence is the natural one, that is, a point $\{x, x^\sigma\}$ is incident with a line $\{L, L^\sigma\}$ iff either $x \in L$ or $x \in L^\sigma$, ... Clearly $N^-O(7, q) = (Q^-(7, q) \setminus Q(6, q))/\langle\sigma\rangle$. The following arguments, suggested to us by the anonymous referee, provide a quick proof of Theorem 1.1, and we include it for the sake of completeness. In [1] it is explained that the affine polar space $Q^-(7, q) \setminus Q(6, q)$ is the universal cover of the geometry $N^-O(7, q)$. Now the full automorphism group of $Q^-(7, q) \setminus Q(6, q)$ is the stabilizer G_Q of $Q(6, q)$ in $P\Gamma O^-(8, q)$. It is well known that the center of G_Q has order two and is exactly $\langle\sigma\rangle$ and that $G_Q/\langle\sigma\rangle \cong P\Gamma O(7, q)$. Because $N^-O(7, q) = (Q^-(7, q) \setminus Q(6, q))/\langle\sigma\rangle$ and because of the fact that $Q^-(7, q) \setminus Q(6, q)$ is the universal cover of the geometry $N^-O(7, q)$ it now follows that $\mathrm{P\Gamma O}(7, q)$ is the full automorphism group of $N^-O(7, q)$. At this point we remark that, for $q > 2$, the planes of $N^-O(7, q)$ are exactly the subspaces of $\Gamma_1(q)$ that determine a clique of size q^2 of $\Gamma(q)$. Hence it is possible to recover $N^-O(7,q)$ from $\Gamma_1(q)$, $q > 2$. Theorem 1.1 follows.

2. Proof of the main theorem

We first remark that the case $q = 3$ has also been settled by Devillers [2] by computer. Note also that the case $q = 2$ is trivial, since $\Gamma(2)$ is a complete graph, and hence has the symmetric group on 28 letters as full automorphism group.

We will now turn to the study of 3-cliques in $\Gamma(q)$.

2.1. *Three mutually adjacent elliptic quadrics*

Let Q_1, Q_2 and Q_3 be three distinct elliptic quadrics $Q^-(5,q) \subset Q(6,q)$ such that $Q_1 \sim Q_2 \sim Q_3 \sim Q_1$ in $\Gamma(q)$. It is easily seen that one of the following cases must occur.

a. $Q_1 \cap Q_2 = Q_1 \cap Q_3 = Q_2 \cap Q_3$. We say that our three quadrics are of type **a**. In the alternative description of the graph this situation corresponds to three pairs of points $\{x_1, x_1^\sigma\}$, $\{x_2, x_2^\sigma\}$ and $\{x_3, x_3^\sigma\}$, such that, without loss of generality, x_1, x_2 and x_3 are three points on a line.
b. $Q_1 \cap Q_2 \cap Q_3$ is a line. We say that our three quadrics are of type **b**. In the alternative description of the graph this situation corresponds to three pairs of points $\{x_1, x_1^\sigma\}$, $\{x_2, x_2^\sigma\}$ and $\{x_3, x_3^\sigma\}$, such that, without loss of generality, x_1, x_2 and x_3 are three points spanning a singular plane.
c. $Q_1 \cap Q_2 \cap Q_3$ is an ovoid $\mathcal{O} \cong Q^-(3,q)$. We say that our three quadrics are of type **c**. In this case one can easily see that for every such ovoid $\mathcal{O}$ in $Q_1 \cap Q_2$ there exists a unique elliptic quadric Q such that $Q_1 \cap Q_2 \cap Q = \mathcal{O}$ and $Q_1 \sim Q \sim Q_2$. In the alternative description of the graph this situation corresponds to three pairs of points $\{x_1, x_1^\sigma\}$, $\{x_2, x_2^\sigma\}$ and $\{x_3, x_3^\sigma\}$, such that, without loss of generality, x_1, x_2 are collinear, x_1 and x_3 are collinear and x_2 and x_3^σ are collinear.

Note that it is not possible for three distinct mutually adjacent elliptic quadrics Q_1, Q_2 and Q_3 to intersect in a cone $pQ(2,q)$ (otherwise the point p would be the vertex of two distinct cones $pQ^-(3,q)$ in Q_1, namely of the cones $Q_1 \cap Q_2$ and $Q_1 \cap Q_3$).

2.2. *Recovering $N^-O(7,q)$*

Main Theorem 2.1. *The full automorphism group of the graph $\Gamma(q)$, $q \geq 3$, is isomorphic to $P\Gamma O_7(q)$.*

Proof. In view of Theorem 1.1, it suffices to distinguish the 3-cliques of type **a** from the other ones. We do this by counting the number of elliptic quadrics adjacent to all vertices of a given 3-clique. So let Q_1, Q_2, Q_3 be three mutually adjacent elliptic quadrics, and let $\{x_i, x_i^\sigma\}$ be the pair of points corresponding to Q_i in our alternative description, $i = 1, 2, 3$. Let $\{y, y^\sigma\}$ be a pair of points corresponding to a generic elliptic quadric $Q \notin \{Q_1, Q_2, Q_3\}$ adjacent to each Q_i, $i = 1, 2, 3$.

a. Suppose Q_1, Q_2, Q_3 are of type **a**. We may assume that x_1, x_2, x_3 lie on a common line L, which meets $Q(6,q)$ in a point z. We may assume that y is collinear with x_1, x_2, and hence also with x_3. There are $q - 3$ choices for y on L. Henceforth, we assume $y \notin L$. Then $\langle y, x_1, x_2\rangle$ is a singular plane π meeting $Q(6,q)$ in a singular line S and containing

L. Each such singular plane gives rise to $q(q-1)$ choices for y not on L. Since there are q^2+1 such singular planes in $Q^-(7,q)$ through L, we now see that there are exactly $q(q-1)(q^2+1)+q-3 = q^4-q^3+q^2-3$ elliptic quadrics adjacent to all of Q_1, Q_2, Q_3, and different from Q_1, Q_2, Q_3.

b. Suppose Q_1, Q_2, Q_3 are of type **b**. We may assume that x_1, x_2, x_3 span a plane π on $Q^-(7,q)$, which meets $Q(6,q)$ in a line L. We may also assume that y is collinear with x_1 on $Q^-(7,q)$. There are q^2-3 choices for y in π (and then y ranges through $S_0 := \{x_1, x_2, x_3\}^\perp \setminus Q(6,q)$). Henceforth we assume that y is not contained in π. Then y belongs to either $S_1 := \{x_1, x_2, x_3^\sigma\}^\perp \setminus Q(6,q)$, or $S_2 := \{x_1, x_2^\sigma, x_3\}^\perp \setminus Q(6,q)$, or $S_3 := \{x_1, x_2^\sigma, x_3^\sigma\} \setminus Q(6,q)$. By symmetry, $|S_1| = |S_2| = |S_3|$. So it suffices to count the number of elements y of S_1. Clearly y belongs to a plane $\alpha \neq \pi$ containing x_1, x_2. There are q^2 such planes. Each such plane intersects $(x_3^\sigma)^\perp$ in a line, which is different from both x_1x_2 and L. Hence $|S_1| = q^2 \cdot q$.

We conclude that there are $3q^3+q^2-3$ elliptic quadrics $Q \notin \{Q_1, Q_2, Q_3\}$ adjacent to all of Q_1, Q_2, Q_3.

c. Suppose Q_1, Q_2, Q_3 are of type **c**. We may assume that x_1x_2, x_1x_3 and $x_2x_3^\sigma$ are lines on $Q^-(7,q)$. Let S_i, $i = 0, 1, 2, 3$, be defined as in Case b. We determine the cardinalities of these sets. If y is a generic element of S_0, then y lies in a plane α through x_1x_2 and on the line $\alpha \cap x_3^\perp$, which is incident with x_1 and contains a unique point of $Q(6,q)$. There are q^2+1 choices for α and $q-1$ for y on $\alpha \cap x_3^\perp$, giving rise to $|S_0| = (q^2-1)(q-1)$. Likewise, $|S_1| = |S_2| = (q^2+1)(q-1)$. Concerning S_3, we note that $\{x_1, x_2^\sigma, x_3^\sigma\}^\perp$ intersects $Q(6,q)$ precisely in the ovoid $\mathcal{O} = Q_1 \cap Q_2 \cap Q_3$. Now $\{x_1, x_2^\sigma, x_3^\sigma\}^\perp$ is the intersection of $Q^-(7,q)$ with a 4-space U. Since the plane $\langle x_1, x_2^\sigma, x_3^\sigma\rangle$ is non-singular, the subspace U meets $Q^-(7,q)$ in a non-degenerate quadric $Q(4,q)$, which has exactly $q^3+q^2+q^1$ points in total, and hence exactly q^3+q points off $\mathcal{O}$.

We conclude that in this case there are $3(q^2+1)(q-1)+q^3+q = 4q^3-3q^2+4q-3$ elliptic quadrics $Q \notin \{Q_1, Q_2, Q_3\}$ adjacent to all of Q_1, Q_2, Q_3.

Now for $q \neq 4$, the numbers in a–c all differ from each other (for $q = 4$, the numbers found in a and b are the same).

Hence the block of $\Gamma_1(q)$, $q \neq 4$, through the two adjacent elliptic quadrics Q, Q' of $\Gamma(q)$ is the union of those 3-cliques of $\Gamma(q)$ that contain Q, Q' and that are themselves contained in precisely $q^4-q^3+q^2-3$ 4-cliques.

The following argument which settles the case $q = 4$ does in fact work for all $q \neq 2$. If $q = 4$ then the numbers found in a and b are equal, but differ from the number found in c. Hence the above allows us to recognize those 3-cliques of $\Gamma(q)$ that are coplanar in $N^-O(7,q)$. Now suppose we have a 4-clique such that each 3-clique in it is coplanar in $N^-O(7,q)$. Say, in the alternative representation, that the vertices of the 4-clique are $\{x_i, x_i^\sigma\}$, $i = 1, 2, 3, 4$. We may suppose that x_1, x_2, x_3 are coplanar in $Q^-(7,q)$. If these three points are in fact on a line then it immediately follows that the 4-clique is coplanar in $N^-(7,q)$. So suppose that x_1, x_2 and x_3 are coplanar in $Q^-(7,q)$, but not collinear. Without loss of generality we may assume that x_4 is collinear with x_1 and x_2. Suppose it were not to be collinear with x_3. Then, however, the 3-clique determined by $\{x_i, x_i^\sigma\}$, $i = 2, 3, 4$ can never be coplanar in $N^-O(7,q)$, contradicting our assumptions. Hence a 4-clique each 3-clique of which is coplanar in $N^-O(7,q)$ has to be coplanar in $N^-O(7,q)$. Hence the planes of $N^-O(7,q)$ are exactly those q^2-cliques of $\Gamma(q)$ each 3-clique of which extends in exactly $q^4-q^3+q^2-3$ or $3q^3+q^2-3$ ways to a 4-clique. This shows that it is possible to recover the planes of $N^-O(7,q)$ from $\Gamma(q)$. Since the lines of

$N^-O(7,q)$ are exactly those q-cliques that arise as intersections of planes, we see that we can recover $N^-O(7,q)$ from $\Gamma(q)$. The theorem follows. $\square$

2.3. An alternative approach

The use of the geometry $N^-O(7,q)$ allows one to overcome the difficulties that arise if one wants to reconstruct $\Gamma_1(q)$ directly from $\Gamma(q)$ if $q = 4$. There is however another way to characterize the lines of $\Gamma_1(q)$ directly in $\Gamma(q)$, which works whenever $q > 3$. One can characterize directly those q-cliques that are blocks of $\Gamma_1(q)$ as follows. For $q > 4$, the blocks of $\Gamma_1(q)$ are exactly those q-cliques C of $\Gamma(q)$ satisfying the following Condition (*): if a vertex $v \notin C$ is adjacent to at least three vertices of C, then it is adjacent to all vertices of C. In the case $q = 4$, there are additional 4-cliques satisfying Condition (*), and one can distinguish these from the blocks of $\Gamma_1(q)$ by recognizing the blocks as those 4-cliques B that satisfy Condition (*) and have the additional property that there are exactly 204 vertices v of $\Gamma(q)$ not belonging to B, but such that every pair of vertices of B lies, together with v, in a 4-clique satisfying Condition (*). The proof of this alternative characterization however needs about three pages, whereas the use of $N^-O(7,q)$ provides an elegant way to overcome the difficulties arising when $q = 4$. Hence we omit this proof here.

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