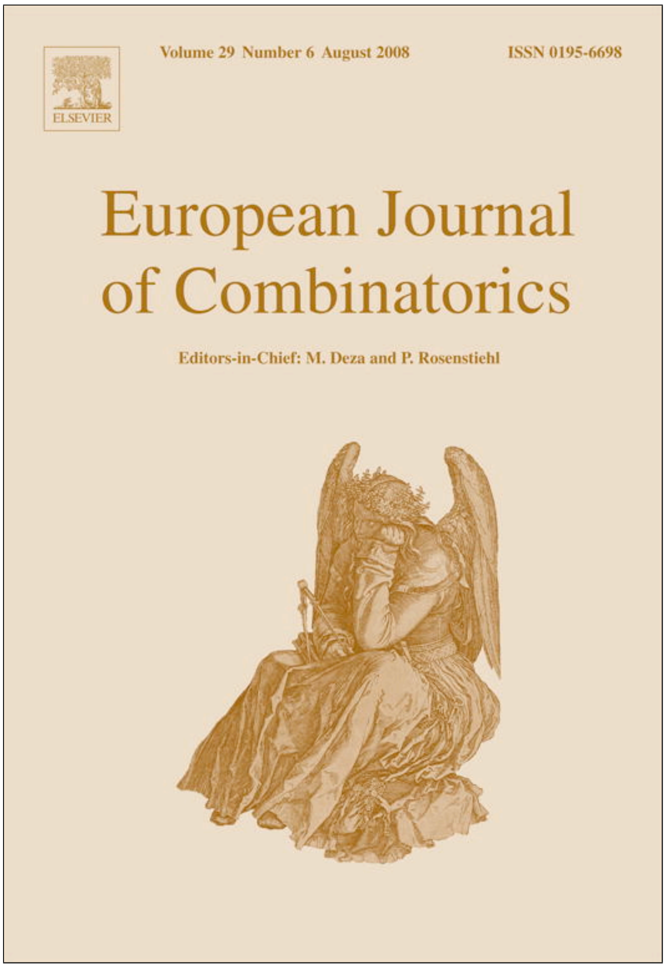
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A characterization of the natural embedding of the split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(6, q)$ by intersection numbers

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Abstract

In this paper, we prove that a set $\mathcal{L}$ of $q^5 + q^4 + q^3 + q^2 + q + 1$ lines of $\mathsf{PG}(6, q)$ with the properties that (1) every point of $\mathsf{PG}(6, q)$ is incident with either 0 or $q+1$ elements of $\mathcal{L}$, (2) every plane of $\mathsf{PG}(6, q)$ is incident with either 0, 1 or $q + 1$ elements of $\mathcal{L}$, (3) every solid of $\mathsf{PG}(6, q)$ is incident with either 0, 1, $q+1$ or $2q+1$ elements of $\mathcal{L}$, and (4) every hyperplane of $\mathsf{PG}(6, q)$ is incident with at most q^3+3q^2+3q members of $\mathcal{L}$, is necessarily the set of lines of a regularly embedded split Cayley generalized hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(6, q)$.

1. Introduction

Recognizing specific geometric structures by certain properties – preferably as weak as possible – is very important in (finite) geometry, since various configurations can turn up unexpectedly in completely different contexts, disguised in an unusual definition. Data that are available for structures in projective spaces are in many situations the intersection numbers with respect to subspaces. A typical example are the conics in Desarguesian planes $\mathsf{PG}(2, q)$, with q odd, which are characterized as sets of $q + 1$ points meeting lines in 0, 1 or 2 points. Other, more involved, examples are the characterizations of quadrics, Hermitian and Veronesean varieties by intersection numbers.

In the present paper, our aim is to characterize the standard embedding of the split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(6, q)$ by intersection numbers with subspaces. Of course, since the points

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of $\mathsf{H}(q)$ are exactly the points of a parabolic quadric in $\mathsf{PG}(6, q)$, such a characterization is impossible if we (only) consider intersections of the point set of the hexagon with subspaces. That is why we consider the intersections of subspaces with the *line set* of $\mathsf{H}(q)$, i.e., we count the number of lines of a given embedded $\mathsf{H}(q)$ in a given subspace and then assume that we have a set of lines with these intersection numbers (in practice, we take weaker assumptions). It is also natural to require that we deal with a tactical configuration, i.e., we assume that each point of the projective space that is incident with al least one line of our set, is incident with exactly $q + 1$ lines of our set.

Before getting down to precise statements, let us recall the definition and construction of the split Cayley hexagons $\mathsf{H}(q)$.

A *point–line geometry* is a triple $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ consisting of a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines, and a symmetric incidence relation $\mathtt{I}$ saying precisely which points are incident with which lines (and conversely). The *incidence graph* of the point–line geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and adjacency relation $\mathtt{I}$. A *generalized hexagon* is a point–line geometry for which the incidence graph has diameter 6 and girth 12, i.e., the maximal distance between two vertices is 6, and the length of a shortest circuit is 12. Whenever each vertex of the incidence graph of a generalized hexagon has valency at least 3, this (bipartite) graph is bi-valent. If the valency of the vertices belonging to $\mathcal{P}$ and $\mathcal{L}$ is equal to $s + 1$ and $t + 1$, respectively, then we say that the generalized hexagon has order (s, t). Distances between elements of a point–line geometry are always measured in the incidence graph. Elements at distance 6 from each other in a generalized hexagon are called *opposite*.

Let q be any prime power. Up to isomorphism, the *split Cayley hexagon* $\mathsf{H}(q)$ is defined as follows (see Tits [4]). Let $\mathsf{Q}(6, q)$ be the parabolic quadric in $\mathsf{PG}(6, q)$ defined by the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. Then the points of $\mathsf{H}(q)$ are the points of $\mathsf{Q}(6, q)$, the lines of $\mathsf{H}(q)$ are the lines of $\mathsf{Q}(6, q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. Incidence is inherited from $\mathsf{PG}(6, q)$. For more details, properties and information about $\mathsf{H}(q)$ we refer to [5].

The above construction, which we call an *embedding* of $\mathsf{H}(q)$ in $\mathsf{PG}(6, q)$ has the following properties, see [5,3]. Recall first that, abstractly, an *embedding* of a point–line geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ in $\mathsf{PG}(d, q)$, for some d, is an injective mapping of $\mathcal{P}$ in the point set of $\mathsf{PG}(d, q)$ inducing an injective mapping from $\mathcal{L}$ into the line set of $\mathsf{PG}(d, q)$ and such that the image of $\mathcal{P}$ generates $\mathsf{PG}(d, q)$.)

First of all, the embedding described above is *polarized*. This means that for every point x of $\mathsf{H}(q)$, the set of points not opposite x is contained in a hyperplane of $\mathsf{PG}(6, q)$. Also, the set of lines of $\mathsf{H}(q)$ incident with x is contained in a plane of $\mathsf{PG}(6, q)$, and we say that the embedding is *flat*. Flat and polarized are abbreviated by the notion *regular embedding*. By [2], up to projectivity, the example above is the only regularly embedded generalized hexagon in $\mathsf{PG}(6, q)$.

We are now ready to state our Main Result.

Let $\mathcal{L}$ be a set of lines of $\mathsf{PG}(6, q)$ and consider the following properties.

(Pt) Every point of $\mathsf{PG}(6, q)$ is incident with either 0 or $q + 1$ elements of $\mathcal{L}$.
(Pl) Every plane of $\mathsf{PG}(6, q)$ is incident with either 0, 1 or $q + 1$ elements of $\mathcal{L}$.
(Sd) Every solid of $\mathsf{PG}(6, q)$ is incident with either 0, 1, $q + 1$ or $2q + 1$ elements of $\mathcal{L}$.
(Hp) Every hyperplane of $\mathsf{PG}(6, q)$ is incident with at most $q^3 + 3q^2 + 3q$ members of $\mathcal{L}$.
(To) The set $\mathcal{L}$ contains $q^5 + q^4 + q^3 + q^2 + q + 1$ lines.

Main Result. *If $\mathcal{L}$ is a set of lines of* $\mathsf{PG}(6, q)$ *satisfying* (Pt), (Pl), (Sd), (Hp) *and* (To)*, then it is the line set of a regularly embedded split Cayley hexagon* $\mathsf{H}(q)$ *in* $\mathsf{PG}(6, q)$*. Conversely, the line set of every regularly embedded split Cayley hexagon* $\mathsf{H}(q)$ *in* $\mathsf{PG}(6, q)$ *satisfies the Properties* (Pt), (Pl), (Sd), (Hp) *and* (To).

Remark. In the proof of our Main Result below, there are only two places where we will use Condition (Sd), namely, in the proof of Lemma 1 and in the proof of Lemma 5. In both cases the solid under consideration contains intersecting lines of $\mathcal{L}$, so that the Condition (Sd) can be replaced by

(Sd′) Every solid of $\mathsf{PG}(6, q)$ containing distinct intersecting elements of $\mathcal{L}$ contains at most $2q + 1$ elements of $\mathcal{L}$.

2. Proof of the Main Result

First let $\mathcal{L}$ be the line set of a regularly embedded split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(6, q)$. Note that the point set of $\mathsf{H}(q)$ is a nonsingular (parabolic) quadric $\mathsf{Q}(6, q)$ in $\mathsf{PG}(6, q)$. Since $\mathsf{H}(q)$ has order q, Condition (Pt) follows. Since the embedding is flat, Condition (Pl) is trivial. Of course, Condition (To) follows from the well known formula of the total number of lines of $\mathsf{H}(q)$.

Consider now Condition (Hp). It is well known that there are three kinds of hyperplanes in $\mathsf{PG}(6, q)$ with respect to the quadric $\mathsf{Q}(6, q)$: tangent hyperplanes — and these contain $q^3 + q^2 + q + 1$ lines of $\mathsf{H}(q)$, namely all those at distance ≤ 3 from the tangent point – elliptic hyperplanes – they meet $\mathsf{Q}(6, q)$ in an elliptic quadric $\mathsf{Q}^-(5, q)$ and contain $q^3 + 1$ lines of $\mathsf{H}(q)$, see [1] – and hyperbolic ones – meeting $\mathsf{Q}(6, q)$ in a hyperbolic quadric $\mathsf{Q}^+(5, q)$ and containing $(q + 1)(q^2 + q + 1)$ lines of $\mathsf{H}(q)$. Whence Condition (Hp).

Finally let Σ be a solid of $\mathsf{PG}(6, q)$ containing at least two lines L, M of $\mathsf{H}(q)$. We may assume that L and M are at maximal distance d in Σ. If $d = 6$, then L and M are at maximal distance in $\mathsf{H}(q)$ and hence Σ meets $\mathsf{Q}(6, q)$ in a nonsingular hyperbolic quadric $\mathsf{Q}^+(3, q)$. So Σ contains $q + 1$ lines of $\mathsf{H}(q)$ (all lines of one regulus of $\mathsf{Q}^+(3, q)$). If $d = 4$, then let N be the line at distance 2 from both L, M. Then Σ contains the $2q + 1$ lines of $\mathsf{H}(q)$ incident with the intersection of N and L, and of N and M. By maximality of d, any other line K of $\mathsf{H}(q)$ in Σ must meet N. But then $\langle L, M, K\rangle$ is 4-dimensional since every three planes of $\mathsf{Q}(6, q)$, containing a common line, generate a 4-space (this can be seen as follows: the 4-dimensional tangent space at N meets $\mathsf{Q}(6, q)$ in a singular quadric with vertex N and base a nonsingular conic). Finally, if $d = 2$, then clearly, by flatness of the embedding, there are exactly $q + 1$ lines of $\mathsf{H}(q)$ in Σ.

For the remainder of this section, we assume now that a set of lines $\mathcal{L}$ in $\mathsf{PG}(6, q)$ satisfies (Pt), (Pl), (Sd), (Hp) and (To).

We prove a sequence of lemmas culminating in a complete proof of our Main Result.

We denote by $\mathcal{P}$ the set of points of $\mathsf{PG}(6, q)$ which are incident with a line of $\mathcal{L}$. Note that an easy counting argument shows that $|\mathcal{P}| = |\mathcal{L}|$.

Lemma 1. *The $q + 1$ lines of $\mathcal{L}$ through a fixed point of $\mathcal{P}$ are all coplanar.*

Proof. Suppose by way of contradiction that there are three lines $L_1, L_2, L_3 \in \mathcal{L}$ incident with a point x of $\mathsf{PG}(6, q)$ and that they span a 3-space. Each of the planes $\langle L_i, L_j\rangle$, $i, j \in \{1, 2, 3\}$, $i \neq j$, contains exactly $q + 1$ members of $\mathcal{L}$, implying that the solid $\langle L_1, L_2, L_3\rangle$ contains at least $3q > 2q + 1$ elements of $\mathcal{L}$. $\square$

Lemma 2. *A hyperplane of* $\mathsf{PG}(6,q)$ *contains either* q^3+1*, or* q^3+q^2+q+1*, or* $(q+1)(q^2+q+1)$ *members of* $\mathcal{L}$*.*

Proof. Let H be any hyperplane, and let h be the number of lines of H in $\mathcal{L}$. Note that by Lemma 1 every point of $H \cap \mathcal{P}$ is incident either with one line of $\mathcal{L}$ inside H, or with $q+1$ such lines. Let a be the number of points of H on exactly one line of $\mathcal{L}$ in H, and let b be the number of points of H on $q+1$ lines of $\mathcal{L}$ lying inside H. A double count of the pairs (x, L), with $x \in H \cap \mathcal{P}$ and $L \in \mathcal{L}$, $x \in L$, leads easily to

$$\begin{cases} a + b(q+1) = h(q+1) \\ q^5 + \cdots + 1 - h = aq, \end{cases}$$

which implies that b is divisible by q^2+q+1, say $b = \ell(q^2+q+1)$. The values $\ell = 0, 1, 2$ lead to the numbers of the lemma, the values $\ell \geq 3$ lead to a contradiction with Condition (Hp). □

Note that the previous proof implies that a hyperplane H contains $0, q^2+q+1$ or $2(q^2+q+1)$ points incident with $q+1$ members of $\mathcal{L}$ belonging to H, depending on whether H contains q^3+1, q^3+q^2+q+1 or $(q+1)(q^2+q+1)$ members of $\mathcal{L}$, respectively.

For any subspace Ω of $\mathsf{PG}(6,q)$, we say that a point x of $\Omega \cap \mathcal{P}$ is *isolated in* Ω if no line of $\mathcal{L}$ through x is contained in Ω.

Lemma 3. *A plane* π *of* $\mathsf{PG}(6,q)$ *containing exactly one line of* $\mathcal{L}$ *contains* ℓq *isolated points, for some nonnegative integer* ℓ*.*

Proof. Let a be the number of solids containing π and containing $q+1$ members of $\mathcal{L}$; let b be the number of solids containing π and containing $2q+1$ members of $\mathcal{L}$. Also, let m be the number of isolated points of π. A double count of the pairs (L, Σ), with L a line of $\mathcal{L}$ meeting π in a unique point, and with Σ a solid containing L and π reveals

$$qa + 2qb = (q^2+q) + m(q+1),$$

implying that m must be divisible by q. □

Lemma 4. *A solid* Σ *of* $\mathsf{PG}(6,q)$ *which contains at least three lines* L_1, L_2, L_3 *of* $\mathcal{L}$ *such that* L_1 *meets* L_2 *in a point* x_{12} *and* L_2 *meets* L_3 *in a point* x_{23}*, with* $x_{12} \neq x_{23}$*, contains at most* $q+1$ *isolated points.*

Proof. Clearly Σ contains the $q+1$ lines of $\mathcal{L}$ through x_{12} and x_{23}, respectively. Since these already account for $2q+1$ elements of $\mathcal{L}$ in Σ, there are no more members of $\mathcal{L}$ in Σ. Let m be the number of isolated points of Σ. Let a be the number of hyperplanes of $\mathsf{PG}(6,q)$ incident with Σ and containing q^3+q^2+q+1 members of $\mathcal{L}$, and let b be the number of hyperplanes of $\mathsf{PG}(6,q)$ incident with Σ and containing $(q+1)(q^2+q+1)$ members of $\mathcal{L}$. Note that by the remark following the proof of Lemma 2 there are no hyperplanes incident with Σ containing exactly q^3+1 elements of $\mathcal{L}$. So we have $a+b = q^2+q+1$. Moreover, a double count of the pairs (L, H), where $L \in \mathcal{L}$ does not belong to Σ, and H is a hyperplane containing Σ and L, shows

$$\begin{aligned} a(q^3+q^2-q) + b(q^3+2q^2) &= (2q^2+q-1)q(q+1) + m(q+1)(q+1) \\ &\quad + (q^5+q^4+q^3+q^2+q+1-(2q^2+q-1)q - m(q+1) - (2q+1)), \end{aligned}$$

which implies $a+m = q+1$ and $b-m = q^2$. The former now reveals that $m = q+1-a \leq q+1$. □

Lemma 5. *The structure $\mathcal{H} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, with $\mathtt{I}$ the appropriate restriction of the natural incidence in $\mathsf{PG}(6, q)$, is a generalized hexagon of order (q, q).*

Proof. Suppose K, L, M is a triangle of lines in $\mathcal{H}$. Lemma 1 implies that the $q+1$ lines of the plane $\langle K, L\rangle$ through the intersection point x of K and L belong to $\mathcal{L}$; hence that plane contains at least $q+2$ lines, a contradiction.

Suppose K, L, M, N form a quadrangle in $\mathcal{H}$. Then these four lines generate a solid Σ. But each of the planes $\langle K, L\rangle$, $\langle L, M\rangle$, $\langle M, N\rangle$ and $\langle N, K\rangle$ contains $q+1$ lines of $\mathcal{L}$, adding up to at least $4q$ members of $\mathcal{L}$ in Σ, contradicting Condition (Sd).

Finally, suppose J, K, L, M, N form a pentagon in $\mathcal{H}$. Similarly as in the previous paragraph, they cannot generate a solid, hence J, K, L, with J and L intersecting K, generate a solid Σ, which contains $q-1$ isolated points arising from the intersection of Σ with the lines of $\mathcal{L}$ incident with the intersection point of M and N. It is easily seen that no two of these isolated points are contained in the same plane through K. But each plane through K containing an isolated point of Σ contains at least q isolated points, by Lemma 3. Hence, if $q > 2$, then Σ contains at least $q(q-1) > q+1$ isolated points, contradicting Lemma 4.

Now suppose $q = 2$. By the previous paragraph, we have at least two isolated points x_1, x_2 in a plane π through K. That plane does not contain J. If $\langle x_1, J\rangle \neq \langle x_2, J\rangle$, these two planes give rise to two more isolated points (using Lemma 3), contradicting Lemma 4. If $\langle x_1, J\rangle = \langle x_2, J\rangle$, we have $\langle x_1, L\rangle \neq \langle x_2, L\rangle$ and we similarly obtain a contradiction.

We have shown that $\mathcal{H}$ does not contain an ordinary n-gon, for $n \leq 5$. Since

$$|\mathcal{P}| = |\mathcal{L}| = q^5 + q^4 + q^3 + q^2 + q + 1$$

and each point is incident with $q+1$ lines, and each line with $q+1$ points, a standard counting argument implies that $\mathcal{H}$ is a generalized hexagon. □

End of the proof of the Main Result. The generalized hexagon $\mathcal{H}$ has order q and is, by Lemma 1, flatly and fully embedded in $\mathsf{PG}(6, q)$. Now (ii) of the Main Result of [3] concludes the proof of our Main Result.

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