Semiaffine Spaces

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Abstract

In this note, we improve on a result of Beutelspacher, De Vito & Lo Re, who characterized in 1995 finite semiaffine spaces by means of transversals and a condition on weak parallelism. Basically, we show that one can delete that condition completely. Moreover, we extend the result to the infinite case, showing that every plane of a planar space with at least two planes and such that all planes are semiaffine, comes from a (Desarguesian) projective plane by deleting either a line and all of its points, a line and all but one of its points, a point, or nothing.

1 Introduction

In [2], Beutelspacher, De Vito and Lo Re prove that linear spaces in which disjoint lines are either "weakly parallel", or admit, through any given point outside the two lines, at most one transversal, and which satisfy a rather strong condition on these weakly parallel lines, are planar spaces all of whose planes are semiaffine. They then use an unpublished result of Teirlinck and the classification of semiaffine planes due to Dembowski & Kuiper [3] to determine all finite such linear spaces. In the present paper we show that one can delete the condition on the parallel lines, and moreover we bypass the classification of finite semiaffine planes and proceed applying Teirlinck's theorem when there are at least two planes, so that also all infinite such linear spaces are classified.

More exactly, we show (but postpone the precise definitions to the next section):

First Main Result. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space such that through every point outside two given disjoint lines either there is at most one transversal, or every line*

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meeting one of the given disjoint lines meets the other. If every line contains at least three points, then $\mathcal{S}$ is a planar space in which every plane is semiaffine.

Second Main Result *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a planar space with at least two planes, such that every line contains at least four points and all planes of which are semiaffine. Then every plane is obtained from a projective plane by deleting either nothing, a single point, a line with all but one of its points, or a line with all its points.*

Main Corollary *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space all lines of which contain at least four points. Then the following are equivalent.*

(SAS1) *Through every point x outside two given disjoint lines L_1, L_2 either there passes at most one transversal, or every line through x meeting $L_1 \cup L_2$ in at least one point meets $L_1 \cup L_2$ in exactly two points. Also, the first possibility really occurs.*

(SAS2) *$\mathcal{S}$ is a planar space with at least two planes such that every plane is a semiaffine plane.*

(SAS3) *$\mathcal{S}$ is an affino-projective semiaffine space (and hence arises from a unique projective space by deleting a set S of points in a hyperplane with the property that, whenever a line meets S in at least two points, then it meets S in either all its points, or all but one of its points).*

This result motivates the definition of a *semiaffine space* as either a semiaffine plane, or a planar space satisfying the condition in (SAS2). It then follows that (SAS1) is a characterization of semiaffine spaces with line size at least three (if one deletes the restriction that the first possibility in (SAS1) must really occur), and that (SAS3) provides a classification of semiaffine spaces with line size at least four, and containing at least two planes. We will give more details concerning (SAS3) below.

Note that our main results are tokens of general and classical geometric occurrences. In particular, the First Main Result is a kind of counterpart for linear spaces of the characterization of polar spaces by Beukenhout & Shult [1] by means of the so-called one-or-all axiom. For linear spaces, a one-or-all axiom cannot involve only a point and one line, but a natural way to state such an axiom is to use one point and two disjoint lines. Here, the possibility of 'zero' must also be considered (if one adds this possibility for linear spaces, then one obtains the definition of a so-called *gamma* space, a notion which plays a central role in the theory of (Grassmannian) point-line geometries from spherical buildings). Hence, the First Main Result characterizes semiaffine spaces by a none-one-or-all axiom. Moreover, if one excludes the possibility of parallel lines, i.e., if one assumes that in (SAS1) only the first possibility can occur, then the condition simplifies to Pasch's famous axiom. So, the First Main Result can also be seen as a generalization of the Veblen & Young [5] characterization for projective space by including the affine spaces and spaces that lie in between an affine and a projective space, and which we called semiaffine spaces. Thirdly, the Second Main Result expresses the idea that, once put in a higher dimensional object, objects of rank two tend to be tame. For instance, projective planes inside projective spaces of dimension at least three are automatically Desarguesian. More generally, generalized polygons in irreducible spherical buildings of rank at least three are Moufang. In our case, semiaffine planes of semiaffine spaces with at least two planes arise automatically from Desarguesian projective planes by deleting either nothing, a point, an affine line and all of its points, or a projective line and all of its points. Note that there are free constructions of semiaffine planes that do not arise from any projective plane in the above mentioned way, see [3].

In our proof, we will use the results of [2]. In fact, we reduce the situation to the hypotheses therein. Also, for our Second Main Result, we assume that lines have infinite size (the finite case following directly from [3]), although our proof almost completely works under the assumption of at least four points per line. We did not try to make it work completely under this assumption (in view of [3]), but this might not at all be a difficult exercise.

2 Preliminaries

A *linear space* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ consists of a set $\mathcal{P}$ of *points* and a family $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, such that each pair of distinct points x, y is contained in a unique line, sometimes denoted by xy. The prominent examples of linear spaces are the projective spaces, which arise from vector spaces by taking for point set the the set of one-dimensional subspaces of a vector space V, and as lines the sets of one-dimensional subspaces contained in a two-dimensional subspace. Let, for a given linear space $(\mathcal{P}, \mathcal{L})$, the set $\mathcal{P}^*$ denote the set of all *pencils*, where a pencil is the set of lines containing a given point. A projective plane is a linear space $(\mathcal{P}, \mathcal{L})$ such that all lines have size at least three, and such that $(\mathcal{L}, \mathcal{P}^*)$ is also a linear space. An *affine plane* is a linear space in which the parallel axiom for lines holds: every point is contained in exactly one line disjoint from or equal to a given line. A *semiaffine plane* is a linear space in which a slightly weaker form of the parallel axiom holds: every point is contained in at most one line disjoint from or equal to a given line. In [3], Dembowski & Kuiper show that every finite semiaffine plane is constructed from a projective plane by deleting either nothing, a point, a line and all of its points, or a line and all but one of its points. Moreover, they show that the finiteness hypothesis is essential by constructing counterexamples in the infinite case.

Now let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space. A *transversal* of two disjoint lines L_1, L_2 is a line meeting $L_1 \cup L_2$ in two points. Consider the following *none-one-or-all axiom*:

(NOA) For every point x and every pair of disjoint lines L_1, L_2 not containing x, exactly one of the following possibilities occurs:

- (i) there is no transversal of L_1, L_2 containing x;
- (ii) there is exactly one transversal of L_1, L_2 containing x;
- (iii) every line through x meeting L_1 or L_2 is a transversal of L_1, L_2.

Let L_1, L_2 be two disjoint lines of $\mathcal{S}$. If, for some x, every line through x meeting L_1 or L_2 is a transversal of L_1, L_2, then Beutelspacher, De Vito & Lo Re [2] say that *L_1 and L_2 are parallel*, and the point x is called a *witness for the parallelity of L_1, L_2*. We shall also adopt this terminology, which allows to rephrase Axiom (NOA) as *if some point x is contained in at least two transversals of some pair of disjoint lines L_1, L_2 not containing x, then L_1 and L_2 are parallel and x is a witness for their parallelity.*

A *subspace* $\mathcal{S}' = (\mathcal{P}', \mathcal{L}')$ of $\mathcal{S}$ consists of a subset $\mathcal{P}' \subseteq \mathcal{P}$ of the point set and a subset $\mathcal{L}' \subseteq \mathcal{L}$ of the line set such that the union of $\mathcal{L}'$ equals $\mathcal{P}'$, and such that $\mathcal{S}'$ is a linear space. Given a family $\mathcal{E}$ of subspaces, we say that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{E})$ is a *planar space* if every triple of noncollinear points is contained in exactly one member of $\mathcal{E}$. The elements of $\mathcal{E}$ are then called *planes*. Concerning terminology, we say that the unique plane containing three given noncollinear points is *spanned* by these points. Similarly, a plane can also be *spanned* by a line and a point not on that line, or by two distinct intersecting lines.

The prototype of planar spaces are the affine and projective spaces of dimension at least three. One can, of course, also consider spaces that somehow "lie between affine and projective spaces". More exactly, we consider the following family of planar spaces.

Let $\mathcal{S}$ be a projective space, viewed as a linear (or planar) space, and let H be a hyperplane (i.e., a subspace intersecting every line nontrivially). If we remove all points and lines (and planes) of H from $\mathcal{S}$, then we obtain an affine linear (planar) space $\mathcal{S}_H$. Also, if $\mathcal{S}'$ is a projective subspace of $\mathcal{S}$, then the intersection $\mathcal{S}'_H$ with $\mathcal{S}_H$ yields an affine subspace of $\mathcal{S}_H$, which we call, by abuse of notation, also an *affine subspace of* $\mathcal{S}$. The subspace $\mathcal{S}'$ will be called the *projective completion of $\mathcal{S}'_H$*. We can now consider a family $\mathfrak{F}$ of affine subspaces of $\mathcal{S}$ with the following properties.

(AP1) $\mathcal{S}_H \in \mathfrak{F}$.

(AP2) The point sets of any two members of $\mathfrak{F}$ are disjoint.

(AP3) If $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{F}$, then, for some $i \in \{1, 2\}$, the point set of $\mathcal{S}_i$ is contained in the point set of the projective completion of $\mathcal{S}_{3-i}$.

If we take the union of the point sets of $\mathfrak{F}$, and endow it with all lines (and planes) induced from $\mathcal{S}$, then we denote the resulting linear (planar) space by $\mathcal{S}_{\mathfrak{F}}$. Condition (AP3) clearly ensures that $\mathcal{S}_{\mathfrak{F}}$ is a semiaffine space, and therefore we call it an *affino-projective* semiaffine space. This construction provides all examples described between parentheses in (SAS3) above, and conversely. This can be easily checked.

We now briefly review some known results. We start with the main point of this paper.

Fact 1 (Beutelspacher, De Vito, Lo Re [2]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space such that through every point outside two given disjoint but nonparallel lines there is at most one transversal. Then $\mathcal{S}$ is a semiaffine space, granted that every point on every transversal of two parallel lines (but not contained in these lines) is a witness for their parallelity.*

Note that the conditions in the previous fact immediately imply that $\mathcal{S}$ satisfies Condition (NOA). But if (NOA) is satisfied, then, in order to obtain the conditions of Fact 1, one has to require that, (*) if for a point x and two disjoint lines L_1, L_1 not containing x, there are at least two transversals of L_1, L_2 through x, this is also true for every other choice of x on any transversal of L_1, L_2 (as long as $x \notin L_1 \cup L_2$). Our main aim is to entirely delete this condition (*), i.e., we will derive (*) from (NOA).

Fact 2 (Teirlinck [4]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{E})$ be a planar space all planes of which are affino-projective semiaffine, and such that every line has at least four points. Then $\mathcal{S}$ is an affino-projective semiaffine space.*

Fact 3 (Dembowski & Kuiper [3]) *Every finite semiaffine plane is affino-projective.*

Corollary 4 (Beutelspacher, De Vito, Lo Re [2]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a finite linear space such that each line contains at least four points and such that through every point outside two given disjoint but nonparallel lines there is at most one transversal. Then $\mathcal{S}$ is an affino-projective semiaffine space, granted that every point on every transversal of two parallel lines (but not contained in these lines) is a witness for their parallelity.*

The aim of the present paper is to considerably weaken the conditions of Fact 1 by virtually deleting the last condition (i.e., we rephrase as done above and delete Condition (*)), and to remove the finiteness condition in Corollary 4. The latter will be done by proving directly that every plane of a semiaffine space containing at least two planes is affino-projective, granted there are infinitely many points per line. Of course, the case where $\mathcal{S}$ is a semiaffine *plane* cannot be included since there are counterexamples, see [3]. Hence we have the following theorem.

Theorem 5 *A linear space in which every line carries at least four points and which satisfies* Condition (NOA) *is either a line, a semiaffine plane, or an affino-projective semi-affine space.*

It is clear that Theorem 5 follows from our First and Second Main Results.

Note that Fact 1 is also true for linear spaces with arbitrary line size, in particular when there are lines of size two. However, this cannot be true for our First Main Result as the following counterexample shows. Let $\mathcal{S}$ consist of a point ∞ and two disjoint families of points $\mathcal{P}'$ and $\mathcal{P}''$. The lines are all pairs of points where one point belongs to $\mathcal{P}'$ and the other to $\mathcal{P}''$, together with the sets $\mathcal{P}' \cup \{\infty\}$ and $\mathcal{P}'' \cup \{\infty\}$. Then $\mathcal{S}$ satisfies the conditions of our First Main Result (except for the line sizes of course), but not of Fact 1. Moreover, it is not a semiaffine plane. So we do need the condition on the size of the lines in our First Main Result.

3 Proofs

3.1 First Main Result

By Fact 1, it suffices to show that, for two parallel lines L_1 and L_2, and any transversal M, every point x on $M \setminus (L_1 \cup L_2)$ is a witness for the parallelity of L_1 and L_2. We prove a few lemmas.

Throughout, we assume that L_1 and L_2 are two given parallel lines, with witness p.

Lemma 6 *Every line parallel to L_1 with witness p is also parallel to L_2 with witness p.*

Proof Suppose, by way of contradiction, that a line L' intersects L_2 in a point a, and that L' is parallel with L_1 with witness p. Let a' be the intersection of L_1 and ap. Pick b on L', with $b \neq a$. Pick b' on L_1, with $b' \neq a'$ and $b' \notin pb$. Note that pb meets L_1. Consider the line bb'.

- Assume bb' meets L_2. Then the two lines bb' and bp through b meet both of L_1 and L_2, hence, by (NOA), b is a witness for the parallelity of L_1 and L_2. Consequently ba, which equals L', must meet L_1, too, a contradiction to our assumption.

- Now assume that bb' is disjoint from L_2, and hence is parallel with it with witness p (since pb meets L_2 because it meets L_1, and pb' meets L_2 since it meets L_1, and L_1 and L_2 are parallel with witness p). This implies that pa meets bb', say in the point c.

  Pick d on L', $d \notin \{a, b\}$. Pick e on L_2, $e \neq a$, and $e \notin pd$ (for instance, to fix the ideas, we can choose e on the line pb). If de met L_1, then, since pd meets L_1 and hence L_2, we see that d would be a witness for the parallelity of L_1 and L_2, implying $da = L'$ meets L_1, a contradiction. Hence de is disjoint from L_1. But both of pd and pe meet L_1; hence p is a witness for the parallelity of L_1 and de. It follows that pa meets de. Now, if de and bb' were disjoint, then they would be parallel with witness a, as L' and ap both meet both of de and bb'. This would imply that also L_2 meets bb', a contradiction. So ed and bb' meet in a point f.

  Since ed and L_1 are parallel with witness p, the line fp meets L_1 and hence also L'. So both lines fp and fb meet both of L' and L_1, hence the latter are parallel with witness f, implying fd meets L_1, the final contradiction.

Hence L' meets L_1 and the lemma is proved. □

We can already conclude the following.

Corollary 7 *Every point not on $L_1 \cup L_2$ on a transversal of L_1 and L_2 that contains p, is a witness for the parallelity of L_1 and L_2.*

Proof If x is such a point, then by the previous lemma, every line through x meeting L_1 meets L_2 and vice versa. □

We now prove that every transversal of L_1 and L_2 contains a witness for the parallelity of L_1 and L_2.

Lemma 8 *Every transversal M of L_1 and L_2 contains a witness for the parallelity of L_1 and L_2.*

Proof Set $x_i := L_i \cap M$, $i = 1, 2$, and set $y_i = px_{3-i} \cap L_i$, $i = 1, 2$. Let z_1 be a point on L_1, $z_1 \notin \{x_1, y_1\}$. If pz_1 meets M, then the result follows from Corollary 7. So we may assume that pz_1 and M are disjoint, in which case they are parallel with witness y_2 (noting that pz_1 meets L_2). Consequently y_2z_1 meets M in some point w. But now clearly w is a witness for the parallelity of L_1 and L_2 and belongs to M. □

Combining Corollary 7 with Lemma 8, the First Main Result follows from Fact 1.

3.2 Second Main Result

Now we assume that $\mathcal{S}$ is a semiaffine planar space, and we may assume that each line has infinitely many points. We forget about the definition of parallelity previously given, and call now two lines L, M *parallel*, in symbols $L \parallel M$, if they are coplanar and disjoint or equal. Our first aim is to prove that parallelism is an equivalence relation.

Lemma 9 *Parallelism in a semiaffine space $\mathcal{S}$ is an equivalence relation.*

Proof Suppose $L_1 \parallel L \parallel L_2$. We clearly may assume that $L_1 \neq L_2 \neq L \neq L_1$. Let π_i be the plane containing L and L_i, $i = 1, 2$. Note that L_1 and L_2 are disjoint as otherwise $\pi_1 = \pi_2$ and so $L_1 = L_2$ by the axioms of semiaffine planes. Pick points $x_i \in L_i$, $i = 1, 2$, and let x be an arbitrary point on L. It is easy to see that there exists a line M meeting $xx_1 \cup xx_2 \cup x_1x_2$ in three distinct points y, y_1, y_2, with $y \in x_1x_2$, $y_i \in xx_i$, $i = 1, 2$. Choose a point $x' \in L$, $x' \neq x$. In π_i, the line $x'y_i$ meets L_i in some point x'_i, $i = 1, 2$ (otherwise there are two lines through x' in π_i parallel to L_i). Consider the plane π generated by x', y_1, y_2.

Suppose first that for some $i \in \{1, 2\}$, the line yx'_i meets the line $x'x'_{3-i}$, say in the point z. Then the plane α generated by y and L_i contains x_{3-i} and z, and so it meets the plane π_{3-i} in the line $K := x_{3-i}z$ (clearly, $x_{3-i} \neq z$). If K met L, then the intersection point would belong to α. Since also L_i belongs to α, this would imply that α coincides with π_i, contradicting $y \notin \pi_i$. Hence K and L are disjoint in π_{3-i}, implying that $K = L_{3-i}$. So α contains the disjoint lines L_1 and L_2; consequently $L_1 \parallel L_2$.

Now suppose that x'_1y is disjoint from $x'x'_2$ and x'_2y is disjoint from $x'x'_1$. We can find $y_0 \in y_1y_2$, $y_1 \neq y_0 \neq y_2$ and $x_0 \in x_1x_2$, $x_1 \neq x_0 \neq x_2$, such that x_0y_0 contains x. Then

$x_i'y$ meets $x'y_0$, say in the point z_i, $i = 1, 2$. The same argument as in the foregoing paragraph now shows that $z_i \neq x_0$ and x_0z_i is parallel to L in the plane π_0 spanned by L and x_0, $i = 1, 2$; this already implies $x_0z_1 = x_0z_2 =: L_0$. Furthermore, as in the previous paragraph, $L_1 \parallel L_0 \parallel L_2$ and L_0, L_i are contained in the plane α_0 spanned by y and L_0, for $i = 1, 2$. Hence α_0 contains both L_1 and L_2 and the lemma is proved. □

The next lemma shows that each parallel class of lines is either trivial or large, in the sense that, if the class is not trivial, each plane through every member of the parallel class contains a lot of other members.

Lemma 10 *Let L be a line of $\mathcal{S}$, and let π_1, π_2 be two distinct planes through L. If some line in π_1 is parallel to L, but distinct from L, then there is at most one point x in π_2 not belonging to a line parallel to L.*

Proof Suppose there are two points x, y in π_2 with the property that no line in π_2 through x or y is parallel to L. Then the line xy is certainly not parallel to L and meets L in some point z. Choose a second point $z' \neq z$ on L. Let $L_1 \neq L$ in π_1 be parallel to L and choose two points z_1, z_1' on L_1, with $z_1 \neq z_1'$, in such a way that zz_1 and $z'z_1'$ meet in a point t. Choose a point u on xy, $u \notin \{x, y, z\}$, such that the line $T := tu$ is not parallel to either xz_1 or yz_1 (here we actually need at least six points on the line xy; in the finite case with at least four points per line, another argument should be included). It follows that xz_1 meets T in some point p_x and yz_1 meets T in some point p_y.

Consider the plane α spanned by T and z', which also contains z_1'. One of the lines p_xz_1', p_yz_1' is not parallel to uz' in α; say p_xz_1' meets uz' in a point x'. Now the plane α_x spanned by p_x and L_1 contains two points (namely, x and x') of π_2, and hence it contains the line xx' of π_2. Clearly xx' is parallel to L_1, hence also to L, a contradiction. □

It now follows that there are two types of lines in $\mathcal{S}$: one type of lines, which we call *projective*, consists of those lines that have no parallel line; the other type, called *affine*, consists of lines that have parallel lines in every plane to which they belong.

Using this lemma, we can now use exactly the same proof as for Lemma 10 to show the following slightly stronger result.

Lemma 11 *Let L be an affine line in some plane π_2. If there exists a point x in π_2 not contained in a line of π_2 parallel to L, then x belongs to every projective line contained in π_2.*

Proof Suppose $x \in \pi_2$ is not contained in a line of π_2 parallel to L, and suppose x does not belong to a given projective line L' of π_2. Then, in the proof of Lemma 10, we can disregard y and we can choose z outside L', we can choose z' in $L \cap L'$, and we can choose u on L' (and $u \in \{x, z\}$, as required). Then, with the above notation, the line p_xz_1' always meets $z'u$ in some point x' and xx' is parallel to L, a contradiction. □

We now handle the case where a plane contains two intersecting affine lines.

Lemma 12 *Let π be a plane of $\mathcal{S}$ and suppose that π contains two intersecting affine lines L_1, L_2. Then either π is an affine plane, or π is an affino-projective semiaffine plane arising from an affine plane by adding just one point at infinity.*

Proof Let π, L_1, L_2 be as described. Since we assume that $\mathcal{S}$ has at least two planes, there must be a point outside π, and hence there exists a line K not belonging to π but intersecting π in $L_1 \cap L_2 =: x$. By Lemma 10, there is at most one point x_i on K such that the plane α_i spanned by L_i and K contains no line through that point parallel to L_i, $i = 1, 2$. Since lines have at least four points (in fact, we assume infinitely many), we find a point x' on K and distinct lines L'_1, L'_2 parallel to L_1, L_2, respectively, and containing x'. Denote the plane spanned by L'_1, L'_2 by π'. Since clearly the lines L'_1 and L'_2 do not have any point in common with π (as K does not belong to π and every triple of noncollinear points is contained in a unique plane), the planes π and π' meet in at most one point (if they met in a line, then this line would meet at least one of L'_1 or L'_2, contradicting our observation just made). Hence there are two possibilities.

- *Suppose $\pi \cap \pi' = \emptyset$.* Let y_i be an arbitrary point on L_i, $i = 1, 2$, with $y_i \neq x$. Let k be a point on K, with $x \neq k \neq x'$. In the plane α_i, the line ky_i is not parallel to L'_i, hence it meets it in a point y'_i, $i = 1, 2$. The lines y_1y_2 and $y'_1y'_2$ belong to the same plane and are disjoint; so they are parallel. Hence y_1y_2 is an affine line. Given z_1 on L_1, with $z_1 \neq x$, we can find a point z'_1 on L'_1, with $z'_1 \neq x'$, such that $y_1z'_1 \cap K \neq \emptyset \neq z_1z'_1 \cap K$. Performing twice the construction above with k substituted with $y_1z'_1 \cap K$ and $z_1z'_1 \cap K$, respectively (and the second time the roles of π and π' interchanged), we see that there is a line in π containing z_1 and parallel to y_1y_2. Replacing L_1 by a parallel line, we see that there is also a line parallel to y_1y_2 containing x. Similarly, we also see that all lines in π must be affine. It now follows easily that through every point of π there is a line parallel to any given line. Hence π is an affine plane.

- *Suppose now $\pi \cap \pi' = \{p\}$, with $p \in \mathcal{P}$.* Similarly as above, one shows that all lines of π not containing p are affine lines, and that for every point $q \neq p$, and every line L of π, with $p \notin L$, there exists a line $L' \subseteq \pi$, with $q \in L' \parallel L$.

  Suppose now, by way of contradiction, that there was an affine line M in π containing p. Then there would be a line M^* parallel to M inside π not containing p. With the notation of the previous paragraphs, this would imply that M^* is parallel to some line M' in π'. By the transitivity of parallelism (see Lemma 9), this yields $M \parallel M'$. But no plane contains both M and M' as the plane generated by M' and $p \in M$ does not contain M, a contradiction.

  Hence all lines in π through p are projective. Removing p from π, we clearly obtain an affine plane.

The lemma is proved. □

From now on, we consider only planes π having no intersecting affine lines, i.e., planes admitting only one parallel class of lines. Of course, if there is no affine line, then the plane is a projective plane, and we are done. Hence we may assume that there is at least one affine line L. Since all lines of π meeting L are projective, Lemma 11 implies that π is the union of all lines of π parallel to L. Adding a point at infinity corresponding with that parallel class of lines in the usual way, we see that π arises from a projective plane by deleting one point and no lines.

Hence we have proved:

Theorem 13 *In a semiaffine planar space with at least two planes, every plane is an affino-projective semiaffine plane.*

Our Second Main Result now follows from Teirlinck's result Fact 2. However, a direct proof is also possible, and we sketch one here (but leave the details to the reader). We add to $\mathcal{S}$ a point at infinity for each parallel class of lines, and add that point to each line of that parallel class. We add lines at infinity, containing points at infinity and possibly ordinary points, corresponding to the lines at infinity of the semiaffine planes of $\mathcal{S}$ which arise from affine planes where at most one point at infinity is added. Hence we have completed every semiaffine plane to a projective plane. It is now trivial to check Pasch's axiom if all lines considered are contained in the completion of a semiaffine plane (belonging to $\mathcal{E}$). So we are reduced to check Pasch's axiom in the case where the elements are not contained in any member of $\mathcal{E}$, but are all "at infinity" of some planes. This case reduces readily to the "three-dimensional" case, i.e., to a subspace of $\mathcal{S}$ generated by two planes intersecting in a line. Using the structure of affino-projective semiaffine planes, the rest of the proof is a tedious but elementary exercise. So one obtains a projective space in which $\mathcal{S}$ is naturally embedded, and now the Second Main result is clear.

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