Moufang sets of type F_4

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Abstract

We give an explicit description of the Moufang sets of type F_4, i.e. the buildings arising from the simple algebraic groups of absolute type F_4 and relative rank one, over an arbitrary field. We use octonion planes and certain polarities to find this description, and we rely on the theory of Albert algebras. We also determine the automorphism groups of the corresponding exceptional unitals, thereby completing the program of J. Tits for these non-abelian Moufang sets. In particular we prove that every automorphism of that unital is induced by a collineation of the ambient projective plane.

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1 Introduction

Moufang sets were introduced by Jacques Tits in [19] as an axiomization of the isotropic simple algebraic groups of relative rank one, and they are, in fact, the buildings corresponding to these algebraic groups, together with some of the group structure (which comes from the root groups of the algebraic group). In this way, the Moufang sets are a powerful tool to study these algebraic groups.

Formally speaking, a *Moufang set* is a set X together with a collection of groups $\big(U_x \leq \mathrm{Sym}(X)\big)_{x\in X}$, such that each U_x acts regularly on $X \setminus \{x\}$, and such that $U_x^\varphi = U_{x^\varphi}$ for all $\varphi \in G^\dagger := \langle U_x \mid x \in X\rangle$. The groups U_x are called the *root groups* of the Moufang set, and the group $G^\dagger$ is called its *little projective group*. An *automorphism* of the Moufang set is an element $\psi \in \mathrm{Sym}(X)$ such that conjugation by ψ restricted to U_x is an isomorphism

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from U_x to U_{x^ψ}, for each $x \in X$. For the Moufang sets arising from algebraic groups of k-rank 1, the set X is the set of all minimal k-parabolics, and each U_x is the root subgroup of x (or, equivalently, the unipotent radical of x) with respect to a fixed maximal k-split torus. The Moufang set of an algebraic group is essentially equivalent to the algebraic group itself; more precisely, the little projective group of the Moufang set is the adjoint representation of the algebraic group.

The classical algebraic groups are very well understood, and the corresponding Moufang sets also have a satisfying and useful description. In contrast, the exceptional groups are much less understood, in particular from an elementary point of view. In this paper, we focus on the algebraic groups of absolute type F_4 and relative rank one (i.e. those of type $F_{4,1}^{21}$ in the notation of [17]). Of course, these groups are known to arise as the automorphism group of certain Albert algebras, but this point of view is often too indirect to be useful. Our aim is to give an elementary description of these Moufang sets, and we immediately illustrate its usefulness by completing Tits' program for these groups (see below).

It was known before (see [1]) that the real algebraic group G of type $F_{4,1}^{21}$ and relative $\mathbb{R}$-rank one, is isomorphic to a centralizer subgroup C of the polarity π of the real octonion projective plane $\mathbb{P}^2(\mathcal{O})$ associated with the standard involution of the real octonions $\mathcal{O}$. Moreover, both points of view define in a natural way a Moufang set as follows. On the one hand, let $B \leq G$ be a Borel subgroup of G and let U be the corresponding unipotent radical. Then the system $(B\backslash G, (U^g)_{Bg \in B\backslash G})$ defines a Moufang set. On the other hand, the group C acts on the set X of incident point-line pairs of $\mathcal{O}$ fixed under the polarity π. For any such pair $P \in X$, the intersection V_P of C with the unipotent radical of the Borel subgroup corresponding with P and related to the algebraic group of relative rank two, defined by the automorphism group of the octonion projective plane $\mathbb{P}^2(\mathcal{O})$, acts sharply transitively on $X \setminus \{P\}$. The corresponding structure $(X, (V_P)_{P \in X})$ is a Moufang set which is isomorphic to $(B\backslash G, (U^g)_{Bg \in B\backslash G})$.

In this paper, we generalize this fact to an arbitrary field of arbitrary characteristic (explicitly allowing characteristic 2), and we use this fact to give an elementary description of these Moufang sets (only using an octonion division algebra over the given field). Not surprisingly, we rely on the theory of Albert algebras to obtain this result.

As an application, we complete Jacques Tits' program on algebraic groups or relative rank one and of type BC_1 (i.e., with non-abelian root groups) for these particular Moufang sets. This program consists of determining the automorphism groups of the associated geometries. These geometries are defined as follows. Let $(X, (U_x)_{x \in X})$ be a Moufang set defined by an algebraic group of relative rank one and of type BC_1. Then the center $Z(U_x)$ of U_x coincides with the commutator subgroup U'_x and we can consider the geometry $(X, \mathcal{B})$, with

$$\mathcal{B} = \{\{x\} \cup y^{U'_x} : x, y \in X, x \neq y\},$$

with $y^{U'_x}$ the orbit of y under the action of U'_x. The conjecture of Tits, yielding a *Fundamental Theorem* for these geometries, states that $\mathrm{Aut}(X, \mathcal{B})$ is precisely equal to $\mathrm{Aut}(X, (U_x)_{x \in X})$. That is exactly what we prove for the algebraic groups of type $F^{21}_{4,1}$. In fact, our proof gives the extra information that every automorphism of the geometry $(X, \mathcal{B})$ arises from a unique collineation of $\mathbb{P}^2(\mathcal{O})$, which is a most satisfying situation.

2 The description

In order to describe the Moufang sets of type F_4, we will make use of the construction of a Moufang set out of a single group and one additional permutation, as introduced in [3]. We briefly repeat this process.

Let U be a group with composition $+$ and identity 0. (The operation $+$ is not necessarily commutative, but it makes sense to choose an additive notation.) Let X denote the disjoint union of U with $\{\infty\}$, where ∞ is a new symbol. For each $a \in U$, we denote by α_a the permutation of X which fixes ∞ and maps x to $x + a$ for all $x \in U$. Let $U_\infty := \{\alpha_a \mid a \in U\}$; this group is naturally isomorphic to U.

Now suppose that τ is a permutation of $U^* := U \setminus \{0\}$. We extend τ to an element of $\mathrm{Sym}(X)$ (which we also denote by τ) by setting $0^\tau = \infty$ and $\infty^\tau = 0$. Next we set $U_0 = U_\infty^\tau$ and $U_a = U_0^{\alpha_a}$ for all $a \in U$. Let

$$\mathbb{M}(U, \tau) := (X, (U_x)_{x \in X}).$$

Of course, this is not always a Moufang set, but it is clear that every Moufang set arises in this way. (In [3], an explicit criterion is given which determines when $\mathbb{M}(U, \tau)$ is a Moufang set.)

We will now describe the group U and the permutation τ for the Moufang sets of type F_4. Let k be an arbitrary commutative field (of any characteristic), let $\mathcal{O}$ be an octonion division algebra over k, and let N and T denote the (reduced) norm and trace from $\mathcal{O}$ to k, respectively. We will denote the standard involution of $\mathcal{O}$ by $x \mapsto \overline{x}$, so that $N(x) = x\overline{x}$ and $T(x) = x + \overline{x}$ for all $x \in \mathcal{O}$. Let

$$U := \{(a, b) \in \mathcal{O} \times \mathcal{O} \mid N(a) + T(b) = 0\}.$$

Then we can make U into a (nonabelian) group by defining the group "addition"

$$(a, b) + (c, d) := (a + c, b + d - \overline{c}a)$$

for all $(a,b),(c,d) \in U$; it is easily checked that this is indeed a group, with neutral element $(0,0)$ and with the inverse given by $-(a,b) = (-a,\overline{b})$. Now we define a permutation τ on U^*, by setting

$$\tau(a,b) = (-ab^{-1}, b^{-1})$$

for all $(a,b) \in U^*$. Then we can state the main result of this paper as follows.

Theorem 2.1. *$\mathbb{M}(U,\tau)$ is a Moufang set; it is the building of an algebraic group of type F_4 whose k-rank is 1, that is, the group $F_{4,1}^{21}$. Conversely, all Moufang sets arising as the building of an algebraic group of type $F_{4,1}^{21}$ can be obtained in this way.*

3 Octonion planes

We now give two descriptions of the Moufang projective planes $\mathbb{P}^2(\mathcal{O})$ defined over the octonion division algebra $\mathcal{O}$. The first description is very simple and direct, whereas the second description is needed to see the relation with the groups of type F_4.

We first describe $\mathbb{P}^2(\mathcal{O})$ as a point-line incidence geometry $(\mathcal{P},\mathcal{L},*)$ obtained by extending the affine plane $\mathbb{A}^2(\mathcal{O})$. The point set $\mathcal{P}$ consists of three types of points. First, we have points of the form (a,b) with $a,b \in \mathcal{O}$; secondly, we have points of the form (c) with $c \in \mathcal{O}$; and thirdly, there is one other point which we denote by (∞). The line set $\mathcal{L}$ is defined in a similar way; it consist of three types of lines: we have lines of the form $[m,k]$ with $m,k \in \mathcal{O}$, lines of the form $[\ell]$ with $\ell \in \mathcal{O}$, and one other line which we denote by $[\infty]$. The incidence relation $*$ is defined as follows.

$$\begin{array}{ll}
(a,b) * [m,k] & \iff ma + b = k \\
(a,b) * [\ell] & \iff a = \ell \\
(c) * [m,k] & \iff c = m \\
(c) * [\infty] & \forall c \in \mathcal{O} \\
(\infty) * [\ell] & \forall \ell \in \mathcal{O} \\
(\infty) * [\infty] &
\end{array}$$

(The three remaining point-line combinations are never incident.) Then $(\mathcal{P},\mathcal{L},*)$ is a Moufang projective plane, which we will denote by $\mathbb{P}^2(\mathcal{O})$.

Remark 3.1. In fact, $*$ is a subset of $(\mathcal{P}\times\mathcal{L}) \cup (\mathcal{L}\times\mathcal{P})$, that is, it consists of the above relations together with their symmetrized versions, so $L * p$ if and only if $p * L$ for all $p \in \mathcal{P}$ and all $L \in \mathcal{L}$.

Remark 3.2. This representation has the disadvantage that the line $[m,k]$ has slope $-m$ rather than m, but for historical reasons, it has become the standard way to represent a projective plane; see, for example, [5, Chapter VI].

We now give another description of $\mathbb{P}^2(\mathcal{O})$, where the points and lines are described by elements of a reduced Albert algebra. We will describe the Albert algebras as cubic norm structures [7, §38], so that our approach will work equally well if $\operatorname{char}(k)$ is 2 or 3. We let $\mathcal{O}_3$ denote the 3×3 matrices with entries in $\mathcal{O}$, and we denote the transpose of such a matrix by $x \mapsto x^t$. Let $g := \left(\begin{smallmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{smallmatrix}\right)$ for some fixed elements $\gamma_1, \gamma_2, \gamma_3 \in k^*$. We set

$$\mathcal{H}(\mathcal{O}_3, g) := \{x \in \mathcal{O}_3 \mid x = g^{-1}\overline{x}^t g \ \wedge \ \operatorname{diag}(x) \in k_3\} \, ;$$

the condition $\operatorname{diag}(x) \in k_3$ is, in fact, only needed if $\operatorname{char}(k) = 2$. We also write $\mathcal{H}(\mathcal{O}_3, (\gamma_1, \gamma_2, \gamma_3))$ for $\mathcal{H}(\mathcal{O}_3, g)$. Then every element $x \in \mathcal{H}(\mathcal{O}_3, g)$ can be written in the form

$$x = \begin{pmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3 \overline{a_2} \\ \gamma_1 \overline{a_3} & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2 \overline{a_1} & \alpha_3 \end{pmatrix} \, ;$$

we will denote this element by $x = (\alpha_1, \alpha_2, \alpha_3; a_1, a_2, a_3)$. Observe that $\mathcal{H}(\mathcal{O}_3, g)$ is a vector space of dimension 27 over k. Moreover, we denote the element $(1, 1, 1; 0, 0, 0)$ simply by 1. Now we set, for every element $x = (\alpha_1, \alpha_2, \alpha_3; a_1, a_2, a_3) \in \mathcal{H}(\mathcal{O}_3, g)$,

$$N(x) := \alpha_1\alpha_2\alpha_3 - \sum_{i=1}^{3} \gamma_j\gamma_k\alpha_i N(a_i) + \gamma_1\gamma_2\gamma_3 T(a_1a_2a_3) \, , \tag{3.1}$$

where (ijk) are cyclic permutations of (123). Then $(k, \mathcal{H}(\mathcal{O}_3, g), N, 1)$ is a cubic norm structure, which we will also denote by $\mathcal{H}(\mathcal{O}_3, g)$. Its corresponding trace and adjoint are given by the formulas

$$T(x, y) = \sum_{i=1}^{3} (\alpha_i\beta_i + \gamma_j\gamma_k T(a_i\overline{b_i})) \, ,$$

where $x = (\alpha_1, \alpha_2, \alpha_3; a_1, a_2, a_3)$ and $y = (\beta_1, \beta_2, \beta_3; b_1, b_2, b_3)$, and

$$x^\sharp = (\delta_1, \delta_2, \delta_3; d_1, d_2, d_3)$$

with $\delta_i = \alpha_j\alpha_k - \gamma_j\gamma_k N(a_i)$ and $d_i = \gamma_i \, \overline{a_j a_k} - \alpha_i a_i$ for each cyclic permutation (ijk) of (123). We will also use the notation $T(x) := T(x, 1)$. The linearization of the adjoint map is denoted by $x \times y := (x + y)^\sharp - x^\sharp - y^\sharp$; this "product" $\times$ is sometimes called the Freudenthal cross product of the cubic norm structure. Note that the quadratic maps

$$U_x : \mathcal{H}(\mathcal{O}_3, g) \to \mathcal{H}(\mathcal{O}_3, g) : y \mapsto T(x, y)x - x^\sharp \times y$$

make $\mathcal{H}(\mathcal{O}_3, g)$ into an (exceptional) quadratic Jordan algebra over k of degree 3, which we will denote by $J(\mathcal{H}(\mathcal{O}_3, g))$.

Remark 3.3. This construction is equally possible if $\mathcal{O}$ is an octonion algebra which is not a division algebra (and is hence split); the corresponding Albert algebra is then also called split. It is also possible to define Albert division algebras, as those division algebras which become isomorphic to a split Albert algebra after scalar extension to the algebraic (or separable) closure $\overline{k}$. These algebras can be described explicitly by the so-called Tits constructions, but we will not need these explicit constructions here.

Now we can define an incidence structure $(\hat{\mathcal{P}}, \hat{\mathcal{L}}, *)$ as follows. We let $\hat{\mathcal{P}} := \{(x) \mid x \in \mathcal{H}(\mathcal{O}_3, g), x \neq 0, x^\sharp = 0\}$; two elements (x) and (x') denote the same point of $\hat{\mathcal{P}}$ if and only if x and x' are proportional, i.e. $kx = kx'$. Similarly, $\hat{\mathcal{L}} := \{[y] \mid y \in \mathcal{H}(\mathcal{O}_3, g), y \neq 0, y^\sharp = 0\}$, where $[y]$ and $[y']$ denote the same line of $\hat{\mathcal{L}}$ if and only if $ky = ky'$.

We define the incidence relation $*$ by

$$(x) * [y] \iff T(x, y) = 0\,.$$

Then $(\hat{\mathcal{P}}, \hat{\mathcal{L}}, *)$ is a projective plane which is isomorphic to $\mathbb{P}^2(\mathcal{O})$; we will denote this incidence geometry by $\mathbb{P}^2(\mathcal{O})_g$.

Remark 3.4. If $\mathcal{O}$ is the split octonion algebra over k, then this incidence structure gives rise to a so-called Hjelmslev-Moufang plane. These are not projective planes (two lines may intersect at more than one point), but by introducing other relations in addition to incidence, some interesting geometric properties arise. (To be correct, the relation defined by $T(x, y) = 0$ is then called connectedness, whereas incidence has a more restrictive definition.) We refer to [13, 16, 21] for more details, and to [4] for a characteristic-free study of these geometries.

Proposition 3.5. *Let* $g := \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$. *The maps*

$$\begin{aligned}
(a, b) &\mapsto (\gamma_1 N(b), \gamma_2 N(a), \gamma_3; a, \overline{b}, b\overline{a}) \\
(c) &\mapsto (\gamma_1 N(c), \gamma_2, 0; 0, 0, -c) \\
(\infty) &\mapsto (1, 0, 0; 0, 0, 0) \\
[m, k] &\mapsto [\gamma_2\gamma_3, \gamma_1\gamma_3 N(m), \gamma_1\gamma_2 N(k); -\gamma_1 \overline{m} k, -\gamma_2 \overline{k}, \gamma_3 m] \\
[\ell] &\mapsto [0, \gamma_3, \gamma_2 N(\ell); -\ell, 0, 0] \\
[\infty] &\mapsto [0, 0, 1; 0, 0, 0]
\end{aligned}$$

form an isomorphism from $\mathbb{P}^2(\mathcal{O})$ *to* $\mathbb{P}^2(\mathcal{O})_g$*; we will denote this isomorphism by* ϕ_g.

Proof. By writing out the conditions for an element $x \in \mathcal{H}(\mathcal{O}_3, g)$ to satisfy $x^\sharp = 0$, it is easily checked that $\phi_g(\mathcal{P}) = \hat{\mathcal{P}}$ and $\phi_g(\mathcal{L}) = \hat{\mathcal{L}}$. An easy calculation then shows that $\phi_g(a, b) * \phi_g[m, k]$ if and only if $N(ma + b - k) = 0$, that is, if and only if $(a, b) * [m, k]$ (since $\mathcal{O}$ is a division algebra). A similar (and even easier) calculation settles the other cases. $\square$

Remark 3.6. Even though the Albert algebra $\mathcal{H}(\mathcal{O}_3, g)$ depends on the choice of g, the projective plane $\mathbb{P}^2(\mathcal{O})_g$ which we have obtained from it, is (up to isomorphism) independent of this choice, as follows from Proposition 3.5.

One of the reasons that the Albert algebras are well studied, is their relation with the exceptional algebraic groups of type F_4.

Theorem 3.7. *Let $\mathcal{H}(\mathcal{O}_3, g)$ be an arbitrary Albert algebra over k (so $\mathcal{O}$ is possibly split). Then* $\mathrm{Aut}(\mathcal{H}(\mathcal{O}_3, g))$ *is an algebraic group of type F_4. Let r be its k-rank; then $r \in \{0, 1, 4\}$. Moreover, $r = 4$ if and only if $\mathcal{H}(\mathcal{O}_3, g)$ is a split Albert algebra, and $r \geq 1$ if and only if $\mathcal{H}(\mathcal{O}_3, g)$ has non-zero nilpotent elements.*

Proof. Let $J := \mathcal{H}(\mathcal{O}_3, g)$. The fact that $\mathrm{Aut}(J)$ is an algebraic group of type F_4 was first proven by Chevalley and Schafer for fields of characteristic different from 2 and 3 in the context of Lie-groups [2], and generalized to arbitrary fields by Springer [14]; see also [7, (25.13) and (26.18)]. The facts about the k-rank can be found, for example, in [17, p. 60–61] for fields of characteristic different from 2 (without proof).

In general characteristic, one could argue as follows. (We are grateful to T. A. Springer for explaining the ideas to us.) Assume that the k-rank of $G = \mathrm{Aut}(J)$ is positive, i.e. G is isotropic, and let T be a maximal k-split torus of G. Consider the natural k-rational representation $\alpha : G \to GL(J_0)$, where J_0 is the subspace of trace zero elements of J (and hence $J = \langle 1 \rangle \oplus J_0$). Let v be a weight vector for T with weight χ; then $\alpha(g)(v) = \chi(g) \cdot v$ for all $g \in T$. Since $\alpha(g)$ induces an element of $\mathrm{Aut}(J)$, this implies $N(1+\chi(g)\cdot v) = N(1+v)$ for all $g \in T$, hence $N(1+tv) = N(1+v)$ for all $t \in \overline{k}^*$; by Zariski density, this constant has to be $N(1) = 1$, and it follows by [14, Prop. 3.15] that v is nilpotent. By the same argument, if $\dim(T) > 1$, there are pairwise nonproportional orthogonal nilpotent elements, which can only happen if J is split.

Conversely, if J is split, then it is k-isomorphic to $\mathcal{H}(\mathcal{O}_3, 1)$ with $\mathcal{O}$ split, and then G is split; see, for example, [15, Theorem 17.6.3(ii)]. Assume finally that J has nilpotent elements; then it follows from [15, Theorem 17.6.5(ii)] that G is isotropic over k. ◻

4 Polarities

We recall that a polarity of an incidence structure $(\mathcal{P}, \mathcal{L}, *)$ is a map π which maps $\mathcal{P}$ to $\mathcal{L}$ and $\mathcal{L}$ to $\mathcal{P}$, which is incidence preserving and of order 2. A point or a line $a \in \mathcal{P} \cup \mathcal{L}$ is called absolute (with respect to π) if $a * \pi(a)$.

The octonion plane $\mathbb{P}^2(\mathcal{O})_g = (\hat{\mathcal{P}}, \hat{\mathcal{L}}, *)$ inside $\mathcal{H}(\mathcal{O}_3, g)$ has a very natural polarity, which is simply given by $\hat{\pi}_g : (x) \leftrightarrow [x]$. It is clear from the definitions of $\mathbb{P}^2(\mathcal{O})_g$ and $\hat{\pi}_g$ that every element of $\mathrm{Aut}(\mathcal{H}(\mathcal{O}_3, g))$ induces a collineation of $\mathbb{P}^2(\mathcal{O})_g$ which commutes with the polarity $\hat{\pi}_g$. More precisely:

Theorem 4.1. *Let G be the collineation group of $\mathbb{P}^2(\mathcal{O})_g$, and let $G^\dagger$ be the little projective group of $\mathbb{P}^2(\mathcal{O})_g$ (that is, the subgroup of G generated by all elations of the projective plane). Let ψ be the natural group morphism from $\mathrm{Aut}(\mathcal{H}(\mathcal{O}_3, g))$ to G. Then ψ is an isomorphism from $\mathrm{Aut}(\mathcal{H}(\mathcal{O}_3, g))$ to $\mathrm{Cent}_{G^\dagger}(\hat{\pi}_g)$, the group consisting of the elements of $G^\dagger$ which commute with $\hat{\pi}_g$.*

Proof. Let $J := J(\mathcal{H}(\mathcal{O}_3, g))$. Following [6], we denote by $M_1(J)$ the group of norm-preserving linear maps from J to itself. (This is, in fact, an algebraic group of type E_6, but we will not need this.) Then for every element $\eta \in M_1(J)$, there is a corresponding element $\hat{\eta} \in M_1(J)$ such that $T(a, b) = T(a^\eta, b^{\hat{\eta}})$ for all $a, b \in J$; we have that $\eta = \hat{\eta}$ if and only if $\eta \in \mathrm{Aut}(J)$. Every such an $\eta \in M_1(J)$ induces a collineation of $\mathbb{P}^2(\mathcal{O})_g$ by the maps

$$\mathcal{P} \to \mathcal{P} : (x) \mapsto (x^\eta) ; \qquad \mathcal{L} \to \mathcal{L} : [y] \mapsto [y^{\hat{\eta}}] ; \tag{4.1}$$

so the map ψ extends to a map from $M_1(J)$ to G. In fact, we have that $\psi(M_1(J)) = G^\dagger$. In characteristic different from two, this is [6, Chap IX, Sect. 8, Thm. 13]; in general characteristic, this follows from [4, Theorem 4.3 together with the argument preceding Lemma 4.9]. Note that Jacobson uses the term "elation" where Faulkner uses "transvection".

On the other hand, it follows from (4.1) that $\eta = \hat{\eta}$ if and only if $\psi(\eta)$ commutes with $\hat{\pi}_g$, and hence $\psi(\mathrm{Aut}(\mathcal{H}(\mathcal{O}_3, g))) = \mathrm{Cent}_{G^\dagger}(\hat{\pi}_g)$.

It remains to show that ψ is injective. But this follows immediately from [6, Chap IX, Sect. 8, Lemma 1]. (Jacobson's proof works in all characteristics. Note that his cross product $\times$ is half of ours.) $\square$

Using Proposition 3.5, it is easily verified that this polarity $\hat{\pi}_g$ corresponds to the polarity π_g of $\mathbb{P}^2(\mathcal{O})$ given by the maps

$$\begin{aligned}
(a, b) &\leftrightarrow [\gamma_1^{-1}\gamma_2\ \overline{ab^{-1}}, -\gamma_1^{-1}\gamma_3\ \overline{b^{-1}}] && (b \neq 0) \\
(a, 0) &\leftrightarrow [-\gamma_2^{-1}\gamma_3\ \overline{a^{-1}}] && (a \neq 0) \\
(0, 0) &\leftrightarrow [\infty] \\
(c) &\leftrightarrow [-\gamma_1^{-1}\gamma_2\ \overline{c^{-1}}, 0] && (c \neq 0) \\
(0) &\leftrightarrow [0] \\
(\infty) &\leftrightarrow [0, 0] \ .
\end{aligned}$$

We would now like to know under which conditions these polarities have absolute points, and we will need the quadratic Jordan algebra $J =$

$J(\mathcal{H}(\mathcal{O}_3, g))$ for this. Even though there is no multiplication on J, it makes sense to define powers of elements of J, by setting $x^0 := 1$, $x^1 := 1$, and then recursively $x^{n+2} := U_x(x^n)$ for all natural numbers n; in particular, $x^2 = U_x(1)$ and $x^3 = U_x(x)$. We first recall some well known properties of quadratic Jordan algebras of degree 3.

Lemma 4.2. *For all $x, y \in J$, we have*

(i) $x^3 - T(x)x^2 + T(x^\sharp)x - N(x)1 = 0$ *;*

(ii) $x^\sharp = x^2 - T(x)x + T(x^\sharp)1$ *;*

(iii) $2T(x^\sharp) = T(x)^2 - T(x^2)$ *;*

(iv) $T(x \times y) = T(x)T(y) - T(x, y)$ *.*

Proof. See, for example, [10]. □

Lemma 4.3. *The polarity $\hat{\pi}_g$ has absolute points if and only if there exists an element $x \in J(\mathcal{H}(\mathcal{O}_3, g))$ such that $x \neq 0$ and $x^2 = 0$.*

Proof. We first note that the Jordan algebra $J = J(\mathcal{H}(\mathcal{O}_3, g))$ is power associative (see [8] for a precise characteristic-free formal definition of this notion). This follows from [8, Proposition 1], since an Albert algebra never contains non-zero absolute zero divisors (i.e. elements $z \neq 0$ such that $U_z = 0$); see, for example, [9, p. 209]. In particular, if $z^n = 0$ for some $z \in J$ and some natural number n, then $z^m = 0$ for every natural number $m \geq n$.

Now let $(x) \in \mathcal{P}$; then $x \neq 0$ and $x^\sharp = 0$ by the definition of $\mathcal{P}$. By definition, (x) is absolute with respect to $\hat{\pi}_g$ if and only if $(x) * [x]$, that is, if and only if $T(x, x) = 0$. Since $x \times x = 2x^\sharp = 0$, it follows from Lemma 4.2(iv) that $T(x, x) = T(x)^2$, and hence (x) is absolute if and only if $T(x) = 0$. Also, Lemma 4.2(ii) simplifies to $x^2 = T(x)x$, and hence (x) is absolute if and only if $x^2 = 0$.

It remains to show that, if $x \in J$ is such that $x \neq 0$ and $x^2 = 0$, then $x^\sharp = 0$. Because of the power associativity, $x^2 = 0$ implies that $x^3 = 0$ as well. Hence Lemma 4.2(i) implies that $T(x^\sharp)x = N(x)1$, and since x cannot be a multiple of 1, this implies that $T(x^\sharp) = 0$. By Lemma 4.2(iii), $T(x) = 0$, and by Lemma 4.2(ii) we can conclude that $x^\sharp = 0$. □

Lemma 4.4. *The polarity $\hat{\pi}_g$ has absolute points if and only if $J(\mathcal{H}(\mathcal{O}_3, g))$ has non-zero nilpotent elements.*

Proof. Since $J = J(\mathcal{H}(\mathcal{O}_3, g))$ is power associative, $x^n = 0$ for some $x \in J \setminus \{0\}$ and some natural number n implies $y^2 = 0$ for $y = x^{\lceil n/2 \rceil}$, and hence the result follows from the previous lemma. □

Theorem 4.5. *Let $\mathcal{H}(\mathcal{O}_3, g)$ be a non-split Albert algebra over k. Then the following are equivalent.*

(i) *The polarity $\hat{\pi}_g$ has absolute points;*

(ii) *$J(\mathcal{H}(\mathcal{O}_3, g))$ has non-zero nilpotent elements;*

(iii) *$\mathcal{H}(\mathcal{O}_3, g) \cong \mathcal{H}(\mathcal{O}_3, (1, -1, 1))$;*

(iv) *$\mathrm{Aut}(\mathcal{H}(\mathcal{O}_3, g))$ has k-rank 1.*

Proof. The equivalence (i)$\Leftrightarrow$(ii) is Lemma 4.4; the equivalence (ii)$\Leftrightarrow$(iv) follows from Theorem 3.7. The equivalence (ii)$\Leftrightarrow$(iii) (for fields of arbitrary characteristic) follows from Springer's Theorem [12, 11]; see also [10]. $\square$

It is clear from (iv) of this theorem that this is the case we are interested in; we will assume from now on that $\hat{\pi}_g$ has absolute points. By (iii), we may in fact assume that $g = \mathrm{diag}(1, -1, 1)$. The polarity π_g now gets the easy form

$$\begin{array}{llll} (a,b) \leftrightarrow [-\overline{ab^{-1}}, -\overline{b^{-1}}] & (b \neq 0)\,; & (c) \leftrightarrow [\overline{c^{-1}}, 0] & (c \neq 0)\,; \\ (a,0) \leftrightarrow [\overline{a^{-1}}] & (a \neq 0)\,; & (0) \leftrightarrow [0]\,; & \\ (0,0) \leftrightarrow [\infty]\,; & & (\infty) \leftrightarrow [0,0]\,. & \end{array} \tag{4.2}$$

Clearly, the flags $(1,0) * [1]$ and $(1) * [1,0]$ are absolute with respect to π_g. It will often be more convenient to have the flags $(\infty) * [\infty]$ and $(0,0) * [0,0]$ as absolute flags, so we will apply a transformation which maps the former to the latter. It turns out that the resulting polarity π then gets the very easy form

$$(a,b) \leftrightarrow [\overline{a}, -\overline{b}]\,; \qquad (c) \leftrightarrow [\overline{c}]\,; \qquad (\infty) \leftrightarrow [\infty]\,. \tag{4.3}$$

The absolute points of this polarity π are given by the set

$$X = \{(a,b) \in \mathcal{P} \mid N(a) + T(b) = 0\} \cup \{(\infty)\}\,.$$

Observe that $X \setminus \{(\infty)\}$ is precisely the set U as in section 2.

5 Moufang structure

We would now like to recover the Moufang structure on the set X arising from the rank one group $\mathrm{Aut}_k(\mathcal{H}(\mathcal{O}_3, g))$. By Theorem 4.1, this group acts on X, and its action is obtained by restricting the action of the little projective group $G^\dagger$ of the projective plane $\mathbb{P}^2(\mathcal{O})$ to the action of $H^\dagger := \mathrm{Cent}_{G^\dagger}(\pi)$ on the set X. In particular, the action of the unipotent radical of a minimal parabolic k-subgroup of the group $H^\dagger$ (which is, by Theorem 3.7 and Theorem 4.1, an algebraic group of absolute type F_4 of k-rank 1) is obtained by the restriction to X of the action of the unipotent radical of a minimal parabolic k-subgroup of the group $G^\dagger$ (which is an algebraic group of absolute type E_6 of k-rank 2).

We will now explicitly compute this action. We fix an arbitrary minimal k-parabolic subgroup of $H^\dagger$, that is to say, we fix an arbitrary maximal flag of X; see, for example, [18, Theorem 5.2]. Recall that $H^\dagger$ has k-rank one, so maximal flags of X are simply elements of X. We choose the element (∞) as our maximal flag; the corresponding maximal flag of $\mathbb{P}^2(\mathcal{O})$ is $\big((\infty),(\infty)^\pi\big) = \big((\infty),[\infty]\big)$. The minimal k-parabolic subgroup P_H corresponding to (∞) in $H^\dagger$ is a k-subgroup of the minimal k-parabolic subgroup P_G corresponding to $\big((\infty),[\infty]\big)$ in $G^\dagger$. We may now choose an arbitrary maximal k-split torus T_G of P_G (all maximal k-split tori are k-conjugate). Again by [18, Theorem 5.2], this amounts to choosing an arbitrary apartment containing the flag $\big((\infty),[\infty]\big)$; we choose the apartment X through the points (0), $(0,0)$ and (∞). The corresponding unipotent radical U_G of P_G is equal to the product of the three root groups corresponding to the pair $\big(X,\big((\infty),[\infty]\big)\big)$. Let U_1 be the group of collineations fixing all points on the line $[0]$ and all lines through the point (∞); let U_2 be the group of collineations fixing all lines through the point (∞) and all points on the line $[\infty]$; let U_3 be the group of collineations fixing all points on the line $[\infty]$ and all lines through the point (0). Then $U_G = U_1U_2U_3$. More explicitly, we have $U_1 = \{x_1(M) \mid M \in \mathcal{O}\}$ where

$$x_1(M): \begin{array}{ccc} (a,b) & \mapsto & (a, b - Ma) \\ [m,k] & \mapsto & [m+M, k] \end{array} ;$$

$U_2 = \{x_2(B) \mid B \in \mathcal{O}\}$ where

$$x_2(B): \begin{array}{ccc} (a,b) & \mapsto & (a, b + B) \\ [m,k] & \mapsto & [m, k+B] \end{array} ;$$

$U_3 = \{x_3(A) \mid A \in \mathcal{O}\}$ where

$$x_3(A): \begin{array}{ccc} (a,b) & \mapsto & (a+A, b) \\ [m,k] & \mapsto & [m, k+mA] \end{array} .$$

(We have omitted the actions on the points (c) and the lines $[\ell]$; it is clear how to extend these maps.) It follows that $U_G = \{x(A,B,M) \mid A,B,M \in \mathcal{O}\}$ where $x(A,B,M) := x_1(M)x_2(B)x_3(A)$, and hence

$$x(A,B,M): \begin{array}{ccc} (a,b) & \mapsto & (a+A, b+B-Ma) \\ [m,k] & \mapsto & [m+M, k+B+mA+MA] \end{array} . \tag{5.1}$$

The subgroup of U_G consisting of the elements which stabilize the set X, will be the unipotent subgroup U_H of P_H corresponding to its k-split torus which is the stabilizer of the apartment $\{(0,0),(\infty)\}$ of the Moufang set with underlying set X (i.e. the rank 1 building of the group $H^\dagger$ over k).

Since the k-rank of $H^\dagger$ is 1, U_H will be equal to the root group U_∞ of the Moufang set X.

So let $\varphi = x(A,B,M)$ be an element of U_G stabilizing X. In particular, by (5.1), φ maps the flag $(0,0) * [0,0]$ to the flag $(A,B) * [M, B+MA]$, and this flag has to be fixed under π. Hence $M = \overline{A}$ and $B + MA = -\overline{B}$, or equivalently, $(A,B) \in X$ and $M = \overline{A}$. We will write $x(A,B)$ for $x(A,B,\overline{A})$, for all $(A,B) \in X$. Conversely, a simple calculation using (5.1) shows that every element of the form $x(A,B)$ with $(A,B) \in X$, stabilizes X. It follows that $U_\infty = U_H = \{x(A,B) \mid (A,B) \in X\}$ and, again by (5.1), that its action on X is given by

$$x(A,B) : (a,b) \mapsto (a + A, b + B - \overline{A}a) \ . \tag{5.2}$$

In particular,

$$x(A,B) \cdot x(C,D) = x(A + C, B + D - \overline{C}A) \ ,$$

and therefore U_∞ is indeed the group U as we have described in section 2.

It only remains to find a collineation σ of $\mathbb{P}^2(\mathcal{O})$ which commutes with π and interchanges the points $(0,0)$ and (∞); let τ be the restriction to X of such a map σ. Since σ is a collineation, $U_\infty^\tau = U_H^\sigma$ will be a root group again, and will therefore coincide with the root group $U_{(0,0)}$ of the Moufang set X. It follows that the construction of $\mathbb{M}(U_\infty, \tau)$ as described in section 2 will then indeed yield the Moufang set which is the rank 1 building corresponding to $H^\dagger$.

Such a collineation σ is given by the map

$$\sigma : \begin{array}{ccc} (a,b) & \mapsto & (-ab^{-1}, b^{-1}) \\ [m,k] & \mapsto & [k^{-1}m, k^{-1}] \end{array} \ ,$$

and its restriction τ to X is indeed the map τ as described in section 2. This finishes the proof of Theorem 2.1.

6 Tits' program

Let X be the set of absolute points of the polarity π of $\mathbb{P}^2(\mathcal{O})$. We endow X with all subsets of X obtained by intersecting X with lines of $\mathbb{P}^2(\mathcal{O})$ that meet X in at least two points. We denote this family of subsets by $\mathcal{B}$ and call its elements *blocks*. As we will show in Lemma 6.1 below, the incidence structure $\mathbb{U}(\mathcal{O}) := (X, \mathcal{B})$ is the *Tits unital* corresponding to the Moufang set $\mathbb{M}(U_\infty, \tau)$ *relative to* $Z(U_\infty)$, i.e. the elements of $\mathcal{B}$ are precisely the images under $\langle U_\infty, U_0 \rangle$ of the subset $\{(\infty)\} \cup (0,0)^{Z(U_\infty)} = [0] \cap X$ (where we use the standard notation x^G for the orbit of the element x under the action of the group G). In particular, the groups U_∞ and U_0, and hence also the little projective group $S := \langle U_\infty, U_0 \rangle$, are automorphism groups of $\mathbb{U}(\mathcal{O})$, and hence the full automorphism group of $\mathbb{U}(\mathcal{O})$ acts doubly transitively on X.

Lemma 6.1. *The elements of $\mathcal{B}$ are precisely the images under $\langle U_\infty, U_0 \rangle$ of the subset $\{(\infty)\} \cup (0,0)^{Z(U_\infty)} = [0] \cap X$.*

Proof. We have to show that an arbitrary block of $\mathbb{U}(\mathcal{O})$ is the intersection of $\mathbb{U}(\mathcal{O})$ with a line of $\mathbb{P}^2(\mathcal{O})$. Since both the centralizer of the polarity π in the automorphism group of $\mathbb{P}^2(\mathcal{O})$ and the little projective group S of the Moufang set $\mathbb{M}(U_\infty, \tau)$ act doubly transitively on X, it suffices to show this for the block defined by the orbit under $Z(U_\infty)$ of the point $(0,0)$. An element $x(a,b) \in U_\infty$ lies in the center if and only if $\overline{c}a = \overline{a}c$ for all $c \in \mathcal{O}$, and this happens precisely when $a = 0$; it follows that the orbit of $(0,0)$ under $Z(U_\infty)$ is equal to $\{(0,b) \mid T(b) = 0\}$, and this is precisely the intersection of the line $[0]$ with the set X. $\square$

We now address the question whether the full automorphism group of $\mathbb{U}(\mathcal{O})$ is induced by automorphisms of $\mathbb{P}^2(\mathcal{O})$. This contributes to the fundamental program explained by Jacques Tits in his lectures at Collège de France, see [20], which has as aim to determine the full automorphism group of all unital-like geometries defined by Moufang sets with nonabelian root groups.

We will answer the question affirmatively. In the following theorem, $\mathrm{Aut}(X, \mathcal{B})$ denotes the set of permutations of X that preserve the family $\mathcal{B}$. Also, $\mathrm{Aut}(X, (U_x)_{x\in X})$ is the set of permutations of X that, under conjugation, preserve the family $\{U_x : x \in X\}$.

Theorem 6.2. *Let $(X, (U_x)_{x\in X})$ be an arbitrary Moufang set of type F_4, and let*

$$\mathcal{B} = \{\{x\} \cup y^{Z(U_x)} : x, y \in X, x \neq y\} .$$

Then

$$\mathrm{Aut}(X, \mathcal{B}) = \mathrm{Aut}(X, (U_x)_{x\in X}) .$$

Moreover, if we consider an arbitrary embedding of the Tits unital $\mathbb{U}(\mathcal{O}) = (X, \mathcal{B})$ in $\mathbb{P}^2(\mathcal{O})$ (as set of absolute points of an associated polarity ρ), then every element of $\mathrm{Aut}(X, \mathcal{B})$ arises from a unique collineation of $\mathbb{P}^2(\mathcal{O})$ which stabilizes the unital $\mathbb{U}(\mathcal{O})$.

Since it is clear that every collineation of $\mathbb{P}^2(\mathcal{O})$ which stabilizes $\mathbb{U}(\mathcal{O})$ induces an element of $\mathrm{Aut}(X, (U_x)_{x\in X})$, and also that every element of $\mathrm{Aut}(X, (U_x)_{x\in X})$ induces an element of $\mathrm{Aut}(X, \mathcal{B})$, it suffices to show the last part. We start with some lemmas.

Lemma 6.3. *Let $a, b \in X$, $a \neq b$. For a block B containing b, let $S_a(B)$ be the set of blocks containing a and meeting B. If B, B' are blocks containing b, then $S_a(B) = S_a(B')$ if and only if $B = B'$.*

Proof. By the double transitivity, we may assume $a = (\infty)$ and $b = (0, 0)$. The other points of X are (x, y), with $x\overline{x} + y + \overline{y} = 0$, $x, y \in \mathcal{O}$. Now let B and B' be two blocks of $\mathbb{U}(\mathcal{O})$ subtended from the lines L, L', respectively. Clearly, if $B = B'$, then $S_a(B) = S_a(B')$. Hence, we assume from now on that $S_a(B) = S_a(B')$ and show that $B = B'$. Let L and L' have coordinates $[\ell, 0]$ and $[\ell', 0]$, respectively. The line with coordinates $[x]$ subtends a block of $\mathbb{U}(\mathcal{O})$ that meets B if and only if there exists $y \in \mathcal{O}$ with $x\overline{x} + y + \overline{y} = 0$ and $\ell x + y = 0$. This is equivalent with $x\overline{x} = \ell x + \overline{x}\overline{\ell}$. Our assumption now implies that

$$\Xi := \{x \in \mathcal{O} \mid x\overline{x} = \ell x + \overline{x}\overline{\ell}\} = \{x \in \mathcal{O} \mid x\overline{x} = \ell' x + \overline{x}\overline{\ell}'\} =: \Xi' .$$

Hence all $x \in \Xi$ satisfy the relation $\ell x + \overline{x}\overline{\ell} = \ell' x + \overline{x}\overline{\ell}'$, which can be written as $T((\ell - \ell')x) = 0$. Since we assume $\ell \neq \ell'$, and since not all elements of $\mathcal{O}$ have trace zero, we may select an element $y \in \mathcal{O}$ such that $T((\ell - \ell')y) \neq 0$. Let now p vary over the center k of $\mathcal{O}$, and express that py belongs to Ξ. We obtain the condition $p^2 y\overline{y} = p(\ell y + \overline{y}\overline{\ell})$. Hence there is a unique nonzero solution p_0 such that $p_0 y \in \Xi$. But clearly $T((\ell - \ell')p_0 y) = p_0 T((\ell - \ell')y) \neq 0$, a contradiction. □

This has the following consequence. We denote by G the full automorphism group of $\mathbb{U}(\mathcal{O})$.

Corollary 6.4. *The little projective group S of the Moufang set $\mathbb{M}(X_\infty, \tau)$ is a normal subgroup of G. Also, S is uniquely determined by $\mathbb{U}(\mathcal{O})$.*

Proof. We show that $Z(U_\infty)$ is equal to the set of elements of G which fix all blocks through (∞); the result will then follow from the fact that $S = \langle Z(U_a) \mid a \in X \rangle$.

It is easy to check that $Z(U_\infty)$ indeed fixes all these blocks, and acts sharply transitively on the points of any of these blocks except for the point (∞). Now suppose that some element $g \in G$ does not belong to $Z(U_\infty)$ and fixes all blocks that contain (∞). Upon replacing g by a suitable element of $gZ(U_\infty)$ we may suppose that g fixes some point b. Now put $(\infty) = a$ and apply the previous lemma. We obtain that g must fix all blocks through b, but since there is a unique block through every two points, it must then clearly fix all points of $\mathbb{U}(\mathcal{O})$. □

We now consider an arbitrary embedding (as set of absolute points of an associated polarity ρ) of $\mathbb{U}(\mathcal{O})$ in $\mathbb{P}^2(\mathcal{O})$. Every block B is contained in some line L_B, and the blocks C for which L_C is incident with L_B^ρ will be called *conjugate to* B. Our next aim is to prove that we can recognize this conjugacy relation in the abstract Tits unital.

But before we proceed, we construct a collection of collineations of $\mathbb{P}^2(\mathcal{O})$ which lie in the little projective group.

Lemma 6.5. *Let $r, s \in \mathcal{O}^*$. Then the maps*

$$\begin{aligned} (a,b) &\mapsto \big(s(a\cdot sr), rb\cdot sr\big)\,, \text{ for all } a,b\in\mathcal{O}\,, \\ [m,k] &\mapsto \big[r\cdot ms^{-1}, rk\cdot sr\big]\,, \text{ for all } m,k\in\mathcal{O}\,, \end{aligned} \tag{6.1}$$

form a collineation of $\mathbb{P}^2(\mathcal{O})$ which lies in the little projective group. Moreover, if $N(r)=N(s)=1$, then this collineation commutes with the polarity $\pi_{(1,-1,1)}$ as given in (4.2), so in particular, it stabilizes the unital $\mathbb{U}(\mathcal{O})$ (with respect to this embedding), and therefore induces an element of S.

Proof. Let $J := \mathcal{H}(\mathcal{O}_3,(1,-1,1))$, and consider the map $\chi_{r,s}$ from J to itself given by

$$\begin{aligned} \chi_{r,s} : (\alpha,\beta,\gamma;a,b,c) \mapsto (&\alpha N(r), \beta N(s), \gamma N(r)^{-1}N(s)^{-1}; \\ &N(r)^{-1}N(s)^{-1}s(a\cdot sr), N(r)^{-1}N(s)^{-1}\overline{r}\,\overline{s}\cdot b\overline{r}, r\cdot c\overline{s}) \end{aligned}$$

for all $\alpha,\beta,\gamma\in k$ and all $a,b,c\in\mathcal{O}$. This map is clearly k-linear. Using the formula (3.1) for the norm N of J, and recalling that $\mathcal{O}$ is "commutative and associative under the trace", it is not very hard to calculate that $\chi_{r,s}$ is norm-preserving; therefore, by (4.1), it induces an element of the little projective group of $\mathbb{P}^2(\mathcal{O})$. Its companion automorphism $\hat{\chi}_{r,s}$ is given by

$$\begin{aligned} \hat{\chi}_{r,s} : (\alpha,\beta,\gamma;a,b,c) \mapsto (&\alpha N(r)^{-1}, \beta N(s)^{-1}, \gamma N(r)N(s); \\ &N(s)^{-1}s(a\cdot sr), N(r)^{-1}\overline{r}\,\overline{s}\cdot b\overline{r}, N(r)^{-1}N(s)^{-1}r\cdot c\overline{s}) \end{aligned}$$

for all $\alpha,\beta,\gamma\in k$ and all $a,b,c\in\mathcal{O}$. Indeed, it is easily checked that $T(x^\eta, y^{\hat{\eta}}) = T(x,y)$ for all $x,y\in J$.

Moreover, if $N(r)=N(s)=1$, then $\chi_{r,s}=\hat{\chi}_{r,s}$, and hence these maps are automorphisms (alternatively, these maps then fix the element $1=(1,1,1;0,0,0)$); by Theorem 4.1, the induced collineation then commutes with the polarity $\pi_{(1,-1,1)}$.

It remains to check (for arbitrary r and s again) that the collineation induced by the pair $(\eta_{r,s},\hat{\eta}_{r,s})$ as in (4.1) is given by the formulas (6.1). It follows from Proposition 3.5 with $\gamma_1=-\gamma_2=\gamma_3=1$ that the isomorphism ϕ from $\mathbb{P}^2(\mathcal{O})$ to $\mathbb{P}^2(\mathcal{O})_{(1,-1,1)}$ is given by the maps

$$\begin{aligned} (a,b) &\mapsto \big(N(b), -N(a), 1; a, \overline{b}, b\overline{a}\big)\,, &&\text{for all } a,b\in\mathcal{O}\,, \\ [m,k] &\mapsto \big[-1, N(m), -N(k); -\overline{m}k, \overline{k}, m\big]\,, &&\text{for all } m,k\in\mathcal{O}\,; \end{aligned}$$

recall that the coordinates of the elements of $\mathbb{P}^2(\mathcal{O})_{(1,-1,1)}$ are only defined up to scalar multiplication. Now one can compute from these maps, together with the formula for $\chi_{r,s}$ for the points (where it suffices to look at the third, fourth and fifth coordinates) and for $\hat{\chi}_{r,s}$ for the lines (where it suffices to look at the first, fifth and sixth coordinates) that the induced collineation is given by

$$\begin{aligned}
(a,b) &\mapsto \big(s(a \cdot sr), \overline{\overline{r}\,\overline{s} \cdot \overline{b}\,\overline{r}}\big)\,, \qquad \text{for all } a,b \in \mathcal{O}\,,\\
[m,k] &\mapsto \big[N(s)^{-1}r \cdot m\overline{s}, \overline{\overline{r}\,\overline{s} \cdot \overline{k}\,\overline{r}}\big]\,, \text{ for all } m,k \in \mathcal{O}\,;
\end{aligned}$$

using the fact that $s^{-1} = N(s)^{-1}\overline{s}$, we get the formulas (6.1) as required. □

Lemma 6.6. *Let B be any block. Then the stabilizer in S of B acts transitively on the set of blocks conjugate to B.*

Proof. We consider the embedding as given at the end of Section 4, so that the polarity ρ takes the form (4.3). We may choose $L_B = [0]$; then $L_B^\rho = (0)$. A line $[0,k]$ through (0) meets the unital nontrivially (meaning: in at least two points) if and only if $-T(k)$ is a nonzero norm n in $\mathcal{O}$. Let r be an arbitrary nonzero element of $\mathcal{O}$. Consider the map g_r, $r \in \mathcal{O}^*$, from $\mathbb{P}^2(\mathcal{O})$ to itself, given by

$$\begin{aligned}
(a,b) &\mapsto (\overline{r}r^{-1} \cdot a\overline{r}, rb \cdot \overline{r})\,, \quad \text{for all } a,b \in \mathcal{O}\,,\\
[m,k] &\mapsto [r(m \cdot r\overline{r}^{-1}), rk \cdot \overline{r}]\,, \text{ for all } m,k \in \mathcal{O}\,.
\end{aligned}$$

By Lemma 6.5 with $s = \overline{r}r^{-1}$, we see that g_r is a collineation of $\mathbb{P}^2(\mathcal{O})$ which belongs to the little projective group. It is easy to check, using the fact that any subalgebra of $\mathcal{O}$ which is generated by two elements is associative, that g_r commutes with the polarity ρ and hence is an automorphism of the unital $\mathbb{U}(\mathcal{O})$; since it lies in the little projective group of $\mathbb{P}^2(\mathcal{O})$, it follows that $g_r \in S$ for all $r \in \mathcal{O}^*$.

Now let $[0,k']$ be a second line meeting the unital nontrivially, and let $-T(k') = n'$, with n' a nonzero norm in $\mathcal{O}$. Then $n'n^{-1}$ is also a nonzero norm in $\mathcal{O}$, say $n'n^{-1} = N(r)$. So g_r maps $[0,k]$ onto some line $[0,k'']$ with $T(k'') = n'n^{-1}T(k) = T(k')$ and hence $T(k'-k'') = 0$.

Now the mapping $E(k'-k'') \in Z(U_\infty)$ defined by

$$\begin{aligned}
(a,b) &\mapsto (a, b+k'-k'')\,, \text{ for all } a,b \in \mathcal{O}\,,\\
[m,z] &\mapsto [m, z+k'-k'']\,, \text{ for all } m,z \in \mathcal{O}\,,
\end{aligned}$$

maps $[0,k'']$ to $[0,k']$ and so $g_sE(k'-k'')$ maps $[0,k]$ onto $[0,k']$ and belongs to S. □

This now implies the following:

Lemma 6.7. *The stabilizer in S of any block B acts transitively on the points of $\mathbb{U}(\mathcal{O})$ off B.*

Proof. In view of the previous lemma, we only need to show that the stabilizer in S of two conjugate blocks B and B' acts transitively on B'.

To show this, consider the first embedding of the unital, where the unital is given by the polarity (4.2). More exactly, the point set consists of the points with coordinates (a, b), with $N(a) - N(b) = 1$, together with the "points at infinity" (c), with $N(c) = 1$. We may take for B the block consisting of the points at infinity, and for B' the block consisting of the points $(a, 0)$, with $N(a) = 1$. Let $x, x' \in \mathcal{O}$ have norm 1, then put $r = x^{-1}x'$ (hence $N(r) = 1$), and consider the map h_r from $\mathbb{P}^2(\mathcal{O})$ to itself defined by

$$\begin{aligned}(a, b) &\mapsto (ar, rbr)\,, \text{ for all } a, b \in \mathcal{O}\,,\\ [m, k] &\mapsto [rm, rkr]\,, \text{ for all } m, k \in \mathcal{O}\,.\end{aligned}$$

Then h_r maps $(x, 0)$ to $(x', 0)$ and by Lemma 6.5 with $s = 1$, $h_r \in S$. $\square$

The next lemma characterizes and recognizes conjugate blocks.

Lemma 6.8. *For every block B and every point P of the unital off B, there is a unique block B' through P which is invariant under the stabilizer in S of B and P. The blocks B and B' are conjugate.*

Proof. We consider the embedding used in the previous lemma, and we may take, by the previous lemma, for B the block of points at infinity, and for P the point $(1, 0)$. Let s be an arbitrary element of $\mathcal{O}$ with norm 1. Then by Lemma 6.5 with $r = s^{-2}$, the map i_s from $\mathbb{P}^2(\mathcal{O})$ to itself defined by

$$\begin{aligned}(a, b) &\mapsto (s \cdot as^{-1}, s^{-2}b \cdot s^{-1})\,, \qquad \text{for all } a, b \in \mathcal{O}\,,\\ [m, k] &\mapsto [s^{-2} \cdot ms^{-1}, s^{-2}k \cdot s^{-1}]\,, \text{ for all } m, k \in \mathcal{O}\,.\end{aligned}$$

belongs to S, and it clearly fixes both B and P. Let B' be the block through $(1, 0)$ conjugate to B, and suppose that some block $C \neq B'$ containing $(1, 0)$ is fixed by all such i_s. Since the line (1) does not define a block (it is a *tangent line*, the absolute line equal to the polar image of $(1, 0)$), there exists a nonzero element $m \in \mathcal{O}^*$ such that $C = [m, m]$. Hence we have the identity

$$s^{-2}ms^{-1} = m\,,$$

for all $s \in \mathcal{O}$ with $N(s) = 1$. Equivalently, $s^2ms = m$, for all norm 1 elements s. Putting $s = -1$, we see that $m = -m$, hence the characteristic of k is equal to 2. It now follows that, for all elements $x \in \mathcal{O}$, the identity

$$x^2mx = \overline{x}^2m\overline{x}$$

holds (putting $s = x\overline{x}^{-1}$).

Substituting $x + 1$ for x and cancelling out the common terms m and $x^2mx = \overline{x}^2m\overline{x}$, we obtain

$$(x + \overline{x})^2 m = m(x + \overline{x})$$

for all $x \in \mathcal{O}$, and hence $T(x) \in \{0, 1\}$ for all $x \in \mathcal{O}$ with $T(x) \neq 0$, which is a contradiction since the octonions over the field of two elements is not s division algebra. $\square$

From now on, we consider the embedding of $\mathbb{U}(\mathcal{O})$ in $\mathbb{P}^2(\mathcal{O})$ where the unital is given by the polarity (4.3). We show that every $g \in G$ is induced by a collineation of $\mathbb{P}^2(\mathcal{O})$. Since G acts two-transitively on X, we may assume that g fixes the points (∞) and $(0, 0)$.

A block defined by the intersection of a line L with the point set X of the unital will be denoted by $B(L)$.

Every line $[a]$ defines a block through (∞); hence there is a permutation φ of $\mathcal{O}$ such that the image under g of the block $B([a])$ is the block $B([a^\varphi])$. We have $0^\varphi = 0$; let $c := 1^\varphi$.

If $m \neq 0$, then the block $B([\overline{m}, 0])$ is the conjugate to $B([m])$ through $(0, 0)$; hence $B([\overline{m}, 0])^g = B([\overline{m^\varphi}, 0])$.

Let (a, b) be a point of $\mathbb{U}(\mathcal{O})$ distinct from (∞). Then (a, b) lies on the block $B([0, b])$. This block is conjugate to $B([0])$, and is hence mapped onto some block $B([0, b^\alpha])$. This defines a permutation α of all elements $x \in \mathcal{O}$ with $-T(x)$ a norm in $\mathcal{O}$. Note $(a, b)^g = (a^\varphi, b^\alpha)$. If $a = 0$, then $T(b) = 0$ and g maps the point $(0, b)$ onto the point $(0, b^\alpha)$. Hence α permutes the elements with trace zero amongst themselves.

Since if $a \neq 0$, the point (a, b) is also incident with the block $B([\overline{-ba^{-1}}, 0])$, we deduce that $(a, b)^g = (a^\varphi, -\overline{(-\overline{a}^{-1}\overline{b})^\varphi} a^\varphi)$ and hence $x^\alpha = -\overline{(-\overline{a}^{-1}\overline{x})^\varphi} a^\varphi$, for all $a \in \mathcal{O}^*$ with $N(a) + T(x) = 0$.

Now consider the line $[\overline{m}, -\overline{l}]$ with $m \neq 0$. This line defines a block if and only if $N(m)+T(l) \neq 0$ and there is some $(a, b) \in X$ with $\overline{m}a+b+\overline{l} = 0$. The latter is equivalent with

$$\begin{aligned} 0 &= N(a) - T(\overline{m}a + \overline{l}) \\ &= N(a - m) - N(m) - T(l). \end{aligned}$$

Thus, $[\overline{m}, -\overline{l}]$ defines a block if and only if $N(m) + T(l) \in N(\mathcal{O}^*)$. All lines defining a block conjugate to $B([\overline{m}, -\overline{l}])$ are incident with (m, l). Amongst these are the lines $[m]$ and $[-lm^{-1}, 0]$. We deduce that

$$B([\overline{m}, -\overline{l}])^g = B([\overline{m^\varphi}, \overline{m^\varphi}(-\overline{m}^{-1}\overline{l})^\varphi]) \,.$$

Note that, if $T(l) = 0$ and $m \neq 0$, then $[\overline{m}, -\overline{l}]$ always defines a block. Since its image under g is incident with $(0,l)^g = (0,l^\alpha)$, we deduce that

$$l^\alpha = \overline{m^\varphi}(-\overline{m}^{-1}\overline{l})^\varphi = \overline{m^\varphi}(\overline{m}^{-1}l)^\varphi\,, \tag{6.2}$$

for all l with $T(l) = 0$ and all $m \in \mathcal{O}^*$. In particular we can choose $m = 1$ to obtain

$$l^\alpha = c(-\overline{l})^\varphi = cl^\varphi\,, \tag{6.3}$$

for all $l \in \mathcal{O}$ with $T(l) = 0$.

Let us denote the set of all elements with trace zero by $\mathcal{O}_0$. Note that $\mathcal{O}_0$ is a hyperplane in the 8-dimensional k-space $\mathcal{O}_0$. Also, we remarked above that α permutes the elements of $\mathcal{O}_0$ (so $\mathcal{O}_0^\alpha = \mathcal{O}_0$).

The next lemma proves the additivity of φ.

Lemma 6.9. *For all $x, y \in \mathcal{O}$ we have $(x+y)^\varphi = x^\varphi + y^\varphi$. In particular, $(-x)^\varphi = -(x^\varphi)$.*

Proof. We start by choosing an arbitrary element $t \in \mathcal{O}_0$ such that $tx^{-1} \cdot y \in \mathcal{O}_0$. Such t certainly exists since the sets $\mathcal{O}_0$ and $\{u \in \mathcal{O} \,:\, T(ux^{-1} \cdot y) = 0\}$ are hyperplanes in the eight-dimensional k-vector space $\mathcal{O}$, and hence these hyperplanes meet nontrivially. We now construct the following sequence of blocks, starting from $B^0_{x,y,t} = B([\overline{x}, 0])$.

- The unique block $B^1_{x,y,t}$ through (∞) conjugate to $B^0_{x,y,t}$ is $B([x])$.
- The unique block $B^2_{x,y,t}$ through $(0,t)$ conjugate to $B^1_{x,y,t}$ is $B([\overline{x}, -\overline{t}])$.
- The unique block $B^3_{x,y,t}$ through $(0,0)$ conjugate to $B^2_{x,y,t}$ is $B([-tx^{-1}, 0])$.
- The unique block $B^4_{x,y,t}$ through (∞) conjugate to $B^3_{x,y,t}$ is $B([-\overline{tx^{-1}}])$.
- The unique block $B^5_{x,y,t}$ conjugate to both $B^4_{x,y,t}$ and $B([\overline{y}, o])$ is $B([-tx^{-1}, -(tx^{-1})y])$.
- The unique block $B^6_{x,y,t}$ through $(0,t)$ conjugate to $B^5_{x,y,t}$ is $B([\overline{x+y}, -\overline{t}])$.
- The unique block $B^7_{x,y,t}$ through (∞) conjugate to $B^6_{x,y,t}$ is $B([x+y])$.

We now apply the map g to this sequence of blocks. By the consecutive uniqueness of the defined blocks (and noting that we substitute x^φ, y^φ and $t^\alpha \in \mathcal{O}_0$ for x, y and t, respectively), the block $(B^7_{x,y,t})^g$ coincides on the one hand with $B([(x+y)^\varphi])$, and on the other hand with $B^7_{x^\varphi,y^\varphi,t^\alpha} = B([x^\varphi + y^\varphi])$.

Note that we did not need the condition $t^\alpha (x^\varphi)^{-1} \cdot y^\varphi \in \mathcal{O}_0$. The reason is that the condition $tx^{-1} \cdot y \in \mathcal{O}_0$ is only needed to be sure that the line $[-tx^{-1}, -tx^{-1} \cdot y]$ defines a block. When we construct the above sequence of blocks using x^φ, y^φ and t^α, we are sure that at every step we get the image of the corresponding block under g. Hence, in Step 5 above, we *know* that there is a unique block conjugate to both $B^4_{x^\varphi,y^\varphi,t^\alpha}$ and $B([y^\varphi,0])$, and we *know* that it contains a point of the block $B([0])^g = B([0])$, and so we *know* that $-t^\alpha(x^\varphi)^{-1}\cdot y^\varphi \in \mathcal{O}_0$. $\square$

We now extend Equation (6.2) to all elements l for which l^α is defined.

Lemma 6.10. *For every $b \in \mathcal{O}$ with $T(b) \in -N(\mathcal{O})$, and every $m \in \mathcal{O}^*$, we have*

$$b^\alpha = \overline{m^\varphi}(\overline{m}^{-1}b)^\varphi = cb^\varphi.$$

Proof. Let $a \in \mathcal{O}^*$ be such that $T(b) = -N(a)$. We may assume that $a \neq m$ by replacing a by $\overline{x}x^{-1}$, with $x \notin k$, if necessary. Putting $l = -(\overline{m}a+b)$, we deduce from the above that $[\overline{m},-\overline{l}]$ defines a block through the point (a,b) of $\mathbb{U}(\mathcal{O})$. Hence $(a,b)^g = (a^\varphi,b^\alpha)$ is contained in $[\overline{m},-\overline{l}] = [\overline{m^\varphi},\overline{m^\varphi}(-\overline{m}^{-1}\overline{l})^\varphi]$. This means that

$$\overline{m^\varphi}a^\varphi + b^\alpha = \overline{m^\varphi}(-\overline{m}^{-1}\overline{l})\varphi\,,$$

and substituting l by $-(\overline{m}a+b)$ and using the additivity of φ, we get $b^\alpha = \overline{m^\varphi}(\overline{m}^{-1}b)^\varphi = cb^\varphi$ (the latter by putting $m=1$). $\square$

We now extend α to $\mathcal{O}$ by defining $b^\alpha = cb^\varphi$. By Lemma 6.10, this is well defined and by Lemma 6.9, α is additive.

We now prove that Lemma 6.10 holds for all $b \in \mathcal{O}$.

Lemma 6.11. *For every $b \in \mathcal{O}$ and every $m \in \mathcal{O}^*$, we have $b^\alpha = \overline{m^\varphi}(\overline{m}^{-1}b)^\varphi$.*

Proof. By the additivity of both φ and α, it suffices to show that every element b of $\mathcal{O}$ is the sum of elements whose trace is plus or minus a norm. Let $x \in \mathcal{O}$ have trace 1, i.e., $T(x) = 1$. Then

$$b = (b - T(b)x) + N(1+b)x - x - N(b)x$$

is the sum of four elements with respective traces $T(b - T(b)x) = 0 = N(0)$, $T(N(1+b)x) = N(1+b)$, $T(-x) = -1 = -N(1)$ and $T(-N(b)x) = -N(b)$. $\square$

Putting $m' = \overline{m}^{-1}b$ in the previous lemma, we obtain the identity

$$(\overline{m}m')^\varphi = \overline{c}^{-1}(\overline{m^\varphi}m'^\varphi)\,, \tag{6.4}$$

for all $m, m' \in \mathcal{O}$.

We can now finish the proof of the fact that g is a collineation of $\mathbb{P}^2(\mathcal{O})$.

Lemma 6.10 implies that the action of g on the points of the unital is given by $(x,y) \mapsto (x^\varphi, \overline{c}y^\varphi)$. Using the calculation rules given in Lemma 6.11 and Identity 6.4, one easily checks that the mapping

$$\begin{aligned}(x,y) &\mapsto (x^\varphi, \overline{c}y^\varphi)\,, \text{ for all } x,y \in \mathcal{O}\,,\\ [m,t] &\mapsto [\overline{\overline{m}^\varphi}, \overline{c}t^\varphi]\,, \text{ for all } m,t \in \mathcal{O}\,,\end{aligned}$$

is a collineation of $\mathbb{P}^2(\mathcal{O})$ and extends g.

This shows the main Theorem 6.2 of this section.

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