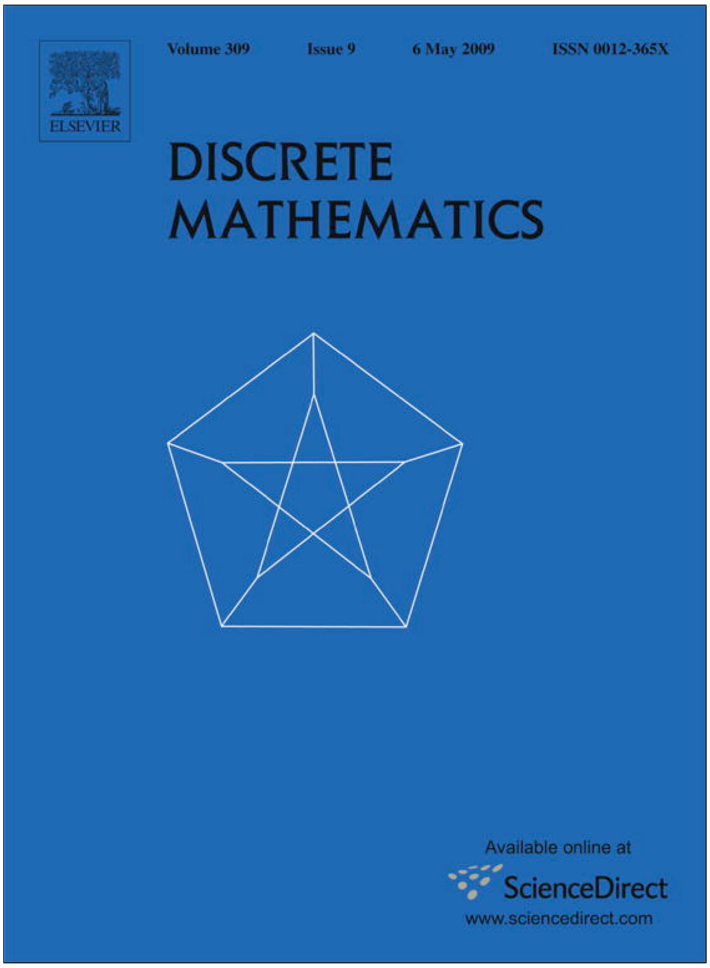
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Planar and affine spaces

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ABSTRACT

In this note, we characterize finite three-dimensional affine spaces as the only linear spaces endowed with set Ω of proper subspaces having the properties (1) every line contains a constant number of points, say n, with $n > 2$; (2) every triple of noncollinear points is contained in a unique member of Ω; (3) disjoint or coincide is an equivalence relation in Ω with the additional property that every equivalence class covers all points. We also take a look at the case $n = 2$ (in which case we have a complete graph endowed with a set Ω of proper complete subgraphs) and classify these objects: besides the affine 3-space of order 2, two small additional examples turn up. Furthermore, we generalize our result in the case of dimension greater than three to obtain a characterization of all finite affine spaces of dimension at least 3 with lines of size at least 3.

1. Introduction

Linear spaces are the fundamental objects of incidence geometry. Projective and affine spaces are the prominent examples of linear spaces (although there are other very important cases like unitals, nets, etc.). They have been extensively studied and a lot of characterizations are known. For an overview, see [4]. In this note, we add a characterization of affine spaces of dimension 3 in terms of *planar spaces*, which are also well-studied objects, see for instance [1,7,5,8], or more recent work of Biondi and Durante (and co-authors), e.g., [2,6], etc.

The most prominent examples of planar spaces are the projective and affine spaces. Many characterizations of projective and affine spaces in terms of planar spaces have been obtained. For instance, if all planes are projective planes, then we have a projective space (Veblen–Young [9]), or if all planes are affine planes and lines have size at least four, then we have an affine space (Buekenhout [3]). Other conditions considered before were of local nature (looking at the geometry of lines and planes through a point) or numerical. In the present paper we consider a global condition, namely, the parallel axiom for planes. Of course, there are many examples of planar spaces satisfying the parallel axiom for planes: just remove an appropriate set of points from a three-dimensional affine space. But these spaces rarely have constant line size, and hence we will also assume constant line size.

This way we characterize finite affine 3-spaces with line size at least 3. If the line size is 2, then some additional examples turn up, which also come out of our classification.

Note that we do not assume that our planes are minimal subspaces (a condition that is sometimes required for planar spaces).

A similar characterization for affine spaces of higher dimension using planar spaces and parallelism seems hard to find, since parallel planes in this case are no longer recognized as the disjoint ones. But if we slightly generalize the notion of a planar space, then we are able to characterize all finite affine spaces of dimension at least 3 having lines of size at least 3

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as the only linear spaces with constant line size $n \geq 3$ endowed with a family of *hyperplanes* satisfying the parallel axiom and such that every three noncollinear points are contained in a constant number of hyperplanes. We will also show that relaxing the condition $n \geq 3$ to $n \geq 2$ creates a number of rather wild examples that might be hard to classify.

We now get down to precise definitions and statements of our main results.

2. Main results

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a linear space, i.e., $\mathcal{P}$ and $\mathcal{L}$ are two disjoint sets whose members are called points and lines, respectively, and $\mathtt{I}$ is a symmetric (incidence) relation between $\mathcal{P}$ and $\mathcal{L}$ with the properties that every pair of distinct points is incident with a unique common line and every line is incident with at least two points. It is an immediate property that every line is determined by the set of points incident with it, and therefore, one usually views the members of $\mathcal{L}$ as subsets of $\mathcal{P}$.

We will be interested in linear spaces with constant line size. There is also a definition of *order* of a linear space, but this does not agree with the usual definitions of order of a projective and/or affine plane, and so we will not use it. We will, though, introduce and use the usual definitions of order of a projective or affine plane (see below).

A *subspace* of a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a linear space $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$, with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathtt{I}'$ the restriction of $\mathtt{I}$ to $\mathcal{P}'$ and $\mathcal{L}'$, such that all points of $\mathcal{S}$ incident with a member of $\mathcal{L}'$ are also points of $\mathcal{S}'$. The subspace $\mathcal{S}'$ is called *proper* if $\mathcal{P}' \neq \mathcal{P}$.

A linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with line size n for which $\mathcal{S}^* = (\mathcal{L}, \mathcal{P}, \mathtt{I})$ is also a linear space is a *projective plane of order* $n-1$. A projective plane of order n from which we delete a line together with all its points is an *affine plane of order n*. *Affine spaces of order n* and of dimension >2 are linear spaces arising from vector spaces of dimension >2 over the field of order n by taking as lines all translates of the one-dimensional subspaces. *Projective spaces of order* $n-1$ and of dimension $d > 2$ arise from vector spaces of dimension $d+1$ over the field with $n-1$ elements by taking as points the one-dimensional subspaces and as lines the two-dimensional subspaces of the vector space.

The linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ endowed with a family Ω of proper subspaces is called a *planar linear space* (or an ℓ-*hyperplanar linear space*, for some positive integer ℓ, respectively) if every triple of noncollinear points (i.e., points that are not contained in a common line) is contained in exactly one (or exactly ℓ, respectively) member(s) of Ω. We briefly say that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ *is a* ($n\ell$-*hyper*) *planar space* and the members of Ω are called *(hyper)planes*. The *parallel axiom for (hyper)planes* in the (ℓ-hyper)planar space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ says that

[ParAx] "have disjoint point sets or coincide" is an equivalence relation in Ω and the union of every equivalence class is $\mathcal{P}$.

This may obviously be rephrased as

[ParAx′] for every point x and every member $\mathcal{S}'$ of Ω not containing x as a point, there exists a unique member of Ω containing x as a point and having a point set disjoint from the point set of $\mathcal{S}'$.

Examples of planar spaces satisfying Axiom [ParAx] arise from ordinary affine spaces of dimension ≥ 3 by deleting a number of points in such a way that one does not delete an entire line. If we identify the affine space with a vector space, then the (hyper)planes are the translates of the next-to-maximal-dimensional subspaces. This provides a wealth of examples. It also shows that a classification of such spaces will be very difficult. However, if we require constant line size n, $n > 2$, then we can classify, and only the affine spaces themselves emerge.

We first consider the case of planar spaces.

Theorem 1. *Every planar space with constant line size n, $n > 2$, in which the parallel axiom for planes holds is a three-dimensional affine space of order n.*

When $n = 2$, then our technique fails, and the difficulty is certainly that planes may have different sizes. Moreover, there is an example of such a situation. Let us give all known examples.

(Ex1) Let $\mathcal{P}$ be a set of six elements, $\mathcal{L}$ is the set of all pairs of $\mathcal{P}$, incidence is the natural one. The family Ω is induced by all 3-subsets of $\mathcal{P}$.

(Ex2) Let $\mathcal{P}$ be the set of points of a Fano plane (a projective plane of order 2), $\mathcal{L}$ is again the set of pairs of $\mathcal{P}$, with natural incidence. The family Ω is induced by the set of lines of that plane, together with the set of complements of lines (or, equivalently, *hyperovals*) of that plane. Here, the planes have different sizes.

(Ex3) The affine 3-space of order 2.

We will prove the following result.

Theorem 2. *The only planar spaces with constant line size* 2 *in which the parallel axiom for planes holds are the examples* (Ex1), (Ex2) *and* (Ex3).

In general, we can state:

Theorem 3. *Every ℓ-hyperplanar space with constant line size n, $n > 2$, in which the parallel axiom for hyperplanes holds is a d-dimensional affine space of order n, with $d \geq 3$.*

When $n = 2$, then the examples (Ex1), (Ex2) and (Ex3) may be generalized, but there are also additional examples showing that a classification will be very unlikely.

(Ex4) Let $\mathcal{P}$ be a set of $2k$ elements, $\mathcal{L}$ is the set of all pairs of $\mathcal{P}$, incidence is the natural one. The family Ω is induced by all k-subsets of $\mathcal{P}$. Then $(\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ is an ℓ-hyperplanar space with $l = \binom{2k-3}{k-3}$.

(Ex5) Let $\mathcal{P}$ be the set of points of a projective space of order 2 and dimension $d \geq 2$, $\mathcal{L}$ is again the set of pairs of $\mathcal{P}$, with natural incidence. The family Ω is induced by the set of hyperplanes of that space, together with the set of complements of hyperplanes. Then $(\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ is an ℓ-hyperplanar space with $\ell = 2^{d-1} - 1$. Here, the hyperplanes have different sizes.

(Ex6) The affine space of order 2 and dimension d, $d \geq 3$, where the members of Ω are the $(d-1)$-dimensional subspaces. Here $\ell = 2^{d-2} - 1$.

(Ex7) Consider an affine space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ of order 2 and dimension 3, viewed as planar space. Consider two disjoint lines L_1, L_2 that are not contained in a common plane. Let σ be the involution on $\mathcal{P}$ interchanging the two points of L_1 and interchanging the two points of L_2, keeping the other four points of $\mathcal{P}$ fixed. Then one easily checks that $\mathcal{S}' = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega \cup \Omega^\sigma)$ is a 2-hyperplanar space with constant line size 2.

(Ex8) Consider the 3-hyperplanar space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ with constant line size 2 arising from a projective space of order 2 and dimension 3 as in Example (Ex5). Consider two skew lines L_1 and L_2 of the projective space (hence both lines have three points). Let σ be an arbitrary permutation of $\mathcal{P}$ fixing all points off $L_1 \cup L_2$, and inducing 3-cycles on both L_1 and L_2. Then one checks that $\mathcal{S}' = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega \cup \Omega^\sigma)$ is a 6-hyperplanar space with constant line size 2, and $\mathcal{S}'' = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega \cup \Omega^\sigma \cup \Omega^{\sigma^2})$ is a 9-hyperplanar space with constant line size 2.

(Ex9) Consider the Steiner system $S(5, 6, 12)$ related to the sporadic simple group M_{12}. Every block has a unique complementary block and three points are contained in exactly 12 blocks. Hence this defines a 12 -hyperplanar space with constant line size 2.

It is clear that (Ex7) and (Ex8) can be generalized to higher dimensions to obtain additional examples, which may be hard to classify since rather wild permutations σ may be considered.

However, the lack of additional examples, and our inability to find them, leads us to the following conjecture.

Conjecture. *In every ℓ-hyperplanar space with constant line size 2 in which the parallel axiom for hyperplanes holds, all parallel classes have exactly two elements. Moreover, the size of any hyperplane differs by at most one half from half the size of the point set.*

In fact, Theorem 1 is obviously a special case of Theorem 3. Yet, we choose to prove Theorem 1 separately since it might appeal more to readers interested rather in planar spaces than in affine spaces.

3. Proof of Theorem 1

We first remark that in a planar space which satisfies the parallel axiom for planes, no plane coincides with a line. Indeed, let L be a line which is also a plane and choose a point x outside L (it exists since every plane is a *proper* subspace). Let π be the unique plane through L and x, and let π' be parallel to π and distinct from it (again, π' exists for the same reason). Then both L and π are parallel to π', but not disjoint and not the same, a contradiction.

Suppose henceforth that the space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a planar space with constant line size n, $n > 2$, satisfying the parallel axiom for planes.

Lemma 3.1. *The number of points in each plane is one more than a multiple of $n - 1$.*

Proof. Fix a point x in a plane π. Then the lines in π through x partition the point set of π without x. All these lines contain $n - 1$ points distinct from x. □

Lemma 3.2. *Each line intersecting a plane π_1 in a point intersects every parallel plane π_2 in a point.*

Proof. Let π_1 and π_2 be two parallel but distinct planes, and let L_1 be a line meeting π_1 in a point. We assume, by way of contradiction, that L_1 does not meet π_2. Let π_1' be a plane containing L_1 and let π_2' be a plane parallel to π_1' but distinct from it. Let there be exactly k planes through L_1. All of these k planes meet π_2, and exactly $k - 1$ of them meet π_2'. Suppose that ℓ of these planes meet π_2 in a line, then π_2 contains $\ell n + k - \ell$ points, which is equal to k modulo $n - 1$. By Lemma 3.1, k is equal to 1 modulo $n - 1$. Suppose also that ℓ' of the former considered set of $k - 1$ planes (through L_1 that meet π_2') meet π_2' in a line, then the number of points of π_2' equals $\ell' n + (k - 1 - \ell')$, and this equals $k - 1$ modulo $n - 1$. Lemma 3.1 implies that $k - 1$ is equal to 1 modulo $n - 1$, which is a contradiction to the above. □

This implies easily:

Corollary 3.3. *Given a plane π and a line L not meeting π, then there exists a unique plane π' parallel to π and containing L.*

Proof. If the plane π' parallel to π and incident with some fixed but arbitrary point x on L did not contain L, then L would intersect π by the previous lemma. □

Lemma 3.4. *Each plane is an affine plane of order n or a projective plane of order $n-1$.*

Proof. It suffices to show that in each plane π, there is at most one line through a given point x parallel to a given line L. Assume, by way of contradiction, that there were at least two such lines L_1, L_2. Let π' be a plane containing L, with $\pi' \neq \pi$. Clearly, both of L_1, L_2 do not meet π', and hence Corollary 3.3 implies that L_i is contained in a plane π_i parallel to π', $i = 1, 2$. But then x is contained in at least two planes parallel to π', a contradiction.

The lemma is proved. □

Lemma 3.5. *Two distinct planes are either parallel or meet in a line.*

Proof. Suppose that two planes π and π' meet in a point x. Then we can find two intersecting lines in π not incident with x. Both lines must be contained in some respective plane parallel to π', according to Corollary 3.3. This is again a contradiction as otherwise the intersection point is contained in more than one plane parallel to π'. □

Lemma 3.6. *All planes are affine planes.*

Proof. Suppose that π is a projective plane. Consider two planes α, α' not parallel to π, but parallel to one another. They have to meet π in two non-intersecting lines, according to Lemma 3.5. But in π, every two lines meet in a point. This is a contradiction. □

We can now finish the proof of our theorem. In fact, it follows readily from a more general result by Buekenhout (see [3]) for $n \geq 4$, or we can appeal to the main result of Doyen and Hubaut [5] or of Teirlinck [8], but we can give an easy and self-contained proof in the above spirit.

We define a parallelism between lines as follows. Two lines L and L' are parallel if they are parallel in some plane of $\mathcal{S}$.

Lemma 3.7. *Parallelism is an equivalence relation in the set of lines $\mathcal{L}$ in $\mathcal{S}$.*

Proof. Suppose that L is parallel to L' and L' is parallel to L'', with $L, L', L'' \in \mathcal{L}$, and with $L \neq L''$. We may assume that the planes π and π' through L, L' and through L', L'', respectively, are distinct (which also implies that L and L'' do not meet). Choose some point x on L and let α be the plane through x and L''. Clearly L' does not meet α (because otherwise $\pi = \alpha = \pi'$). By Corollary 3.3, there is some plane α' parallel to α and containing L'. If L intersected α' nontrivially, then $\alpha' = \pi = \alpha = \pi'$, a contradiction. Hence L and L'' are contained in α by Lemma 3.2, so L and L'' are parallel. If L were not parallel to L'' in α, then π and π' would both coincide with the plane through L' and the intersection point of L and L'', a contradiction. Hence L and L'' are parallel. □

Here we could actually refer to Theorem 2.7 of [4] to conclude the proof. But the proof is also easily concluded with independently as follows.

Lemma 3.8. *There are n^3 points in $\mathcal{S}$.*

Proof. Let α and α' be two planes meeting in a line L. Every line in α parallel to L is contained in a plane parallel to α', and there are exactly n of these. Clearly, there are no more planes parallel to α'. Since every plane contains n^2 points, the lemma follows. □

It now follows that there are n^2+n+1 lines through a point, and hence there are precisely n^2+n+1 parallel classes of lines. Let $\mathcal{P}'$ be the set of all points of $\mathcal{S}$ and all parallel classes of lines of $\mathcal{S}$. Let $\mathcal{L}'$ be the set of all lines of $\mathcal{S}$ together with all sets of $n+1$ parallel classes of lines that contain lines of some plane. Then $\mathcal{S}' = (\mathcal{P}', \mathcal{L}')$, with natural incidence, is a linear space, which satisfies the axiom of Veblen–Young (all planes are projective), and in which the intersection of any two distinct planes is a line. Hence $\mathcal{S}'$ is a three-dimensional projective space of order n, and so $\mathcal{S}$ is a three-dimensional affine space of order n.

The theorem is proved.

4. Proof of Theorem 2

We start with a special case of Theorem 2.

Lemma 4.1. *The only planar spaces with constant line size* 2 *in which the parallel axiom for planes holds and which have constant plane size are the examples* (Ex1) *and* (Ex3).

Proof. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ be a planar space with constant line size 2 satisfying the parallel axiom for planes. Let v be the number of points of $\mathcal{S}$ and suppose that all members of Ω have same number k of points. We may assume that $k \geq 3$ to avoid trivial cases. Since every three distinct points are necessarily noncollinear, the number b of planes is equal to

$$|\Omega| = b = \frac{v(v-1)(v-2)}{k(k-1)(k-2)}.$$

We claim that $k^2 > v$. Indeed, if, on the contrary, k^2 did not exceed v, then a plane α would meet every plane of a parallel class of planes in at most one point (since there are $v/k \geq k$ planes in a parallel class), and so every two nonparallel planes would meet in a single point, contradicting the fact that there exist planes meeting in exactly two points (considering the unique planes through two fixed and one varying point). The claim follows.

Now fix a parallel class $C \subseteq \Omega$ of planes. It contains v/k elements. Let π be any plane not contained in C. Since π contains k points and meets every member of C in one or two points, it meets exactly $\frac{k^2-v}{k} > 0$ members of C in two points, and $\frac{2v-k^2}{k}$ members in one point. Hence every plane outside C meets at least one member of C in two points. The number of planes not in C meeting a member α of C in two fixed points is $\frac{v-k}{k-2}$. Consequently there are $\frac{k(k-1)}{2}\frac{v-k}{k-2}$ planes not belonging to C meeting α in two points. If we vary α over C, then we count every plane exactly $\frac{k^2-v}{k}$ times. Hence there are exactly

$$\frac{\frac{k(k-1)}{2}\frac{v-k}{k-2}\cdot\frac{v}{k}}{\frac{k^2-v}{k}} = \frac{vk(k-1)(v-k)}{2(k-2)(k^2-v)}$$

planes not belonging to C. So we obtain the equality

$$\frac{vk(k-1)(v-k)}{2(k-2)(k^2-v)} + \frac{v}{k} = \frac{v(v-1)(v-2)}{k(k-1)(k-2)},$$

which, after an elementary calculation, simplifies to

$$2v^2 - 2(k^2-k+3)v + (k^4-4k^3+7k^2) = 0.$$

Since v is a positive integer, the discriminant D of the latter equation, where v is viewed as the unknown, must be a nonnegative, and a perfect square. We compute easily $D = -k^4 + 6k^3 - 7k^2 - 6k + 9$, which is nonnegative only for $k = 3, 4$ (remember also k is nonnegative and $k \geq 3$). If $k = 3$, then $D = 9$ and $v \in \{3, 6\}$; if $k = 4$, then $D = 1$ and $v \in \{7, 8\}$. Hence for $k = 3$, we must have $v = 6$ (otherwise there is only one plane) and Example (Ex1) arises. If $k = 4$, then $v \neq 7$ as k must divide v. Hence $v = 8$ and it is easily shown that the unique example in this case is Example (Ex3).

The lemma is proved. □

Now we prove the following lemma, which is again a special case of Theorem 2.

Lemma 4.2. *The only planar spaces with constant line size* 2 *in which the parallel axiom for planes holds, and which are the union of two parallel planes, are the examples* (Ex1), (Ex2) *and* (Ex3).

Proof. Let $\alpha_1\alpha_2$ be two parallel planes, with $\alpha_1 \cup \alpha_2 = \mathcal{P}$, the point set of the planar space $\mathcal{S}$ with constant line size 2. Suppose $|\alpha_1| \geq 5$.

We claim that $|\alpha_2| > 3$. In deed, if not, then $|\alpha_2| = 3$. There is a plane $\pi \neq \alpha_2$ containing two points of α_2. It meets α_1 in at most two points. Every plane parallel to π meets both α_1 and α_2. But there is only one point in α_2 not in π, and hence there is only one plane π' distinct from π and parallel to it. Clearly, π' meets α_1 in at most two points, and since $\pi \cup \pi' = \mathcal{P}$, we see that α_1 contains at most 4 points, a contradiction.

Hence $|\alpha_2| \geq 4$. Again let π be a plane meeting α_2 in exactly 2 points x, x'. Let y, y' be two distinct points of α_2 different from x, x'. Since α_1 has at least 5 points, and every plane through y, y' meets α_1 in at most 2 points, there are at least three planes through y, y'. They cannot all meet π nontrivially since there are only at most two candidates left (the intersection of α_1 with π). Hence there is some plane π' parallel to π containing y, y'. Varying y' keeping y fixed, we immediately see that $\alpha_2 = \{x, x', y, y'\}$ (using the uniqueness of the parallel through the point y). But now we see that π and π' are the unique planes in their parallel class and so their union must be $\mathcal{P}$. But they cannot cover all points of α_1, a contradiction.

So we conclude $|\alpha_1| \leq 4$ and likewise $|\alpha_2| \leq 4$. Hence $|\mathcal{P}| \in \{6, 7, 8\}$. If $|\mathcal{P}| = 8$, then it is easily seen that every parallel class contains exactly two planes of size 4, and so all planes have the same size. The result follows easily (or from Lemma 4.1). Likewise, if $|\mathcal{P}| = 6$, then it is easily seen that every parallel class contains exactly two planes of size 3, and so all planes have the same size. The result follows easily (or again from Lemma 4.1). Finally, if $|\mathcal{P}| = 7$, then each parallel class of planes contains a plane of size 4 and one of size 3. Hence all planes of size 3 meet each other. Suppose that two such planes meet in two points. Then the respective parallel planes of size 4 must meet in $7 - 1 - 1 - 2 = 3$ points, a contradiction. Hence the planes of size 3 define a projective plane, which must be the unique projective plane of order 2.

The lemma is proved. □

The next lemma reduces the general case to almost constant plane size.

Lemma 4.3. *If every parallel class of planes in a planar space with constant line size* 2 *in which the parallel axiom for planes holds, has at least three members, then there is a number* k *such that each plane has size* k *or* $k+1$.

Proof. Suppose that π, π' are two planes of the planar space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Omega)$ with constant line size 2 of different size. We may assume that $|\pi| > |\pi'| =: k$. We remark that $k > 3$ for otherwise any plane meeting π' in two points has a unique disjoint parallel plane. Hence we can choose two points x, x' in π' which are not contained in π. Note that there exists a point y not in $\pi \cup \pi'$ (indeed, if π is parallel to π', then this follows by assumption; if π is not parallel to π' then obviously any parallel plane to π contains a point not contained in π'). We claim that we may assume that the plane α through x, x', y is not parallel to π. Indeed, suppose it were. Then π' is not parallel to π and hence must contain at least one point x'' not contained in $\pi \cup \alpha$ (x'' is a point in a third plane parallel to both π and α). Then we can substitute α with the plane through x, x'', y and the claim follows.

Hence α meets π in at least one point y' and we can consider the set S of planes through y, y'. The plane π' meets every member of S in at least one point, except possibly one of them (a parallel one), and it meets at least one member in two points (namely, α). So $|S| \leq k$. The plane π meets every member in at most one point different from y, and hence $|\pi| - 1 \leq |S|$. Combining these inequalities, the assertion follows. □

We can now finish the proof of Theorem 2. There is an approach similar to the one used in the proof of Lemma 4.1, but there is also another possibility, which we shall present now, and which, so we believe, is slightly more efficient.

In the situation of the proof of Lemma 4.3, and with the same notation, we deduces that $|S| = k$. Hence, if there are precisely a planes of size k in S, then there are $k-a$ planes of size $k+1$ in S and $|\mathcal{P}| = v = a(k-2) + (k-a)(k-1) + 2 = k^2 - k - a + 2$. We deduce easily that every parallel class of planes contains at least $k-2$ (because $k-3$ planes can cover at most $(k-3)(k+1) = k^2 - 2k - 3 \leq k^2 - k - a - 3 < v$ points of $\mathcal{S}$) and at most $k-1$ (because k disjoint planes cover at least $k^2 = k^2 - k + k > k^2 - k + 2 - a$ points) members. Also, if some parallel class contains $k-2$ members, then it contains $a-4$ planes of size k and $k-a+2$ planes of size $k+1$. Hence, in this case, $a \geq 4$. If some parallel class contains $k-1$ members, then it contains $k+a-3$ planes of size k and $2-a$ planes of size $k+1$. Hence, in that case, $a \leq 2$. We conclude that either all parallel classes of planes contain exactly $k-2$ members, or all such classes contain exactly $k-1$ members. Consequently, we can break up the proof in two cases.

Case A: *All parallel classes of planes contain exactly* $k-2$ *members.* Notice that this implies $a \geq 4$, as remarked above. We first show that there must be two points contained in exactly $k-1$ planes.

Let S' be the set of planes through two points z, z'. Suppose that there are $a+b$ planes of size k in S', with b an integer. Then the number of planes of size $k+1$ in S' is equal to $\frac{(v-2)-(a+b)(k-2)}{k-1} = k - a - \frac{b(k-2)}{k-1}$, which implies that $k-1$ divides b, and that $b/(k-1) \leq (k-a)/(k-2) \leq (k-4)/(k-2) < 1$. Hence $b \in \{1-k, 0\}$. If $b = 1-k$, then there are $a-k+1$ planes of size k containing z, z', and there are $2k-a-2$ planes of size $k+1$ through z, z'. Hence, in total, there are $k-1$ planes through z, z'. We conclude that, if there are never exactly $k-1$ planes through two points, then there are always exactly k planes through two points, a of which have size k, and the remaining ones have size $k+1$.

So we may assume that there are always k planes through two points. Let X be the number of planes of size k and let Y be the number of planes of size $k+1$. Counting in two different ways the ordered triples (π, p, p') of distinct points p, p' and planes π of size k and $k+1$, respectively, with $p, p' \in \pi$, we obtain

$$X = \frac{v(v-1)a}{k(k-1)}; \qquad Y = \frac{v(v-1)(k-a)}{(k+1)k}.$$

Since every parallel class of planes contains exactly $a-4$ planes of size k and exactly $k-a+2$ planes of size $k+1$, we deduce that the number of parallel classes is equal to $X/(a-4) = Y/(k-a+2)$. Using the above expressions for X and Y in terms of v, a and k, we obtain, after rewriting slightly,

$$1 < \frac{a}{a-4} = \frac{k-a}{k-a+2} \cdot \frac{k(k-1)}{(k+1)k} < 1,$$

a contradiction. We conclude that there must exist two points z, z' contained in exactly $a-k+1$ planes of size k and $2k-a-2$ planes of size $k+1$. Since $0 \leq a \leq k$, we obtain the condition $a \in \{k-1, k\}$.

- Suppose first that $a = k-1$. Then there are no planes of size k through z, z', and so there must exist a pair of points x, x' contained in $a = k-1$ planes of size k and $k-a = 1$ plane of size $k+1$. Let z'' be a point distinct from z, z'. Then the plane through z, z', z'' has size $k+1$. Let π be a plane through z, z'' not containing z'. Since π meets every plane through z, z' in at most one point distinct from z, we see that $|\pi| \leq 1 + (k-1) = k$. Hence through z, z', there are exactly $k-1$ planes of size k. Since from the previous counting it follows that every plane of size k through z meets every plane of size $k+1$ through z, z' in a unique point distinct from z, z', we see that there are exactly $(k-1)^2$ planes of size k containing z.
  Note that $v - 2 = (k-1)^2$, implying $v = k^2 - 2k + 3$.
  Now let y be any point. Then the previous arguments imply that there is at least one point y' with the property that there are no planes of size k through y, y'. Suppose, by way of contradiction, that there is no such point y'. Then any point $y'' \neq y$ is contained, together with y, in a unique plane of size $k+1$. The number of such planes is equal to $(v-1)/k$, which implies that $k = 2$, a contradiction.
  Hence every point "behaves" like z and so there are exactly $k-1$ planes of size $k+1$ and $(k-1)^2$ planes of size k through every point. Let X be the number of planes of size k and let Y be the number of planes of size $k+1$. Counting in two different ways the ordered pairs (π, p) of points p and planes π of size k and $k+1$, respectively, with $p \in \pi$, we obtain

  $$X = \frac{v(k-1)^2}{k}; \qquad Y = \frac{v(k-1)}{k+1},$$

  which implies that v is divisible by k, so $k = 3$. But then a parallel class contains only one member, a contradiction.
- Suppose now that $a = k$. Then there are exactly $k-2$ planes of size $k+1$ through z, z', and exactly one plane of size k. It follows that $v = (k-2)k + 2$. Let z'' again be a point distinct from z, z', but not contained in the unique plane of size k through z, z'. As above, every plane through z, z'' must have size k (as otherwise it must meet some plane through z, z' in at least three points, including z, a contradiction), except for the one through z'. But then the number of planes through z, z'' equals $((v-2) - (k-1))/(k-2) + 1 = k - 1/(k-2)$, implying $k = 3$, a contradiction.

So we have shown that Case A cannot occur.

Case B: *All parallel classes of planes contain exactly* $k-1$ *members.* Here, there are $a+k-3$ planes of size k, and $2-a$ planes of size $k+1$ in every parallel class of planes. If $a=2$, then there would only be planes of size k and we can appeal to Lemma 4.1 to conclude that $k=3$, contradicting our assumption that every parallel class contains at least 3 members. Hence $a \in \{0, 1\}$.

Let z, z' be any two different point. Similarly as in the beginning of Case A, one shows that there are two possibilities: either z, z' are contained in exactly a planes of size k, and in $k-a$ planes of size $k+1$, or z, z' are contained in exactly $a+k-1$ planes of size k, and in $2-a$ planes of size $k+1$. Suppose first that the latter possibility never occurs. With the same notation as before, we have again

$$X = \frac{v(v-1)a}{k(k-1)}; \qquad Y = \frac{v(v-1)(k-a)}{(k+1)k}.$$

Hence the number of parallel classes equals

$$\frac{X}{a+k-3} = \frac{Y}{2-a} \Rightarrow k^3 - 4k^2 + (3+a)k + 2a^2 - 5a = o.$$

For $a=0$, this implies $k \in \{1, 3\}$, and for $a=1$, this implies $k=3$. This always implies that there are only two members in a parallel class, a contradiction.

Hence there are points z, z' contained in exactly $a+k-1$ planes of size k, and in $2-a$ planes of size $k+1$. Since $a+k-1 \neq 0$ and $k \geq 4$, we can consider two points y, y' in a plane α of size k through z, z', with $\{z, z'\} \cap \{y, y'\} = \emptyset$. If there were a second plane α' of size k through y, y', then, since it meets each of the k planes through z, z' different from α in at most one point, there would be at least two planes through z, z' disjoint from (and hence parallel to) α', a contradiction. Hence $a=1$ (since clearly $k+a-1>1$). Replacing α with a plane of size $k+1$ through z, z', we similarly deduce $a=0$. This is a contradiction.

We conclude that Case B cannot occur and the proof of Theorem 2 is complete.

5. Proof of Theorem 3

Suppose henceforth that the space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is an ℓ-hyperplanar space with constant line size n, $n>2$, satisfying the parallel axiom for hyperplanes. The following lemma has a similar proof as Lemma 3.1, and shall therefore be omitted.

Lemma 5.1. *The number of points in each hyperplane is one more than a multiple of* $n-1$. *Also, the number of points in the intersection of two nonparallel hyperplanes is one more than a multiple of* $n-1$.

Lemma 5.2. *Each line intersecting a hyperplane* ξ_1 *in a point* x_1 *intersects every parallel hyperplane* ξ_2 *in a point.*

Proof. Let ξ_1 and ξ_2 be two parallel but distinct hyperplanes, and let L_1 be a line meeting ξ_1 in a point x. We assume, by way of contradiction, that L_1 does not meet ξ_2. Let ξ_1' be a hyperplane containing L_1 and let ξ_2' be a hyperplane parallel to ξ_1' but distinct from it. Let there be exactly k hyperplanes through L_1. All of these k hyperplanes meet π_2, and exactly $k-1$ of them meet π_2'. We count in two ways the pairs (ξ, y), where ξ is a hyperplane through L_1 and y is a point in ξ and in ξ_2. Lemma 5.1 implies that $\ell|\xi_2|$ is the sum of k natural numbers every one of which is one modulo $n-1$. Reading this modulo $n-1$, we see that ℓ is equal to k modulo $n-1$. Similarly, $\ell|\xi_2'|$ is the sum of $k-1$ natural numbers every one of which is one modulo $n-1$. So ℓ is equal to $k-1$ modulo $n-1$, a contradiction. □

We now introduce planes as follows. Let x_1, x_2, x_3 be three noncollinear points. Then the intersection of all hyperplanes containing x_1, x_2 and x_3 is called a *plane*. The set of planes is denoted by Δ.

Lemma 5.3. *The structure* $\mathcal{S}' = (\mathcal{P}, \mathcal{L}, \mathtt{I}; \Delta)$ *is a planar space.*

Proof. Let π be a plane. It suffices to prove that for every three noncollinear points x_1, x_2, x_3 of π, the intersection of all hyperplanes through x_1, x_2, x_3 is precisely π. By definition of π, this holds for some three points y_1, y_2, y_3 of π. Since there are precisely ℓ hyperplanes through y_1, y_2, y_3, and since all these hyperplanes contain x_1, x_2, x_3, we see that these ℓ hyperplanes are exactly the set of hyperplanes through x_1, x_2, x_3 and so π is also the intersection of all hyperplanes through x_1, x_2, x_3. This shows the lemma. □

We can now show the counterpart of Lemma 3.6.

Lemma 5.4. *All planes are affine planes.*

Proof. Let π be any plane and let L be a line contained in π (since π contains at least three noncollinear points, it contains at least three lines). We first claim that not all hyperplanes that contain L contain π. Clearly π does not coincide with $\mathcal{P}$, and so there is some point $x \in \mathcal{P}$ outside π. There is a unique plane π' containing L and x, and it differs from π. Note that it cannot contain π (otherwise there are at least two planes through three noncollinear points of π). Since π' is the intersection of all hyperplanes containing L and x, there must be a hyperplane, say ξ, through L and x not containing π and the claim follows.

Next we claim that ξ meets π in L. Indeed, we already have that L is contained in $\xi \cap \pi$. Suppose that some point $y \in \xi \cap \pi$ is not incident with L. Then the plane through L and y coincides with π, but must also be contained in ξ, by the definition of planes. This contradicts the choice of ξ. The claim follows. Note that the same argument shows that hyperplanes and planes meet in either the empty set, or a point, or a line. Consider now a point z on L. Since $n > 2$, it is easy to see that there are at least two lines L_1, L_2 through z in π distinct from L. Every hyperplane ξ' parallel to ξ meets these two lines in two (different points), and hence meets π in a line L'. We conclude that there are exactly n hyperplanes in the parallel class of ξ, each of them intersecting π in n points. So π contains n^2 points. But a linear space of size n^2 where each line has size n is an affine plane. □

We now define a parallelism between lines, just like in the case $\ell = 1$, as follows. Two lines L and L' are parallel if they are parallel in some plane of $\mathcal{S}$, and we can prove:

Lemma 5.5. *Parallelism is an equivalence relation in the set of lines $\mathcal{L}$ in $\mathcal{S}$.*

Proof. Suppose that L is parallel to L' and L' is parallel to L'', with $L, L', L'' \in \mathcal{L}$, and with $L \neq L''$. We may assume that the planes π and π' through L, L' and through L', L'', respectively, are distinct (which also implies that L and L'' do not meet). Choose some point x on L and let ξ be any hyperplane through x and L'' not containing π (this exists because otherwise the plane through L'' and x would contain π, a contradiction). We first claim that ξ meets π in a line. Indeed, as in the proof of Lemma 5.4, we see that every parallel class of hyperplanes contains exactly n members. But every member of that parallel class meets π in at most n points, and the union of these intersections must be π, which contains n^2 points. Hence all intersections must contain n points. In particular, ξ meets π in a line. If this line does not coincide with L, then it meets L' and so ξ would contain π' and hence also π, a contradiction. The claim follows. It now also follows that every hyperplane containing L'' and x contains L. So the plane through L'' and x contains L and obviously L and L'' are parallel in that plane. Hence L and L'' are parallel and the lemma is proved. □

Similarly as in the case $\ell = 1$, one can now complete the proof of Theorem 3. We can also refer to Theorem 2.7 of [4]. This finished the proof of Theorem 3.

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